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ON THE CONVERGENCE OF AVERAGE PRODUCTIVITY OF LABOUR AMONG ECONOMIES

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Abstract. This article deals with the various approaches to the verification of absolute convergence and conditional convergence of average productivity of labour among economies. In our contribution hypothesis of absolute and conditional convergence is originally formulated and original verification of the both convergences within the augmented Solow-Swan model is conducted.

In the paper two concepts of the convergence of average productivity of labour are distinguished: absolute and conditional. The hypothesis that poor economies tend to grow faster per capita than rich ones - without conditioning on any other characteristics of economies - is referred to as absolute convergence. If steady states differ, then we consider a concept of conditional convergence ([1]).

The concept of productivity convergence was derived from the Solow-Swan neoclassical growth model (Solow, 1956; Swan, 1956). This model examines stability of GDP per capita around the steady state assuming exogenously given constant growth rate of population. In the following years the model was augmented by the assumption of the constant rate of depreciation of capital goods within economy and exogenous technical progress.

We will consider the augmented Solow-Swan model of economic growth given by the following conditions:

$$I = i.Y, i > 0, \tag{1}$$

$$I = K' + \delta K, \delta > 0, \tag{2}$$

$$Y = F\left(K, A\left(t\right).L\right), \tag{3}$$

$$\frac{L'}{L} = n, n > 0, \tag{4}$$

where Y - production, I - investments, K - capital, L - labour, $A\left(t\right) = e^{x.t}$ - rate of technological progress, i - marginal rate to investment, δ - depreciation rate, α - elasticity of change in capital with respect to change in production, n - the growth rate of labour, $' = \frac{d}{dt}$, t - time.

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Production function $F(.)$ is homogeneous of the first degree and satisfies the following conditions:

$$\frac{\partial F}{\partial L} > 0, \frac{\partial^2 F}{\partial L^2} < 0, \frac{\partial F}{\partial K} > 0, \frac{\partial^2 F}{\partial K^2} < 0 \cdot \tag{5}$$

It is also assumed that $F(.)$ satisfies the Inada conditions:

$$\lim_{K \to 0} F_K = \lim_{L \to 0} F_L = \infty, \lim_{K \to \infty} F_K = \lim_{L \to \infty} F_L = 0. \tag{6}$$

We will work with the variable average productivity of labour defined by $y = \frac{Y}{L}$ and capital-labour ratio defined by $k = \frac{K}{L}$. If we differentiate $K = k.L$ with respect to time and utilize (4), we obtain

$$K' = k'L + k.nL. \tag{7}$$

Putting (1) and (3) into (2) the condition for change in capital stock can be written in the formula:

$$K' = i.F(K, A(t).L) - \delta K. \tag{8}$$

If we utilize $K = k.L$ and apply the fact that $F(.)$ is a homogeneous function of the first degree, we can rewrite (8):

$$K' = i.L.F(k, A(t)) - \delta.k. \tag{9}$$

Comparing right-hand sides of (7) and (8) we obtain an expression for the growth rate of capital-labour ratio:

$$\frac{k'}{k} = \frac{i.F(k, A(t))}{k} - (n + \delta). \tag{10}$$

Because $F(.)$ is a homogeneous function of the first degree, new formula for average productivity of labour can be derived:

$$y = \frac{Y}{L} = \frac{F(K, A(t).L)}{L} = F(k, A(t)). \tag{11}$$

Effective labour is defined by $\hat{L} \equiv A(t).L$. R. J. Barro and X. X. Sala-i-Martin ([1]) work with the variable capital per effective labour defined by $\hat{k} \equiv k/A(t)$ and with the variable average productivity of effective labour defined by $\hat{y} \equiv y/A(t)$. Because $F(.)$ is a homogeneous function of the first degree, (11) can be re-written in the following formula:

$$\hat{y} = F\left(\hat{k}, 1\right) \equiv f\left(\hat{k}\right), \tag{12}$$

where $f(.)$ is the intensive production function. Putting $\hat{k} = k/A(t)$, (11) and (12) into (10) we have:

$$\frac{\hat{k}'}{\hat{k}} = \frac{i.f\left(\hat{k}\right)}{\hat{k}} - (x + n + \delta). \tag{13}$$

We will consider a Cobb-Douglas production function:

$$F(K, A(t).L) = K^{\alpha}.[A(t).L]^{1-\alpha}.$$

Utilizing Cobb-Douglas production function in (11) and (12) we obtain $f\left(\hat{k}\right) = \hat{k}^{\alpha}$, i.e. $\hat{y} = \hat{k}^{\alpha}$. Differentiating of $\hat{y} = \hat{k}^{\alpha}$ with respect to time we can write:

$$\frac{\hat{y}'}{\hat{y}} = \alpha.\frac{\hat{k}'}{\hat{k}}. \tag{14}$$

In [1] it is shown that $i.\frac{f(\hat{k})}{\hat{k}}$ is a decreasing function of time. As a consequence, according to (13) and (14) $\hat{y}'/\hat{y}$ and $\hat{k}'/\hat{k}$ are also decreasing functions of time. Utilizing these information hypotheses of absolute convergence and conditional convergence is verified on the basis of graphical analysis (see [1]).

G. Gandolfo ([2]) verified hypothesis of absolute convergence within the simple Solow-Swan model of economic growth (technological progress $A(t)$ is not considered). He proved that

$$\frac{\partial(y'/y)}{\partial y_0} < 0, \tag{15}$$

where y_0 is the initial level of average productivity of labour. The growth rate of average productivity of labour is inversely related to the initial level of average productivity of labour.

We will formulate hypothesis of absolute convergence and hypothesis of conditional convergence of average productivity of labour among economies precisely and find conditions of their validity. We will consider a Cobb-Douglas production function

$$F(K, A(t).L) = e^{\mu.t}K^{\alpha}L^{1-\alpha},$$

where $\mu \equiv x.(1-\alpha)$ is a measure of technological progress. Because $F(.)$ is a homogeneous function of the first degree, new formula for average productivity of labour can be derived:

$$y = \frac{Y}{L} = \frac{F(K, A(t).L)}{L} = F(k, A(t)) = e^{\mu.t}.k^{\alpha}. \tag{16}$$

We will distingusih 2 types of economies. We say that economy is less developed economy if it has lower initial level of capital-labour ratio. Developed economy is an economy with higher initial level of capital-labour ratio. In the paper we will denote less developed economies by subindex 1 and developed economies by subindex 2.

Hypothesis of Absolute Convergence of Average Productivity of Labour among Economies

Consider two groups of economies with the same marginal rate to investment, i, the same depreciation rate, δ, the same growth rate of labour force, n, but different initial levels of capital-labour ratio, $k_{0,1} < k_{0,2}$. Under the certain circumstances the difference between average productivity of labour of less developed economies and average productivity of labour of developed economies is a decreasing function of time and it converges to zero.

We have just stated hypothesis of absolute convergence of average productivity of labour among economies. Putting (16) into (10) and multiplying the both sides of this equation by k, we obtain a basic differential equation of the Sollow-Swan model:

$$k' + (n+\delta)\,k = ie^{\mu t}k^{\alpha}. \tag{17}$$

Equation (17) is a Bernoulli differential equation. Utilizing $q = r^{1-\alpha}$ we can transfer it into a homogeneous linear differential equation:

$$q' + (n+\delta)(1-\alpha)\,q = ie^{\mu t}(1-\alpha)\,. \tag{18}$$

We can find its particular solution in the form $q_p = ae^{\mu t}$:

$$q_p = \frac{i(1-\alpha)}{\mu + (n+\delta)(1-\alpha)}e^{\mu t}. \tag{19}$$

The solution of equation (17) given by the initial condition: $q(0) = q_0$ is:

$$q(t) = \left(q_0 - \frac{i(1-\alpha)}{\mu + (n+\delta)(1-\alpha)}\right)e^{-(n+\delta)(1-\alpha)t} + \frac{i(1-\alpha)}{\mu + (n+\delta)(1-\alpha)}e^{\mu t}. \tag{20}$$

Putting $k = q^{1/(1-\alpha)}$, $k(0) = k_0$ and $q_0 = k_0^{1-\alpha}$ into (20) we can come back to variable $k(t)$:

$$k(t) = \left\{\left(k_0^{1-\alpha} - \frac{i(1-\alpha)}{\mu + (n+\delta)(1-\alpha)}\right)e^{-(n+\delta)(1-\alpha)t} + \right.$$

$$\left. + \frac{i(1-\alpha)}{\mu + (n+\delta)(1-\alpha)}e^{\mu t}\right\}^{\frac{1}{1-\alpha}}. \tag{21}$$

We will utilize the following denotations: $A = \frac{\mu}{1-\alpha}$, $B_1 = k_{0,1}^{1-\alpha} - \frac{i(1-\alpha)}{\mu+(1-\alpha)(n+\delta)}$, $B_2 = k_{0,2}^{1-\alpha} - \frac{i(1-\alpha)}{\mu+(1-\alpha)(n+\delta)}$, $C = (n+\delta)(1-\alpha) + \mu$, $D = \frac{i(1-\alpha)}{\mu+(1-\alpha)(n+\delta)}$, $E = B_1e^{-Ct} + D$ and $F = B_2e^{-Ct} + D$, where A, B_1, B_2, C, D, E and F are positive constants. Now, formula (22) can be written for less developed and for developed economies:

$$y_1(t) = e^{At}\left(B_1e^{-Ct} + D\right)^{\frac{\alpha}{1-\alpha}}, \tag{23}$$

$$y_2(t) = e^{At}\left(B_2e^{-Ct} + D\right)^{\frac{\alpha}{1-\alpha}}. \tag{24}$$

Because $k_{0,1} < k_{0,2}$, it holds: $B_1 < B_2$. Comparing (23) and (24) under assumption $B_1 < B_2$ we find out that average productivity of labour of less developed economies is in every moment lower than average productivity of labour of developed ones:

$$\forall t \geqslant 0; y_1(t) < y_2(t)\,. \tag{25}$$

Utilizing (23) and (24) we can count limits:

$$\lim_{t\to\infty}\frac{y_1(t)}{y_2(t)} = \lim_{t\to\infty}\frac{\left(B_1e^{-Ct} + D\right)^{\frac{\alpha}{1-\alpha}}}{\left(B_2e^{-Ct} + D\right)^{\frac{\alpha}{1-\alpha}}} = \frac{D^{\frac{\alpha}{1-\alpha}}}{D^{\frac{\alpha}{1-\alpha}}} = 1, \tag{26}$$

$$\lim_{t\to\infty} y_1(t) = \infty, \tag{27}$$

$$\lim_{t\to\infty} y_2(t) = \infty. \tag{28}$$

We are going to count a limit:

$$\lim_{t\to\infty} \left(y_1(t) - y_2(t)\right),$$

which is $\infty - \infty$ type. We will transform it into the limit of $\frac{0}{0}$ type:

$$\lim_{t\to\infty} \left(y_1(t) - y_2(t)\right) = \lim_{t\to\infty} \frac{\frac{y_1}{y_2} - 1}{\frac{1}{y_2}}. \tag{29}$$

Because

$$\left(\frac{y_1}{y_2} - 1\right)' = \frac{\alpha}{1-\alpha} \frac{E^{\frac{2\alpha-1}{1-\alpha}} Ce^{-Ct}\left(B_2E - B_1F\right)}{F^{\frac{1}{1-\alpha}}}, \tag{30}$$

$$\left(\frac{1}{y_2}\right)' = -e^{-At} F^{\frac{1}{\alpha-1}} \left(AF - \frac{\alpha}{1-\alpha} B_2 Ce^{-Ct}\right), \tag{31}$$

$\lim\limits_{t\to\infty} \frac{(y_1/y_2-1)'}{(1/y_2)'}$ exists. Utilizing l'Hospital rule, (30) and (31) we will continue counting the limit (29):

$$\lim_{t\to\infty} \left(y_1(t) - y_2(t)\right) = \lim_{t\to\infty} \frac{\left(\frac{y_1}{y_2} - 1\right)'}{\left(\frac{1}{y_2}\right)'} =$$

$$= \lim_{t\to\infty} \frac{-\alpha}{1-\alpha} \frac{\left(B_1 e^{-Ct} + D\right)^{\frac{2\alpha-1}{1-\alpha}} D\left(B_2 - B_1\right) Ce^{(A-C)t}}{\left(AF - \frac{\alpha}{1-\alpha} B_2 Ce^{-Ct}\right)}. \tag{32}$$

We can distinguish three cases:

1. If $\frac{\mu\alpha}{1-\alpha} - (n+\delta)(1-\alpha) > 0$, then

$$\lim_{t\to\infty} \left(y_1(t) - y_2(t)\right) = -\infty.$$

In this case average productivity of labour of developed economies straggles with increasing time from average productivity of labour of less developed ones (Fig. 1).

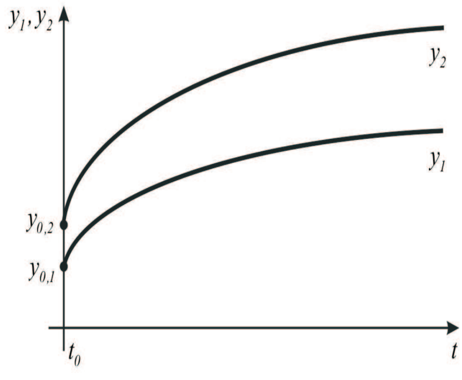

Fig. 1

2. If $\frac{\mu\alpha}{1-\alpha} - (n+\delta)(1-\alpha) = 0$ then

$$\lim_{t\to\infty}(y_1(t) - y_2(t)) = -\frac{\alpha}{\mu}\left(\frac{i(1-\alpha)}{\mu + (n+\delta)(1-\alpha)}\right)^{\frac{2\alpha-1}{1-\alpha}}\left(k_{0,2}^{1-\alpha} - k_{0,1}^{1-\alpha}\right) < 0.$$

It implies that average productivity of labour of less developed economies will not achieve average productivity of labour of developed ones over time (Fig. 2).

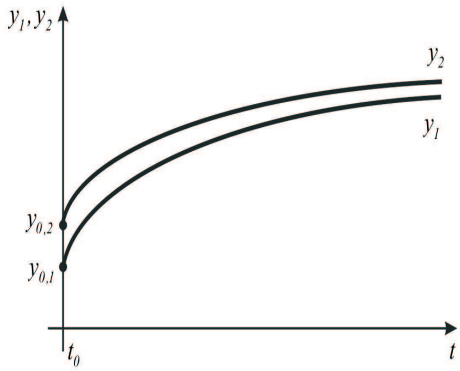

Fig. 2

3. If $\frac{\mu\alpha}{1-\alpha} - (n+\delta)(1-\alpha) < 0$ then

$$\lim_{t\to\infty}(y_1(t) - y_2(t)) = 0.$$

In this case the difference between average productivity of labour of less developed economies and average productivity of labour of developed ones converges to zero over time (Fig. 3).

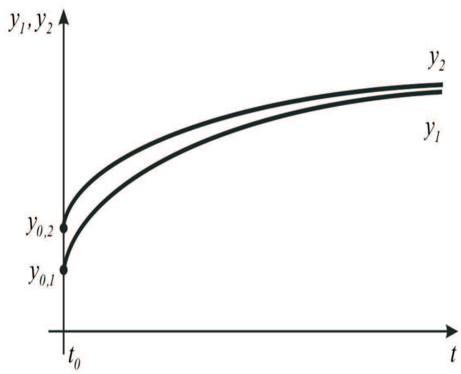

Fig. 3

On the basis of the above analysis we can formulate the following theorem.

Theorem 1. Consider two groups of economies with the same marginal rate to investment, i, the same depreciation rate, δ, the same growth rate of labour force, n, but different initial levels of capital-labour ratio, $k_{0,1} < k_{0,2}$, within the augmented Solow-Swan model given by the conditions (1) - (6), (16). Then:

1. If $\frac{\mu\alpha}{1-\alpha} - (n+\delta)(1-\alpha) = 0$, then $\lim_{t\to\infty}(y_1(t) - y_2(t)) = -\infty$.
2. If $\frac{\mu\alpha}{1-\alpha} - (n+\delta)(1-\alpha) = 0$, then $\lim_{t\to\infty}(y_1(t) - y_2(t)) < 0$.
3. If $\frac{\mu\alpha}{1-\alpha} - (n+\delta)(1-\alpha) < 0$, then $\lim_{t\to\infty}(y_1(t) - y_2(t)) = 0$.

Hypothesis of Conditional Convergence of Average Productivity of Labour among Economies

Consider two groups of economies with the same depreciation rate, δ, the same growth rate of labour, n, but different marginal rate to investment, $i_1 \neq i_2$, and the initial levels of capital-labour ratio, $k_{0,1} < k_{0,2}$. Then: if marginal rate to investment, i_1, in less developed economies exceeds marginal rate to investment, i_2, in developed ones, average productivity of labour of less developed economies will reach average productivity of labour of developed ones.

We have just stated hypothesis of conditional convergence of average productivity of labour among economies. We will utilize the following denotations: $G_1 = k_{0,1}^{1-\alpha} - \frac{i_1(1-\alpha)}{\mu+(1-\alpha)(n+\delta)}$, $G_2 = k_{0,2}^{1-\alpha} - \frac{i_2(1-\alpha)}{\mu+(1-\alpha)(n+\delta)}$, $H_1 = \frac{i_1(1-\alpha)}{\mu+(1-\alpha)(n+\delta)}$, $H_2 = \frac{i_2(1-\alpha)}{\mu+(1-\alpha)(n+\delta)}$, where G_1, G_2, H_1, H_1 and H_2 are positive constants. Applying these denotations in (22) we obtain:

$$y_1(t) = e^{At}\left(G_1 e^{-Ct} + H_1\right)^{\frac{\alpha}{1-\alpha}}, \tag{33}$$

$$y_2(t) = e^{At}\left(G_2 e^{-Ct} + H_2\right)^{\frac{\alpha}{1-\alpha}}. \tag{34}$$

We are going to count a limit $\lim_{t\to\infty}(y_1(t) - y_2(t))$. We will distinguish two cases related to marginal rate to investment.

1. If $i_1 > i_2$, then from (33) and (34) we have:

$$\lim_{t\to\infty}(y_1(t) - y_2(t)) = \infty. \tag{35}$$

Under the assumption that less developed economies invest more in relation to production than developed ones ($i_1 > i_2$) according to (33) and (34) it holds: $y_{0,1} < y_{0,2}$ and as a consequence of (35) average productivity of labour of less developed economies must reach average productivity of labour of developed ones. We will find a time t^* when it happens. Equality $y_1(t) = y_2(t)$ is equivalent to equality $y_1^{\frac{1-\alpha}{\alpha}}(t) = y_2^{\frac{1-\alpha}{\alpha}}(t)$. Utilizing (33) and (34) in this equality we obtain:

$$t^* = \frac{1}{(n+\delta)(1-\alpha)+\mu}\ln\frac{k_{0,2}^{1-\alpha} - k_{0,1}^{1-\alpha} + (i_1 - i_2)z}{(i_1 - i_2)z}, \tag{36}$$

where z denotes $\frac{1-\alpha}{\mu+(1-\alpha)(n+\delta)}$.

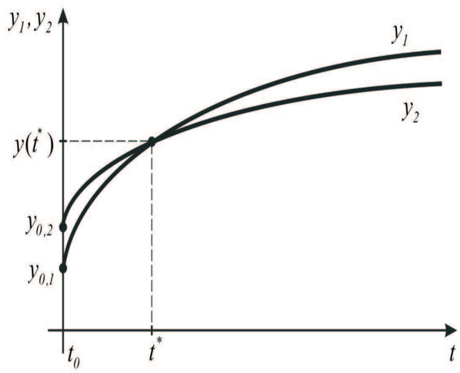

Fig. 4

Formula (36) determines a time, when average productivity of labour of less developed economies reaches average productivity of labour of developed ones (Fig. 4).

2. If $i_1 < i_2$, then from (33) and (34) we obtain:

$$\lim_{t\to\infty} (y_1(t) - y_2(t)) = -\infty. \tag{37}$$

If less developed economies invest less in relation to production than developed ones ($i_1 < i_2$), average productivity of labour of less developed economies will not reach average productivity of labour of developed ones. Under assumptions: $k_{0,1} < k_{0,2}$, $i_1 < i_2$, it holds: $G_1 < G_2, H_1 < H_2$. According to (33), (34) we find out that average productivity of labour of less developed economies is in every moment lower than average productivity of labour of developed ones:

$$\forall t \geqslant 0; y_1(t) < y_2(t). \tag{38}$$

According to (37) and (38) the difference between average productivity of labour of less developed economies and average productivity of labour of developed ones is an increasing function of time (Fig. 5).

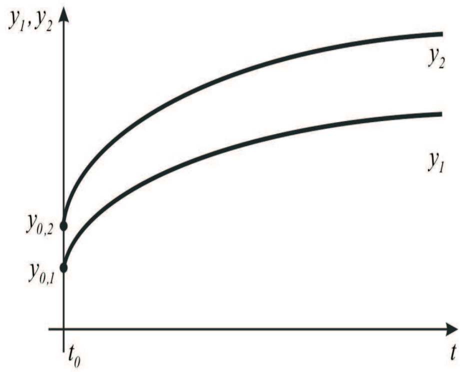

Fig. 5

On the basis of the given analysis we can formulate the following theorem.

Theorem 2. Consider two groups of economies with the same depreciation rate, δ, the same growth rate of labour, n, but different marginal rate to investment, $i_1 \neq i_2$, and different initial levels of capital-labour ratio, $k_{0,1} < k_{0,2}$, within the augmented Solow-Swan model given by the conditions (1) - (6), (16). Then:

1. If $i_1 > i_2$, then $\lim_{t \to \infty} (y_1(t) - y_2(t)) = \infty$ and there exists a time t^*, determined by (36), in which $y_1(t^*) = y_2(t^*)$.
2. If $i_1 < i_2$, then $\lim_{t \to \infty} (y_1(t) - y_2(t)) = -\infty$.

Conclusion

First, we found out the sufficient condition for absolute convergence of the difference between average productivity of labour of less developed economies and average productivity of labour of developed ones to zero (see Theorem 1, part 3.). width Second, we found out a time, in which average productivity of labour of less developed economies reaches average productivity of labour of developed ones which proves conditional convergence of average productivity of labour among economies.

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