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MINIMAL ECCENTRIC SEQUENCES WITH TWO VALUES

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Abstract. An eccentric sequence is called minimal if it has no proper eccentric subsequence with the same number of distinct eccentricities. A graph is said to be a minimal graph if it realizes a minimal eccentric sequence. All minimal eccentric sequences of type $(4^\alpha, 5^\beta)$ are described and a conjecture about all minimal eccentric sequences of type $(r^\alpha, (r+1)^\beta)$ is proposed.

1. Introduction

Characterization of eccentric sequences is considered to be an important problem in graph theory (see Problem 1 in [2]). It is very difficult. The authors of the present paper guess that even finding all minimal eccentric sequences with the least member 4 will probably require several years. In this paper all minimal eccentric sequences of type $(4^\alpha, 5^\beta)$ are described and a conjecture on all minimal eccentric sequences of type $(r^\alpha, (r+1)^\beta)$ is proposed ($\alpha \neq 0$ and $\beta \neq 0$). It is known that there are exactly six minimal eccentric sequences with the least eccentricity at most two ([1], Theorem 9.5) and there are exactly 13 minimal eccentric sequences with the least eccentricity three (see [4]). In [5] some minimal graphs and some minimal eccentric sequences with the least eccentricity r are described (e.g. r^{2r} are all minimal eccentric sequences with one value).

We consider undirected connected finite graphs without loops and multiple edges. We will use standard notations of the graph theory (see for example [3]). We recall some of them. The vertex set of a graph G is denoted by $V(G)$, while the edge set is denoted by $E(G)$. A cycle of length m is denoted by C_m. The subgraph of G induced by the edges of a path or a cycle is also referred to as a path or a cycle of G. The *circumference* of G (denoted by $c(G)$) is the length of any longest cycle of G. We denote by $d_{G'}(u,v)$ the distance between vertices $u, v \in V(G')$ in the subgraph G' of the graph G. The distance between a vertex $u \in V(G)$ and a subgraph G' of G will be denoted by $d_G(v, G')$, i.e. $d_G(v, G') = \min\{d_G(v,x); x \in V(G')\}$.

Denote the degree of a vertex $u \in V(G)$ by $\deg_G(u)$ and the eccentricity of a vertex $u \in V(G)$ by $e_G(u)$. Recall that

$$e_G(u) = \max\{d_G(u,v);\ v \in V(G)\}.$$

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We denote it briefly by $e(u)$ when no confusion can arise. We will use the symbol rad G to denote the radius of the graph G (i.e. the minimum of eccentricities of vertices of G). The symbol diam G is used for the diameter of G (i.e. the maximum of eccentricities of its vertices). We write simply r and d when there is no confusion.

The eccentric sequence of a graph G is a list of the eccentricities of its vertices in nondecreasing order. Since often there are some vertices having the same eccentricity we will denote it simply

$$e(G) = (e_1^{m_1}, e_2^{m_2}, \dots, e_k^{m_k})$$

where e_i are eccentricities for which $e_i < e_{i+1}$ and m_i is the multiplicity of e_i. A sequence of positive integers is called eccentric if there is a graph which realizes the considered sequence. L. Lesniak showed that a sequence S of positive integers is eccentric if and only if some subsequence S' of S with the same number of distinct values is eccentric (see [6]). An eccentric sequence is called minimal if it has no proper eccentric subsequence with the same number of distinct eccentricities. Throughout the paper, any graph which realizes a minimal eccentric sequence is said to be a minimal graph.

We recall terminology which were defined in [5]. A cycle C in a graph G is called a *geodesic cycle* if for each two vertices x, y of C it holds $d_C(x,y) = d_G(x,y)$. A vertex of the cycle C of length $2k$ or $2k+1$ is said to be C-*excited* (in the graph G) if its eccentricity is larger than k. The number of the C-excited vertices of G will be denoted by $\mathrm{exc}_G(C)$. A graph G is said to be a *sun-graph* if it is unicyclic (i.e. it has exactly one cycle C), $\deg_G(u) \le 3$ for $u \in V(C)$ and $\deg_G(u) \le 2$ for $u \in V(G) - V(C)$.

By [5] the following statement holds.

Lemma 1.1. *Let C be a cycle of a graph G and $|V(G)| - |V(C)| = m$. Then*
a) $exc_G(C) \le 2m - 1$ if length of C is even and $m \ge 1$,
b) $exc_G(C) \le 2m$ if length of C is odd,
c) $exc_G(C) \le 2m - n$ if C is an even cycle and there are at least n vertices from $V(G) - V(C)$ such that each of them is adjacent to at least one vertex of C.

2. The Main Results

Theorem 2.1. *Let $r \ge 3$ and $e(G) = (r^\alpha, (r+1)^\beta)$. Then*

a) *there exists a block B of G which contains all cut-vertices of G and moreover with the property that for every $u \in V(G) - V(B)$ it holds $d_G(u, B) = 1$,*
b) *for circumference of G and for the block B from the previous case it holds*
$$c(G) \ge c(B) \ge 2r - 2,$$
c) *if $c(G) < 2r$ then $\alpha \ge 2r - 2$,*
d) *if $|V(G)| \le 3r - 2$ then $c(G) \ge c(B) \ge 2r$.*

Proof. a) The statement evidently holds if there is no cut-vertex of G. Let us denote by A the set of all cut-vertices of G, $A = \{u_1, u_2, \dots, u_n\}$. Let G_1' be a component of graph $G - u_1$ which contains the vertex v_1 such that $d_G(u_1, v_1) \ge r$. Then obviously $d_G(u_1, v_1) = r$ (and so $e_G(u_1) = r$) and for every vertex $x \in V(G) - V(G_1')$ it holds inequality $d_G(x, u_1) \le 1$. Let us denote by G_1 the subgraph of G induced by the vertex set $V(G_1') \cup \{u_1\}$ (i.e. $G_1 = \langle V(G_1') \cup \{u_1\} \rangle_G$). Then u_1 is not a cut-vertex of G_1 and G_1 has the cut-vertices $u_2, u_3 \dots, u_n$ (if $n > 1$). If $n = 1$, i.e. u_1 is the only cut-vertex of G then we have $B = G_1$. If $n > 1$ then u_2 is the cut-vertex of G_1 and

$e_G(u_2) = r$. Let $d_{G_1}(u_2, v_2) = r$ and let us denote by G_2' the component of graph $G_1 - \{u_2\}$ which contains the vertex v_2. Let $G_2 = \langle V(G_2') \cup \{u_2\} \rangle_G$. Obviously, $A \subseteq V(G_2)$ and u_1, u_2 are not the cut-vertices of G_2. If $n = 2$ then $B = G_2$. If $n > 2$ then u_3 is the cut-vertex of G_2 and we can repeat the previous steps. It is clear that we get the block B with the claimed properties after finite number of steps.

Now we prove the cases b), c) and d). Since $r \geq 3$ we have $B \neq K_2$. Let $c(B) = m$ and C is a cycle of length m in B .

b) The inequality $c(G) \geq c(B)$ is obvious. If $m < 2r - 2$ we get $e_G(v) \leq r - 1$ for every vertex $v \in V(B)$ (according to the previous case and the fact that every two vertices of B lie on a common cycle, see Theorem 1.6 in [1]), a contradiction.

c) If $m < 2r$ then $m \in \{2r - 2, 2r - 1\}$, by the case b). According to the case a) no vertex of B (and obviously of C, too) has the eccentricity $r + 1$, therefore $\alpha \geq m \geq 2r - 2$.

d) If $m < 2r$ then $m \in \{2r - 2, 2r - 1\}$. If $m = 2r - 1$ and $|V(G)| \leq 3r - 2$ then $\mathrm{exc}_G(C) \leq 2r-2$ by Lemma 1.1b, a contradiction. If $m = 2r-2$ and $|V(G)| \leq 3r-3$ then $\mathrm{exc}_G(C) \leq 2(r-1) - 1 = 2r - 3$ by Lemma 1.1a, a contradiction again.

Now we can assume that $m = 2r - 2$ and $|V(G)| = 3r - 2$.
If G contains some of the graphs in Figure 2.1 (for simplicity, the vertices of C with degree two are not marked because their position on C is not relevant; this simplification will be used throughout the paper) then $\mathrm{exc}_G(C) \leq 2r-3$ (see Lemma 1.1a,c), a contradiction. So, we can suppose that none of the graphs in Figure 2.1 is a subgraph of G.

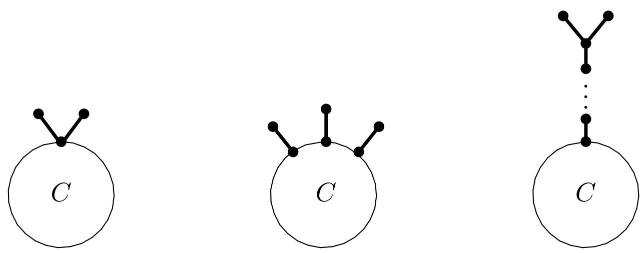


Figure 2.1

If $n = |\{v \in V(G); d_G(v, C) = 1\}|$ then we have two possibilities:

(i) $n = 1$
If $d_G(v, C) = 1$ then v is a cut-vertex of G. By the case a) $v \in B$ and so, we get $e_G(v) \leq r - 1$, a contradiction.

(ii) $n = 2$
In this case G contains a graph H_1 in Figure 2.2.

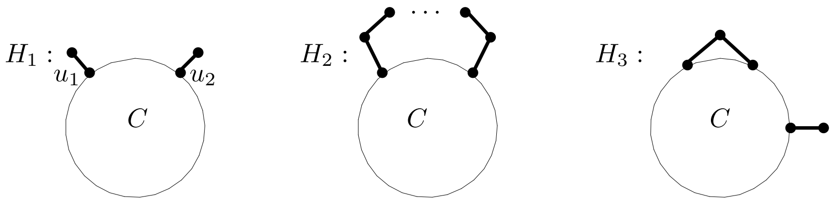


Figure 2.2

Firstly, we suppose that G has a subgraph H_2 too. Since G does not contain any of the graphs in Figure 2.1, $V(G) = V(H_2) = V(B)$, a contradiction (rad $B < r$). Hence G does not contain H_2 and according to Theorem 2.1a both vertices u_1, u_2 cannot be cut-vertices of G (otherwise rad $G < r$, a contradiction). Therefore G has a subgraph H_3 (Figure 2.2) and now we can easily to see that $\mathrm{exc}_G(C) \leq 2r-3$, a contradiction.

□

Corollary. *Let $e(G) = (4^\alpha, 5^\beta)$, $\alpha + \beta \leq 14$ and B be a block with the properties from Theorem 2.1. Then*

a) *each cycle of G with length at least 8 is a subgraph of B,*

b) *if G has a subgraph in Figure 2.3 such that $k \geq 8$ and $d_G(u, C_k) = 2$ then the vertex w is not a cut-vertex of G.*

Proof. a) The statement is evidently true if G has no cut-vertex ($B = G$). So, let us suppose that C_k, $k \geq 8$ be a cycle of a block B' which is different from B. Then $|V(B')| \geq 8$ and according to Theorem 2.1a for each vertex $y \in V(B')$ it holds $d_G(B, y) \leq 1$. Further evidently $|V(B)| \leq 7$ and $|V(B)| \geq 6$ (see Theorem 2.1b) and so G has at most two cut-vertices. If x is the only cut-vertex of G then $e_G(x) \leq 3$ (every two vertices of B lie on a common cycle, see Theorem 1.6 in [1]), a contradiction. If G has exactly two cut-vertices x, y then $|V(B)| = 6$ and $d_G(x, y) \leq 3$ (since diam $G = 5$). Then there exists a vertex $z \in V(B)$ such that $d_G(x, z) \leq 2$ and $d_G(y, z) \leq 2$. So we have $e_G(z) \leq 3$, a contradiction again.

b) The statement is obvious according to the previous case a) and Theorem 2.1a. □

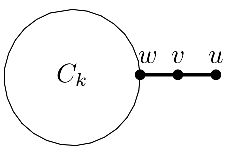


Figure 2.3

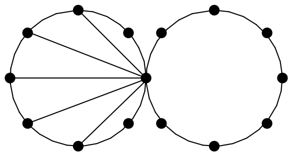


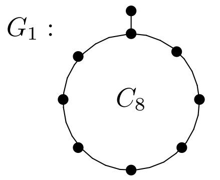
Figure 2.4

Remark. In the corollary the upper bound in the assumption $\alpha + \beta \leq 14$ is the best possible, see Figure 2.4.

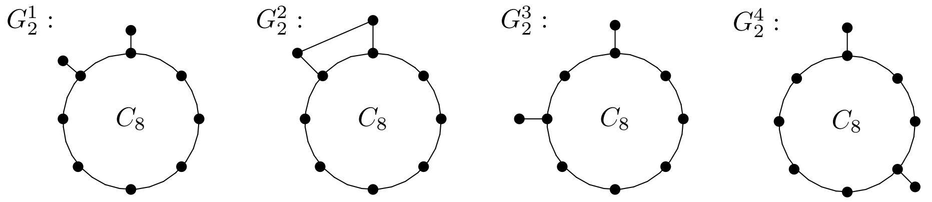
Theorem 2.2. *There are exactly seven minimal eccentric sequences of type* $(4^\alpha, 5^\beta)$, *namely,*

$(4^7, 5^2), (4^6, 5^4), (4^5, 5^6), (4^4, 5^8), (4^3, 5^9), (4^2, 5^{12}), (4, 5^{14})$.

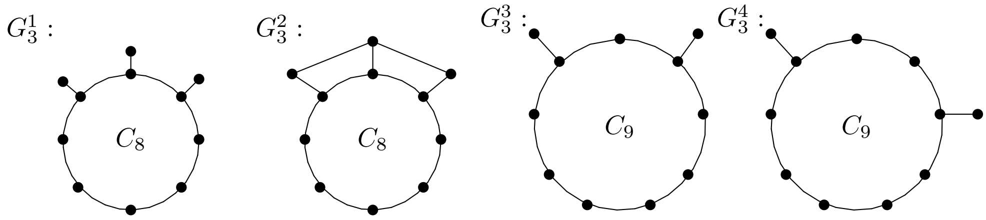
The proof of Theorem 2.2 is rather difficult and long. We will give it in the third section. In Figure 2.5 there are depicted the graphs (not all) which realize minimal eccentric sequences from Theorem 2.2. Note that in most of the cases these graphs are not uniquely given.



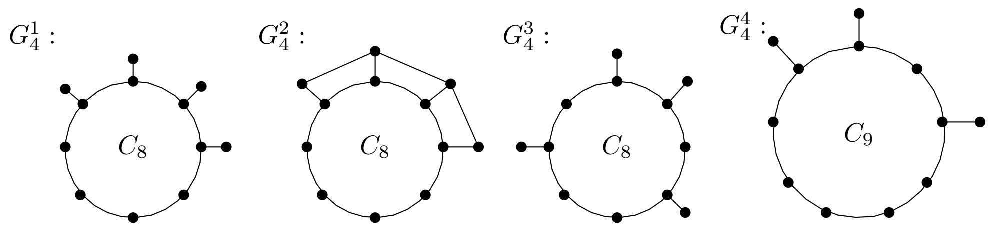
$e(G_1) = (4^7, 5^2)$



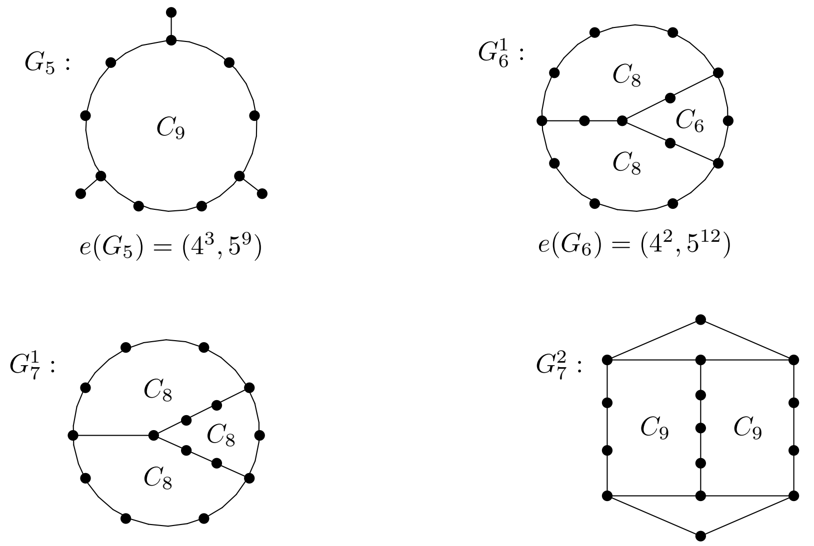
$e(G_2^1) = e(G_2^2) = e(G_2^3) = e(G_2^4) = (4^6, 5^4)$



$e(G_3^1) = e(G_3^2) = e(G_3^3) = e(G_3^4) = (4^5, 5^6)$



$e(G_4^1) = e(G_4^2) = e(G_4^3) = e(G_4^4) = (4^4, 5^8)$



$e(G_5) = (4^3, 5^9)$ $e(G_6) = (4^2, 5^{12})$

$e(G_7^1) = e(G_7^2) = (4, 5^{14})$

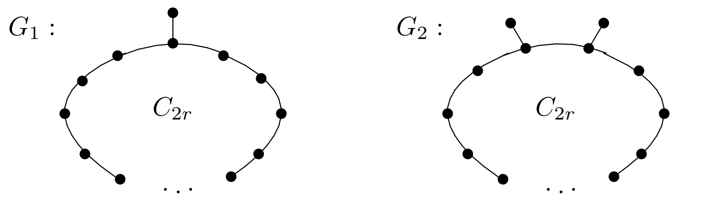
Figure 2.5

Concerning all minimal eccentric sequences of type $(r^\alpha, (r+1)^\beta)$ we propose the following

Conjecture. *For* $r \geq 3$ *there are exactly* $2r-1$ *minimal eccentric sequences of type* $(r^\alpha, (r+1)^\beta)$*, namely,*

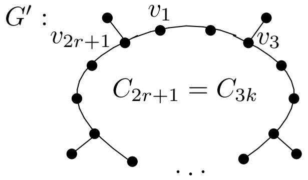
a) $(r^{2r-1}, (r+1)^2)$,
$(r^{2r-2}, (r+1)^4)$,
b_1) $(r^{2r-2i+1}, (r+1)^{3i})$, $i = 2, 3, \ldots, \lfloor \frac{2r+1}{3} \rfloor$,
b_2) $(r^{2r-2i}, (r+1)^{3i+2})$, $i = 2, 3, \ldots, \lfloor \frac{2r-1}{3} \rfloor$,
c) $(r^i, (r+1)^{4r-2i})$, $i = 1, 2, \ldots, \lfloor \frac{2r}{3} \rfloor$.

The above mentioned sequences are eccentric (see Figures 2.6a, b_1), b_2) and c)).



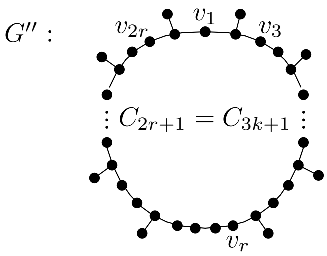
$e(G_1) = (r^{2r-1}, (r+1)^2)$ $e(G_2) = (r^{2r-2}, (r+1)^4)$

Figure 2.6a



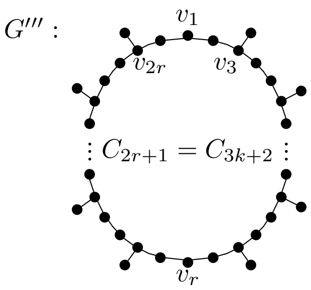
$3|(2r+1)$
$\deg(v_{3j}) = 3$
$j = 1, 2, \dots, \frac{2r+1}{3}$

$e(G') = (r^{2r-2i+1}, (r+1)^{3i}),\ i = \frac{2r+1}{3}$



$3|2r$
$\deg(v_{3j-1}) = \deg(v_{2r+4-3j}) = 3,$
$j = 1, 2, \dots, \frac{r}{3}$

$e(G'') = (r^{2r-2i+1}, (r+1)^{3i}),\ i = \frac{2r}{3}$



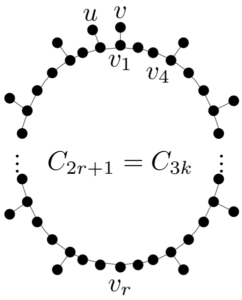
$3|(2r-1)$
$\deg(v_{3j}) = \deg(v_{2r-3j}) = \deg(v_{2r}) = 3,$
$j = 1, 2, \dots, \frac{r-2}{3}$

$e(G''') = (r^{2r-2i+1}, (r+1)^{3i}),\ i = \frac{2r-1}{3}$

Figure $2.6b_1$

To find a graph which realizes the given sequence in the case b_1) it is sufficient to choose a subgraph of one of the graphs in Figure $2.6b_1$ (depending on whether $2r+1 = 3k$, $2r+1 = 3k+1$ or $2r+1 = 3k+2$). The subgraph has to be a sun-graph with the required number of vertices.

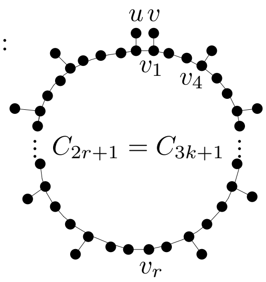
H' :



$$3|2r+1$$
$$\deg(v_{2r+1}) = \deg(v_{3j-2}) = \deg(v_{2r+2-3j}) = 3,$$
$$j = 1, 2, \dots, \tfrac{r-1}{3}$$

$e(H') = (r^{2r-2i}, (r+1)^{3i+2})$, $i = \frac{2r-2}{3}$

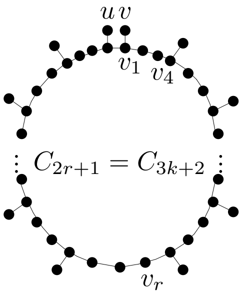
H'' :



$$3|2r$$
$$\deg(v_{2r+1}) = \deg(v_{r-2}) = \deg(v_{3j-2}) = \deg(v_{2r-3j}) = 3,$$
$$j = 1, 2, \dots, \tfrac{r}{3} - 1$$

$e(H'') = (r^{2r-2i}, (r+1)^{3i+2})$, $i = \frac{2r-3}{3}$

H''' :



$$3|2r-1$$
$$\deg(v_{3j+1}) = \deg(v_{2r+1-3j}) = 3,$$
$$j = 0, 1, 2, \dots, \tfrac{r-2}{3}$$

$e(H''') = (r^{2r-2i}, (r+1)^{3i+2})$, $i = \frac{2r-1}{3}$

Figure 2.6b_2

Analogously to the case b_1) to find a graph which realizes the given sequence in the case b_2) it is sufficient to choose a subgraph of one of the graphs in Figure 2.6b_2 (depending on whether $2r+1 = 3k$, $2r+1 = 3k+1$ or $2r+1 = 3k+2$). The subgraph has to be a sun-graph with the required number of vertices containing the vertices u, v.

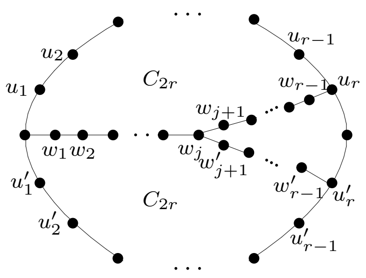
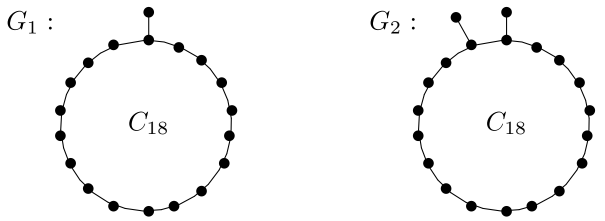


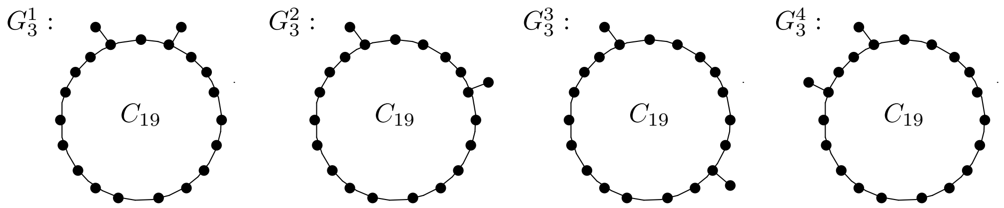
Figure 2.6c

It is known that the conjecture holds for $r = 3$ (see [4]), for $r = 4$ (see the Section 3 in this paper) and for the sequences in the cases a), b_1) and b_2) (see [5]). So, to show that the conjecture is true it is sufficient to prove that the sequences in the case c) are minimal.

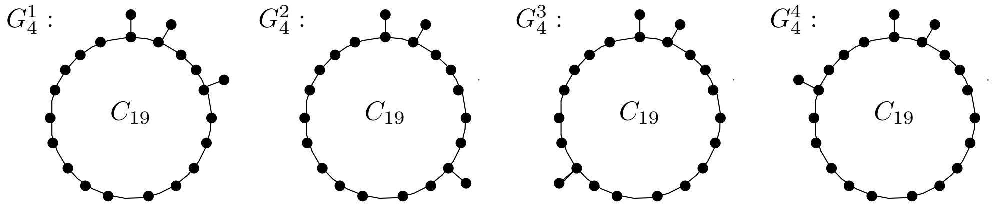
In the Figure 2.7, as illustration, supposed minimal graphs with eccentric sequences $(r^\alpha, (r+1)^\beta)$ for $r = 9$ are depicted.



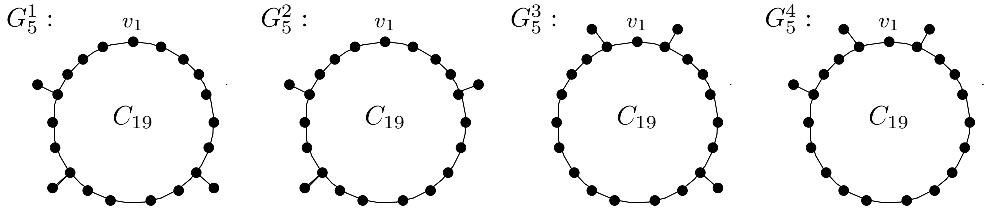
$e(G_1) = (9^{17}, 10^2)$ $e(G_2) = (9^{16}, 10^4)$



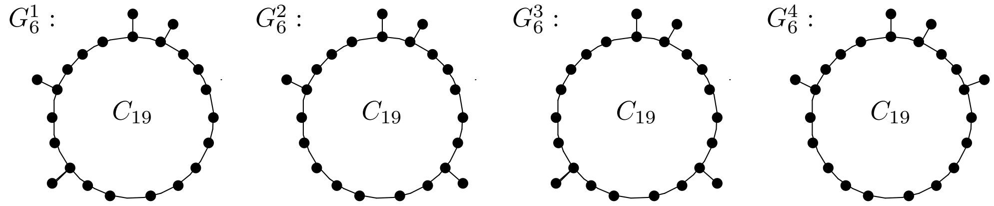
$e(G_3^1) = e(G_3^2) = e(G_3^3) = e(G_3^4) = (9^{15}, 10^6)$



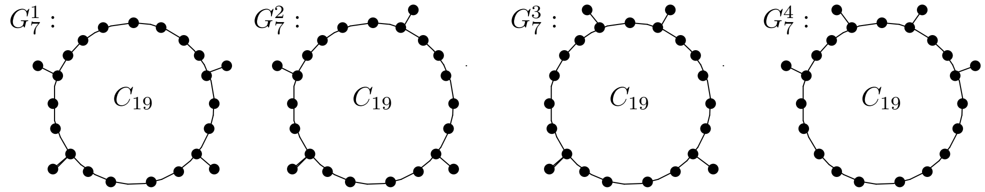
$e(G_4^1) = e(G_4^2) = e(G_4^3) = e(G_4^4) = (9^{14}, 10^8)$



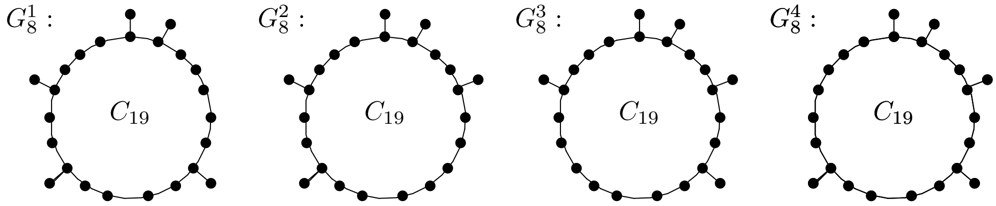
$$e(G_5^1) = e(G_5^2) = e(G_5^3) = e(G_5^4) = (9^{13}, 10^9)$$



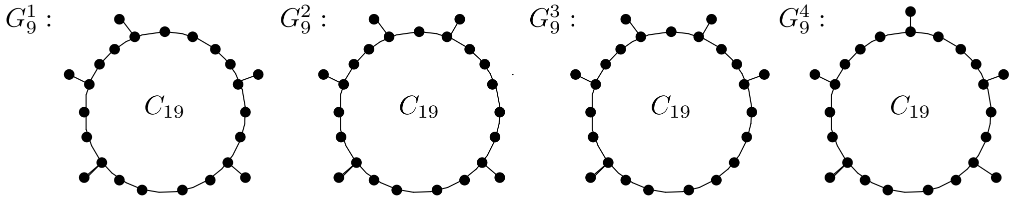
$$e(G_6^1) = e(G_6^2) = e(G_6^3) = e(G_6^4) = (9^{12}, 10^{11})$$



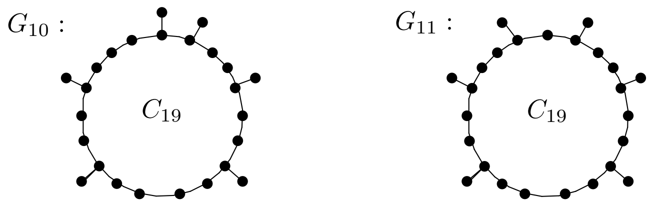
$$e(G_7^1) = e(G_7^2) = e(G_7^3) = e(G_7^4) = (9^{11}, 10^{12})$$



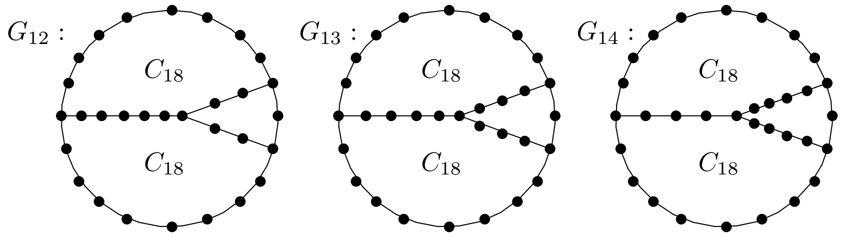
$$e(G_8^1) = e(G_8^2) = e(G_8^3) = e(G_8^4) = (9^{10}, 10^{14})$$



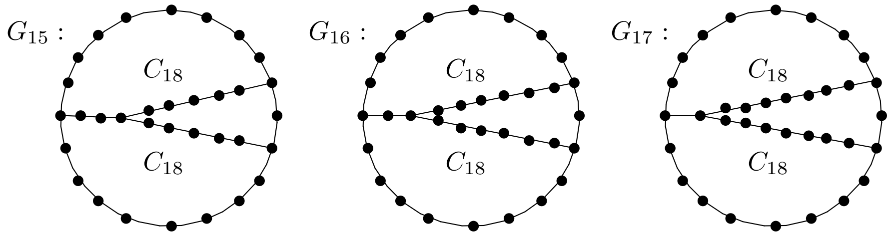
$$e(G_9^1) = e(G_9^2) = e(G_9^3) = e(G_9^4) = (9^9, 10^{15})$$



$e(G_{10}) = (9^8, 10^{17})$ $e(G_{11}) = (9^7, 10^{18})$



$e(G_{12}) = (9^6, 10^{24})$ $e(G_{13}) = (9^5, 10^{26})$ $e(G_{14}) = (9^4, 10^{28})$



$e(G_{15}) = (9^3, 10^{30})$ $e(G_{16}) = (9^2, 10^{32})$ $e(G_{17}) = (9, 10^{34})$

Figure 2.7

3. Minimal eccentric sequences of type $(4^\alpha, 5^\beta)$

Firstly, we give some usefull lemmas which we will need to prove Theorem 2.2.

Lemma 3.1. *Let* $e(G) = (4^\alpha, 5^\beta)$ *and* u *be a cut-vertex of* G. *Then* $e_G(u) = 4$.

Proof. If $e_G(u) \neq 4$ then $e_G(u) = 5$. Hence diam $G \geq 6$, a contradiction. □

Lemma 3.2. *For graphs* H_1 *and* H_2 *in Figure 3.1 it holds* $exc_{H_1}(C_9) \leq 1$ *and* $exc_{H_2}(C_9) \leq 3$.

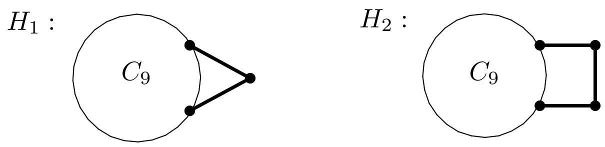


Figure 3.1

Proof. The proof is straightforward. □

Lemma 3.3. *Let rad $G = 4$ and G contain a geodesic cycle C_m of length $m = 10$ or $m = 11$. Let H be a component of the graph $G - C_m = \langle V(G) - V(C_m)\rangle_G$ containing a vertex with eccentricity 4 in G. Then for each path of C_m of length 3 it holds that at least one of its vertices is adjacent to a vertex of H.*

Proof. Since C_m is a geodesic cycle of length at least 10, $e_G(x) \geq 5$ for every vertex $x \in V(C_m)$. Let $u \in V(G) - V(C_m)$ be a vertex for which $e_G(u) = 4$. Suppose contrary to our claim that there is a path $(v_1, v_2, v_3, v_4, v_5, v_6)$ of C_m such that none of its vertices v_2, v_3, v_4, v_5 is adjacent to a vertex of the component H of graph $G - C_m$ containing the vertex u. Since C_m is a geodesic cycle, each cycle of G which contains the vertices $v_1, v_2, \ldots, v_6$ has the length at least 10. It follows easily that $d_G(u, v_i) \geq 5$ for some $i \in \{2, 3, 4, 5\}$. Hence $e_G(u) \geq 5$, a contradiction. □

By Lemma 3.3 it is easy to verify that the next lemma holds.

Lemma 3.4. *Let rad $G = 4$, G contain a geodesic cycle C_m with length $m = 10$ or $m = 11$ and let H be a component of $G - C_m$ containing a vertex with eccentricity 4 in G. Then there exist vertices $u_1, u_2, u_3 \in V(H)$ (not necessarily distinct) and vertices $v_1, v_2, v_3 \in V(C_m)$ such that $u_i v_i \in E(G)$ for $i \in \{1, 2, 3\}$ and $d(v_1, v_2) + d(v_2, v_3) + d(v_3, v_1) = m$.*

Lemma 3.5. *Let the graph in Figure 3.2 be a subgraph of G, $e(G) = (4^\alpha, 5^\beta)$ and $\alpha + \beta = 14$. Then G contains a cycle C_8 or C_9.*

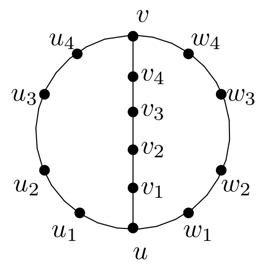


Figure 3.2

Proof. Let G contain neither C_8 nor C_9. Then $\deg_G(u) = \deg_G(v) = 3$ and $\deg_G(u_1) = \deg_G(v_1) = \deg_G(w_1) = \deg_G(u_4) = \deg_G(v_4) = \deg_G(w_4) = 2$. If $\deg_G(u_2) = \deg_G(u_3) = 2$ or $\deg_G(v_2) = \deg_G(v_3) = 2$ or $\deg_G(w_2) = \deg_G(w_3) = 2$ then we get a contradiction by Lemma 3.3. In the opposite case, there is a vertex of G with the eccentricity less than 4 (since G does not contain C_8 or C_9), a contradiction again. □

Lemma 3.6. *Let G contain neither a cycle C_8 nor a cycle C_9, $e(G) = (4^\alpha, 5^\beta)$ and $\alpha + \beta \leq 14$. Then G does not contain a geodesic cycle of length at least 10.*

Proof. Let G contain a geodesic cycle C_m for $m \geq 10$. If $m \geq 12$ then for each vertex $u \in V(C_m)$ holds $e_G(u) \geq 6$, a contradiction. Let $m \in \{10, 11\}$. By Lemma 3.4 there exist vertices $v_i, v_j \in \{v_1, v_2, v_3\}$ such that $d_G(v_i, v_j) \geq \lceil \frac{10}{3} \rceil = 4$. So, it is sufficient to consider two cases only:

a) $d_G(v_i, v_j) = 4$

Since $|V(G - C_m)| \leq 4$, G contains C_8 or C_9, a contradiction.

b) $d_G(v_i, v_j) = 5$

Since C_m is a geodesic cycle we get $m = 10$. Hence the graph in Figure 3.2 is a subgraph of G and we have a contradiction by Lemma 3.5. □

Lemma 3.7.

a) *Let all vertices of a graph H belong to a cycle C_{10} and $|E(H)| = 11$. Then any path of C_{10} of length 4 has at most two vertices with eccentricity 5 (in H).*

b) *Let all vertices of a graph H belong to a cycle C_{11} and H contain a cycle C_k for some $k \in \{7, 8, 9\}$. Then any path of C_{11} of length 4 has at most 3 vertices with eccentricity 5.*

c) *Let all vertices of a graph H belong to a cycle C_{12}, $|E(H)| = 13$ and H contain a cycle C_k for some $k \in \{7, 8, 9\}$ Then any path of C_{12} of length 5 has at most 4 vertices with eccentricity greater than 4.*

d) *Let a connected graph H with 11 vertices contain a cycle C_{10} and the subgraph of H induced by $V(C_{10})$ have at least 11 edges. Then any path of C_{10} of length 4 has at most 3 vertices with eccentricity greater than 4.*

e) *Any path of length 4 in each graph in Figure 3.3 has at most 3 vertices with eccentricity 5.*

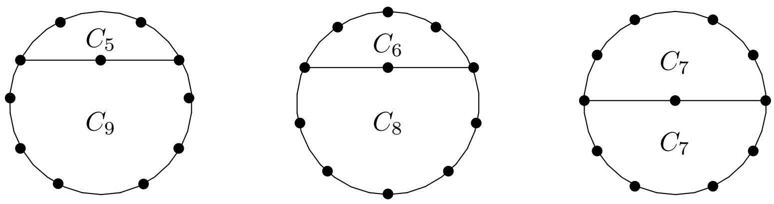


Figure 3.3

Proof. A straightforward verification of cases a) - e) shows that the statements are true. □

Lemma 3.8. *Let the graph in Figure 3.4 be a subgraph of G, $e(G) = (4^\alpha, 5^\beta)$ and $\alpha + \beta = 14$. Then $\alpha \geq 2$.*

Proof. If $\deg_G(u_i) > 2$ for some $i \in \{2, 3, 4\}$ then by Lemma 3.7a,b there are at least 2 vertices of C_9 with eccentricity at most 4 in G, i.e. $\alpha \geq 2$. Let $\deg_G(u_i) = 2$, $i = 2, 3, 4$. By Lemma 3.7a,b we can also suppose that the degree of at least one of the vertices u_1 and u_5 is also 2. Hence we get (by Lemma 3.3) that C_{10} (see Figure 3.4) is not a geodesic cycle and it follows that G contains a cycle C of length at most 9 such that $u_i \in V(C)$ for each $i \in \{1, 2, 3, 4, 5\}$. According to Lemma 3.7a we can suppose that at least 2 vertices of C belong to C_9 (Figure 3.4). It follows that the eccentricities of these vertices are at most 4 (in G) and so we have $\alpha \geq 2$. □

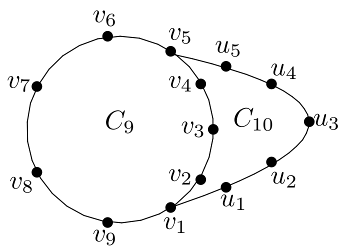


Figure 3.4

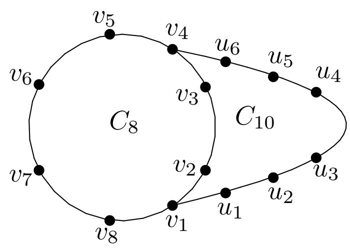


Figure 3.5

Lemma 3.9. *Let the graph in Figure 3.5 be a subgraph of* G, $e(G) = (4^\alpha, 5^\beta)$ *and* $\alpha + \beta = 14$. *Then* $\alpha \geq 2$.

Proof. Let $\alpha = 1$. According to Lemma 3.7a,c we can suppose that $\deg_G(u_3) = \deg_G(u_4) = 2$. Further we can suppose (by Lemma 3.7a,c again) that if $\deg_G(u_5) > 2$ then $u_5v_5 \in E(G)$ and in this case we have $\deg_G(u_1) = \deg_G(u_2) = 2$ (otherwise $e_G(v_6) \leq 4$, $e_G(v_7) \leq 4$, which contradicts our assumption or we get a contradiction by Lemma 3.7a,c). Analogously, we get that if $\deg_G(u_2) > 2$ then $\deg_G(u_5) = \deg_G(u_6) = 2$. Therefore C_{10} (see Figure 3.5) is not a geodesic cycle by Lemma 3.3. Hence G contains a cycle C of length at most 9 such that $u_2, u_3, u_4, u_5 \in V(C)$. Further it is sufficient to consider two cases only:

(i) $u_1, u_6 \in V(C)$
According to Lemma 3.7a,c we can suppose that C contains at least 2 vertices from C_8. Therefore G has at least 2 vertices with eccentricities at most 4, a contradiction.

(ii) $u_1 \notin V(C)$ or $u_6 \notin V(C)$
Without loss of generality we can suppose that $u_6 \notin V(C)$. It was shown above that then $u_5v_5 \in E(C)$ and $u_1, v_1 \in V(C)$. Hence we get $e_G(v_5) \leq 4$ and $e_G(v_1) \leq 4$, a contradiction. □

Lemma 3.10. *Let the graph in Figure 3.6 be a subgraph of* G, $e(G) = (4^\alpha, 5^\beta)$ *and* $\alpha + \beta = 14$. *Then* $\alpha \geq 2$.

Proof. We distiguish two cases.

a) Let the subgraph of G induced by the set $\{u_1, u_2, \ldots, u_6\}$ have 5 edges.
By Lemma 3.7b we can suppose that $\deg_G(u_3) = \deg_G(u_4) = 2$ and the degree of at most one of the vertices u_2, u_5 is greater than 2. Without loss of generality we can suppose that $\deg_G(u_5) = 2$. If $\deg_G(u_2) > 2$ then according to Lemma 3.7b we get $v_1u_2 \in E(G)$ and $\deg_G(u_6) = 2$. By Lemma 3.3 the vertices $u_2, u_3, \ldots, u_6$ cannot belong to a geodesic cycle of length at least 10. Hence these vertices belong to a cycle C of length at most 9. Therefore by Lemma 3.7b we can suppose that at least 2 vertices of C belong to C_8 (see Figure 3.6) and so we get $\alpha \geq 2$.

b) Let the subgraph of G induced by the set $\{u_1, u_2, \ldots, u_6\}$ have more than 5 edges.
By Lemma 3.7b we can suppose that $uv \notin E(G)$ for $u \in \{u_1, \ldots, u_6\}$ and $v \in \{v_2, v_3, v_4, v_6, v_7, v_8\}$. Hence G does not contain any geodesic cycle of length 10 (see Lemma 3.3). Now it is easy to check that the eccentricities of at least 2 vertices of C_8 are at most 4 (in G) and so $\alpha \geq 2$. □

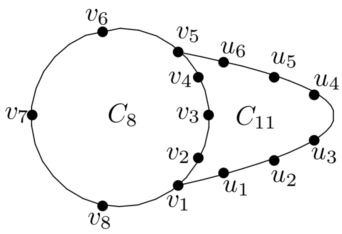


Figure 3.6

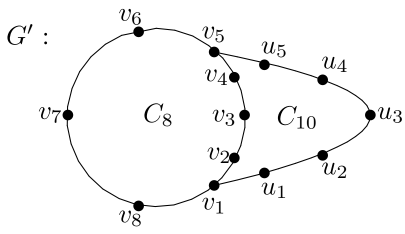


Figure 3.7

Lemma 3.11. *Let the graph G' in Figure 3.7 be a subgraph of G and $e(G) = (4^{\alpha}, 5^{\beta})$.*

a) *If $\alpha + \beta = 13$ then $\alpha \geq 3$.*

b) *If $\alpha + \beta = 14$ then $\alpha \geq 2$.*

Proof. a) If a subgraph of G induced by the set of vertices of a cycle of length 10 has at least 11 edges then G has at least 3 vertices with eccentricity 4 (Lemma 3.7a) and so $\alpha \geq 3$. In the opposite case C_{10} (see Figure 3.7) is not a geodesic cycle (by Lemma 3.3). Hence G contains a cycle C of length at most 9 such that $\{v_1, v_5, u_1, u_2, u_3, u_4, u_5\} \subseteq V(C)$. We get that 4 vertices of C belong to C_8 and so $\alpha \geq 4$.

b) If a subgraph of G induced by the set of vertices of a cycle of length 10 contains at least 11 edges then G contains at least 2 vertices with eccentricity at most 4 (see Lemma 3.7a,d and Lemma 1.1a), i.e. $\alpha \geq 2$. Now we can suppose that each edge of G from $E(G) - E(G')$ is incident with the vertex $v \in V(G) - V(G')$ or it is an edge $v_i v_j$ for $i \in \{2, 3, 4\}$, $j \in \{6, 7, 8\}$. Further we can suppose that none of the graphs in Figure 3.3 is a subgraph of G (Lemma 3.7e). Hence if the vertex v is adjacent to some vertex from the set $\{v_2, v_3, v_4, v_6, v_7, v_8\}$ then v is not adjacent to any vertex from $\{u_2, u_3, u_4\}$ and it is adjacent to at most one of the vertices u_1 and u_5. By Lemma 3.3 G cannot contain a geodesic cycle of length 10. Therefore there is a cycle C of length at most 9 in G with $\{u_1, u_2, u_3, u_4, u_5\} \subseteq V(C)$. If $v \in V(C)$ then C contains (according to Lemma 3.7) at least 3 vertices of C_8 and if $v \notin V(C)$ then C contains (according to Lemma 3.7) at least 4 vertices of C_8. Therefore there are at least 2 vertices of C_8 (see Lemma 1.1a,b) with eccentricity at most 4 in G and so $\alpha \geq 2$. □

Lemma 3.12. *Let a graph H in Figure 3.8 for $k \geq 3$ be a subgraph of G, $|V(G)| = 14$ and $e(G) = (4^{\alpha}, 5^{\beta})$. Then $\alpha \geq 2$.*

Proof. It is easy to see (according to Lemmas 3.8 and 1.1a,b) that the statement holds for $k = 4, 5$. Let $k = 3$ and $V(G) - V(H) = \{u, v\}$. It is easy to verify that at least 5 vertices of C_9 have eccentricity 4 in H, i.e. $\text{exc}_H(C_9) \leq 4$. If $d(u, H) > 1$ then v is a cut-vertex of G and so $e_G(v) = 4$ by Lemma 3.1. Let $wv \in E(G)$ and $w \in V(H)$. If $w \notin V(C_9)$ then at least 2 vertices of C_9 have eccentricity at most 4 in G, i.e. $\text{exc}_G(C_9) \leq 7$ and we get $\alpha \geq 3$. If $w \in V(C_9)$ then $\text{exc}_G(C_9) \leq 8$ (since $\text{exc}_H(C_9) \leq 4$) and we get $\alpha \geq 2$. What is left is to show that the statement holds also for the case $d(u, H) = d(v, H) = 1$. If each of the vertices u, v is adjacent to some vertex from the set $\{v_1, v_2, v_3\}$, then $\text{exc}_G(C_9) \leq 6$. If u is adjacent to some vertex from $\{v_1, v_2, v_3\}$ and $H' = \langle V(H) \cup \{u\} \rangle_G$ then $\text{exc}_{H'}(C_9) \leq 5$. It follows

$\mathrm{exc}_G(C_9) \leq 7$ and so $\alpha \geq 2$. Let none of the vertices u, v be adjacent to a vertex from $\{v_1, v_2, v_3\}$. If the degree of at least one of the vertices u, v is greater than 1 then $\mathrm{exc}_G(C_9) \leq 7$ (using Lemma 3.2 and the fact that $\mathrm{exc}_H(C_9) \leq 4$). Let $\deg_G(u) = \deg_G(v) = 1$. If $d_G(u,v) = 2$ then $\mathrm{exc}_G(C_9) \leq 6$ (using Lemma 1.1b and the fact that $\mathrm{exc}_H(C_9) \leq 4$). Hence we get $\alpha \geq 3$. If $d_G(u,v) > 2$ then two vertices of C_9 are cut-vertices of G and so we have $\alpha \geq 2$. □

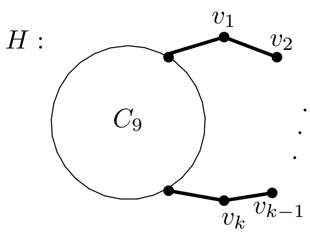


Figure 3.8

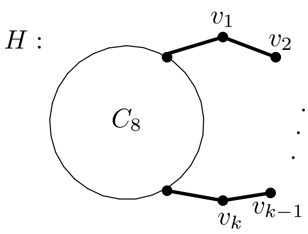


Figure 3.9

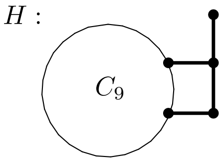
Lemma 3.13. *Let a graph H in Figure 3.9 for $k \geq 3$ be a subgraph of G, let G do not contain C_9, $|V(G)| = 14$ and $e(G) = (4^\alpha, 5^\beta)$. Then $\alpha \geq 2$.*

Proof. a) If we take Lemmas 3.9, 3.10 and 3.11 into account, the statement is easy to verify for $k = 5, 6$.

b) Let $k = 4$ and $V(G) - V(H) = \{u, v\}$.
It is easy to check that at least 4 vertices of C_8 have eccentricity at most 4 in H, i.e. $\mathrm{exc}_H(C_8) \leq 4$. If $d_G(u,H) = d_G(v,H) = 1$ then $\mathrm{exc}_G(C_8) \leq 6$ (i.e. $\alpha \geq 2$). If $d_G(u,H) = 2$ then v is a cut-vertex of G, and so $e_G(v) = 4$. Since $\mathrm{exc}_G(C_8) \leq 7$ we get $\alpha \geq 2$ again.

c) Let $k = 3$ and $V(G) - V(H) = \{w_1, w_2, w_3\}$.
Evidently, $\mathrm{exc}_H(C_8) \leq 3$. If $d_G(w_i, H) = 1$, $i \in \{1,2,3\}$ then $\mathrm{exc}_G(C_8) \leq 6$ and we have $\alpha \geq 2$. Now we suppose that $d_G(w_1,H) = 1$, $d_G(w_2,H) = 2$ and $w_1w_2 \in E(G)$. If w_1 is a cut-vertex of G then $e_G(w_1) = 4$ and $\mathrm{exc}_G(C_8) \leq 7$ (we have $\mathrm{exc}_{H'}(C_8) \leq 6$, where $H' = \langle V(H) \cup \{w_1, w_2\}\rangle_G$). If w_1 is not a cut-vertex of G then $\mathrm{exc}_G(C_8) \leq 6$ (it follows from the fact that $\mathrm{exc}_H(C_8) \leq 3$). □

Lemma 3.14. *If G is a graph with $e(G) = (4, 5^{13})$ then H in Figure 3.10 is not a subgraph of G.*



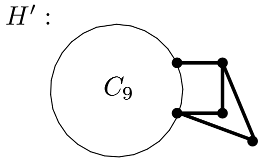


Figure 3.10

Proof. Let H be a subgraph of G. Since $\mathrm{exc}_G(C_9) \leq 8$, no cut-vertex of G belongs to $G - C_9$. According to Lemma 3.12 G contains a graph H' in Figure 3.10 and each vertex of $G - H'$ is adjacent to a vertex of C_9. Evidently, $\mathrm{exc}_{H'}(C_9) \leq 3$ and so $\mathrm{exc}_G(C_9) \leq 7$, a contradiction. □

Lemma 3.15. *Let G contain neither C_8 nor C_9 and $e(G) = (4^\alpha, 5^\beta)$.*

a) *If G contains C_{10} which is not a geodesic cycle then at least one of the graphs H_1, H_2, H_3 in Figure 3.11 (the vertices are numbered with their eccentricities) is a subgraph of G.*

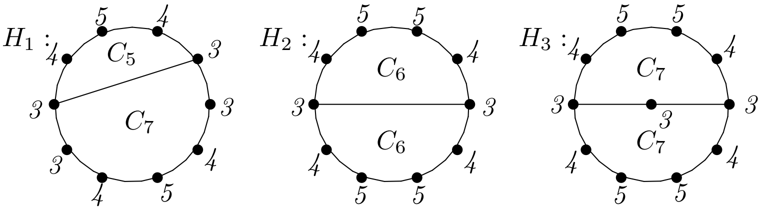


Figure 3.11

b) *If G does not contain C_{10} and it contains C_{11} which is not a geodesic cycle then H_4 in Figure 3.12 is a subgraph of G.*

c) *If G contains neither C_{10} nor C_{11} and it contains C_{12} then H_5 in Figure 3.12 is a subgraph of G.*

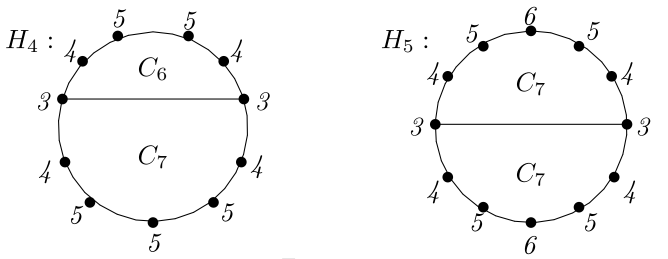


Figure 3.12

Proof. The statements are evident in the cases a), b). In the case c) it is sufficient to realize that C_{12} cannot be a geodesic cycle (diam $G < 6$). $\square$

The proof of Theorem 2.2.

All sequences from Theorem 2.2 are eccentric (see Figure 2.5).

a) We show that the sequence $(4^7, 5^2)$ is a minimal eccentric sequence.
It is sufficient to show that a graph with eccentric sequence neither $(4^6, 5^2)$ nor $(4^7, 5)$ exists. Suppose on the contrary that G is such a graph. By Theorem 2.1d G contains a cycle C_8. Hence diam $G < 5$, a contradiction.

b) We show that the sequence $(4^6, 5^4)$ is a minimal eccentric sequence.
It is sufficient to show that a graph with eccentric sequence neither $(4^5, 5^4)$ nor $(4^6, 5^3)$ exists. Suppose on the contrary that G is such a graph. By Theorem 2.1d G contains a cycle C_8 or C_9. If G contains C_9 then diam $G < 5$, a contradiction. If G contains C_8 then $\mathrm{exc}_G(C_8) \leq 1$. Hence G has at least 7 vertices with eccentricity at most 4, a contradiction.

c) We show that $(4^5, 5^6)$ is a minimal eccentric sequence.
Suppose analogously to the previous cases that there is a graph with eccentric sequence either $(4^4, 5^6)$ or $(4^5, 5^5)$. By Theorem 2.1d G contains a cycle of length at least 8.
Let G contain C_8. If $d_G(u, C_8) \leq 1$ for each vertex $u \in V(G)$ then $\mathrm{exc}_G(C_8) \leq 2$

(Lemma 1.1c). Hence G contains at least 6 vertices with eccentricity at most 4, a contradiction. If for $u \in V(G)$ it holds $d_G(u, C_8) = 2$ then the vertex v adjacent to u is a cut-vertex of G and so $e_G(v) = 4$. Since $\mathrm{exc}_G(C_8) \le 3$ (Lemma 1.3a), at least 5 vertices of C_8 have eccentricity at most 4, a contradiction.
If G contains C_9 then $\mathrm{exc}_G(C_9) \le 2$. Hence G contains at least 7 vertices with eccentricity at most 4, a contradiction.
It is left the case that G contains C_{10} and it contains neither C_8 nor C_9. If $|E(G)| = 10$ then $e(G) = (5^{10})$, a contradiction. If $|E(G)| > 10$ then G has a vertex with eccentricity at most 3, and we have a contradiction again.

d) We show that $(4^4, 5^8)$ and $(4^3, 5^9)$ are minimal eccentric sequences.
It is sufficient to show that a graph G with eccentric sequence neither $(4^3, 5^8)$, $(4^4, 5^7)$ nor $(4^2, 5^9)$ exists. Suppose on the contrary that G is a graph with eccentric sequence $e(G) = (4^\alpha, 5^{11-\alpha})$, $\alpha \in \{2, 3, 4\}$. By Theorem 2.1c G contains a cycle of length at least 8. We distinguish several cases.

d_1) G contains C_8.
If $d(v, C_8) \le 1$ for each vertex $v \in V(G)$ then $\mathrm{exc}_G(C_8) \le 3$. Hence $\alpha \ge 5$, a contradiction. The same conclusion is easy to verify in the case that G contains a graph H in Figure 3.9 for $k = 3$.
Obviously, G does not contain a vertex u such that $d(u, C_8) = 3$ (see Theorem 2.1a). It is left the case that G has a subgraph H in Figure 2.3 for $k = 8$, $d_G(u, C_8) = 2$ and w is not a cut-vertex (see Corollary b) of Theorem 2.1). We can suppose that v is a cut-vertex of G and the vertex $x \in V(G) - V(H)$ is adjacent to a vertex from $V(C_8) \cup \{v\}$. Therefore $e_G(v) = 4$, $\mathrm{exc}_G(C_8) \le 4$ and so $\alpha \ge 5$, a contradiction.

d_2) G contains C_9.
In this case $\mathrm{exc}_G(C_9) \le 4$ (Lemma 1.1b), a contradiction.

d_3) G contains C_{10} and neither C_8 nor C_9.
Obviously, C_{10} cannot be a geodesic cycle. Hence at least one of the graphs in Figure 3.11 is a subgraph of G. So, we get rad $G \le 3$, a contradiction.

d_4) G contains C_{11} and neither C_8, C_9 nor C_{10}.
Since C_{11} cannot be a geodesic cycle, H_4 in Figure 3.12 is a subgraph of G and we have rad $G \le 3$, a contradiction again.

e) We prove that $(4^2, 5^{12})$ is a minimal eccentric sequence. It is sufficient to show that a graph G with eccentric sequence neither $(4, 5^{12})$ nor $(4^2, 5^{11})$ exists. Suppose on the contrary that G is a graph with eccentric sequence $(4^\alpha, 5^{13-\alpha})$, $\alpha \in \{1, 2\}$. By Theorem 2.1c G contains a cycle of length at least 8. Further we distinguish several cases.

e_1) G contains C_8.
By Lemma 3.11a the graph in Figure 3.7 is not a subgraph of G. It is easy to check that if G contains one of the graphs in Figure 3.9 for $k = 4$ or $k = 5$ (except the graph in Figure 3.7) then $\alpha \ge 3$, a contradiction.
Let G contain a graph H in Figure 3.9 for $k = 3$. We have $\mathrm{exc}_H(C_8) \le 3$. If each of the two vertices of $G - H$ is adjacent to a vertex of $V(H)$ then $\mathrm{exc}_G(C_8) \le 5$ and so $\alpha \ge 3$, a contradiction. If $d_G(u, H) = 2$ then $\deg_G(u) = 1$ and the vertex v adjacent to u is a cut-vertex of G and it follows $e_G(v) = 4$. It is easy to see that $\mathrm{exc}_G(C_8) \le 6$ and so $\alpha \ge 3$, a contradiction again.
If $d_G(u, C_8) \le 1$ for each vertex $u \in V(G)$ then $\mathrm{exc}_G(C_8) \le 5$ (Lemma

1.1c), a contradiction. So, let for some vertex $u \in V(G)$ hold $d_G(u, C_8) = 2$ and the graph H in Figure 2.3 for $k = 8$ is a subgraph of G. The vertex w is not a cut-vertex of G (Corollary b) of Theorem 2.1) and we can suppose that none of the graphs represented by Figure 3.9 for $k \geq 3$ is a subgraph of G. It follows that v is a cut-vertex of G. Now it is easy to check that $\alpha \geq 3$ (really, if there is another cut-vertex outside C_8 then $\mathrm{exc}_G(C_8) \leq 7$; otherwise $\mathrm{exc}_G(C_8) \leq 6$), a contradiction.

e_2) G contains C_9.
It is easy to check that if G contains at least one of the graphs in Figures 3.13, 3.14 and 3.8 for $k \geq 3$ then $\mathrm{exc}_G(C_9) \leq 6$, so we have $\alpha \geq 3$, a contradiction. Further we suppose that G does not contain any of these graphs.

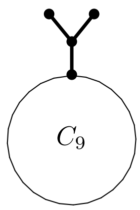


Figure 3.13

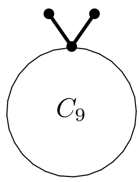


Figure 3.14

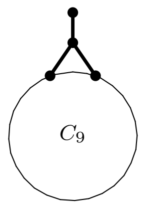


Figure 3.15

Firstly we assume that there is a vertex $u \in V(G)$ such that $d_G(u, C_9) = 2$. Hence the graph in Figure 2.3 for $k = 9$ is a subgraph of G and v is a cut-vertex of G. By Corollary b) of Theorem 2.1 the vertex w is not a cut-vertex of G and consequently a graph in Figure 3.15 is a subgraph of G. Obviously, $\mathrm{exc}_G(C_9) \leq 7$ and since $e_G(v) = 4$, $v \notin V(C_9)$, we have a contradiction ($\alpha \geq 3$).

What is left is to consider the case $d_G(u, C_9) \leq 1$ for every vertex $u \in V(G)$. Let $H = G - C_9$. Since the graph in Figure 3.13 is not a subgraph of G, each component of H has at most 2 vertices. Now it is easy to see that $\mathrm{exc}_G(C_9) \leq 6$, a contradiction. Really, it is sufficient to take into account Lemma 3.2 and the fact that if a vertex with degree 1 is adjacent to the vertex v then $e_G(v) = 4$ (and so, v is not a C_9-excited vertex).

e_3) G contains C_{10} and it contains neither C_8 nor C_9.
By Lemma 3.6 C_{10} is not a geodesic cycle and by Lemma 3.15a G contains at least one of the graphs H_1, H_2 and H_3 in Figure 3.11. If $d_G(x, H_i) \leq 1$ for each vertex x of G then it is easy to check that there are at least 3 vertices of G with the eccentricity at most 4, a contradiction. If G has a subgraph in Figure 3.16 then there is a path of C_{10} of length 4 such that the distance of each of its vertices from every vertex v_1, v_2, v_3 is at most 4. We have $\alpha \geq 3$ (see Figure 3.11), a contradiction. Let none of the graphs represented by Figure 3.16 be a subgraph of G and $u \in V(G)$ be a vertex such that $d_G(u, H_i) = 2$. Therefore the graph H in Figure 2.3 for $k = 10$ is a subgraph of G, w is not a cut-vertex and v is a cut-vertex of G. Since there are at least 2 vertices of C_{10} with the eccentricity at most 4 in G (see Figure 3.11) and $e_G(v) = 4$, we get a contradiction.

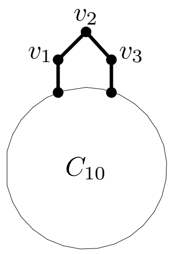


Figure 3.16

e_4) G contains C_{11} and it contains neither C_8, C_9 nor C_{10}.
By Lemma 3.6 C_{11} is not a geodesic cycle. Hence the graph H_4 in Figure 3.12 is a subgraph of G. Let $V(G) - V(H_4) = \{u, v\}$. If we distinguish two cases $d(u, H_4) = d(v, H_4) = 1$ or $d(u, H_4) = 2$ then it is easily seen that there are at least 3 vertices of G with the eccentricity at most 4, a contradiction.

e_5) G contains C_{12} and it does not contain C_k for $k \in \{8, 9, 10, 11\}$.
By Lemma 3.15 the graph H_5 in Figure 3.12 is a subgraph of G and it evidently leads to a contradiction (rad $G < 4$ or $\alpha > 3$).

e_6) G contains C_{13} and it does not contain C_k for $k \in \{8, 9, \ldots, 12\}$.
Obviously, $E(G) = E(C_{13})$ and so $e(G) = (6^{13})$, a contradiction.

f) We prove that $(4, 5^{14})$ is a minimal eccentric sequence.
It is sufficient to show that a graph G such that $e(G) = (4, 5^{13})$ does not exist. Suppose on the contrary that G is such a graph. By Theorem 2.1c G contains a cycle of length at least 8. We distinguish several cases.

f_1) G contains C_9.
(i) If G contains the graph H in Figure 2.3 for $k = 9$ and $d_G(u, C_9) = 2$ then the vertex w is not a cut-vertex of G. By Lemma 3.12 G does not contain a graph H in Figure 3.8 for $k \geq 3$. Hence v is a cut-vertex of G and so $e_G(v) = 4$. If G contains the graph in Figure 3.13 or 3.14 then $\mathrm{exc}_G(C_9) \leq 8$. Hence at least one vertex of C_9 has eccentricity 4 in G ($e_G(v) = 4$, too). We get $\alpha \geq 2$, a contradiction. So, we can suppose that none of the graphs in Figures 3.13 and 3.14 is a subgraph of G. Since G cannot have two cut-vertices, it holds $d_G(x, C_9) = 1$ and $\deg_G(x) > 1$ for each vertex $x \in V(G) - V(H)$. According to Lemma 3.2 we get $\mathrm{exc}_G(C_9) \leq 8$, a contradiction again.
(ii) What is left is to consider the case that for each vertex $x \in V(G)$ it holds $d_G(x, C_9) \leq 1$. If the graph in Figure 3.17 is a subgraph of G then $\mathrm{exc}_G(C_9) \leq 6$, and so $\alpha \geq 3$, a contradiction. Let $G_1 = G - C_9$. Now according to Lemma 3.14 we can suppose that there are at most 2 vertices in every component of G_1.

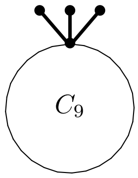


Figure 3.17

Firstly, let each component of G_1 be K_1. If at least 3 vertices of G_1 have degree at least 2 in G then $\mathrm{exc}_G(C_9) \leq 7$ (by Lemma 3.2), a contradiction. So, at least 3 vertices of G_1 have degree 1 in G. It follows $\mathrm{exc}_G(C_9) \leq 7$ (the graph in Figure 3.17 is a subgraph of G or at least two vertices of C_9 are cut-vertices of G), a contradiction.
Let one of the components of G_1 be K_2 and the left three ones be K_1. If the degree of at least two vertices with degree 0 in G_1 is at least 2 in G then $\mathrm{exc}_G(C_9) \leq 7$ (Lemma 3.2), a contradiction. If two vertices with degree 0 in G_1 have degree 1 in G then $\mathrm{exc}_G(C_9) \leq 7$ again (the graph in Figure 3.14 is a subgraph of G or G has 2 cut-vertices).
If G_1 has two components K_2 then obviously $\mathrm{exc}_G(C_9) \leq 8$ and the equality holds if and only if G is the graph in Figure 3.18. In this case G has a vertex with eccentricity 6, a contradiction.

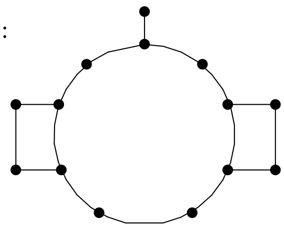


Figure 3.18

f_2) G contains C_8 and it does not contain C_9.
If H in Figure 2.3 for $k = 8$ is a subgraph of G and $d_G(u, C_8) = 2$ then the vertex w is not a cut-vertex of G. By Lemma 3.13 G does not contain a graph H in Figure 3.9 for $k \geq 3$. Hence v is a cut-vertex of G and so $e_G(v) = 4$. Since G cannot contain 2 cut-vertices, for each vertex $x \in V(G) - V(H)$ it holds $d_G(x, C_8) = 1$ or $xv \in E(G)$. Hence $\mathrm{exc}_G(C_8) \leq 7$ and we get $\alpha \geq 2$, a contradiction.
If $d(x, C_8) \leq 1$ for each vertex $x \in V(G)$ then $\mathrm{exc}_G(C_8) \leq 6$ (Lemma 1.1c) and we have $\alpha \geq 2$, a contradiction again.

f_3) G contains C_{10} and neither C_8 nor C_9.
By Lemma 3.6 C_{10} is not a geodesic cycle, whence at least one of the graphs in Figure 3.11 is a subgraph of G. By Lemma 3.5 the graph in Figure 3.2 is not a subgraph of G.

(i) If G contains a graph in Figures 3.19 (different from the graph in Figure 3.2) or 3.20 then there is a path of C_{10} of length 3 such that the distance of each of its vertices from every vertex v_1 v_2, v_3, v_4 is at most 4. Since every path of length 3 in H_i, $i = \{1,2,3\}$ (see Figure 3.11) has at least 2 vertices with eccentricity at most 4 in G (i.e. $\alpha \geq 2$) we have a contradiction.

(ii) If G contains a graph in Figure 3.16, $V(G) - V(C_{10}) = \{v_1, v_2, v_3, v_4\}$ and $d_G(v_4, C_{10}) = 1$ then there is a path of C_{10} of length 4 such that the distance of each of its vertices from every vertex v_1, v_2, v_3 is at most 4. Now it is easily seen that G has at least two vertices with the eccentricity at most 4 (see Figure 3.11), a contradiction.

(iii) If G contains a graph in Figure 3.21 (and none of the previous cases takes place) then $\deg_G(v_4) = 1$ and so we have $e_G(v_2) = 4$. Since there is a path of C_{10} of length 2 such that the distance of each of its vertices from every vertex v_1, v_2, v_3, v_4 is at most 4 we have a contradiction again (see Figure 3.11).

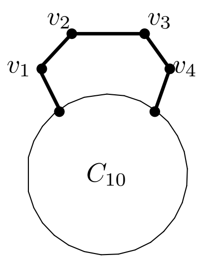


Figure 3.19

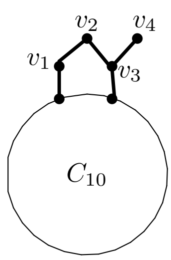


Figure 3.20

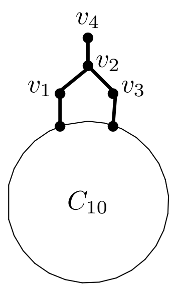


Figure 3.21

(iv) Let $d_G(v, H_i) \leq 1$, $i \in \{1,2,3\}$ for each $v \in V(G)$ (see Figure 3.11). Then vertices of eccentricity 3 in H_i have eccentricity at most 4 in G, a contradiction.

(v) Let none of the previous cases hold. We get that G contains the graph H in Figure 2.3 for $k = 10$, $d_G(u, C_{10}) = 2$ and v is a cut-vertex of G. Hence by Lemma 3.1 $e_G(v) = 4$. Then v is the only cut-vertex of G and so every vertex $x \in V(G) - V(H)$ is adjacent to the vertex v or to a vertex of C_{10}. It follows that there is a vertex x of C_{10} with $e_{H_i}(x) = 3$, $i \in \{1,2,3\}$ (see Figure 3.11) for which $e_G(x) \leq 4$, a contradiction.

f_4) G contains C_{11} and neither C_8, C_9 nor C_{10}.
By Lemma 3.6 C_{11} is not a geodesic cycle. Hence the graph H_4 in Figure 3.12 is a subgraph of G. If G contains a graph in Figure 3.22 then there is a path of C_{11} of length 4 such that the distance of each of its vertices from every vertex v_1, v_2, v_3 is at most 4, and we get a contradiction (see H_4 in Figure 3.12). In the other case G contains the graph in Figure 2.3 for $k = 11$, $d_G(u, C_{11}) = 2$ and v is a cut-vertex of G. Then there is a vertex $x \in V(C_{11})$ for which $e_G(x) \leq 4$, (see Figure 3.12). We have $\alpha \geq 2$, a contradiction.

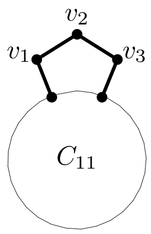


Figure 3.22

f_5) G contains C_{12} and it does not contain C_k for $k \in \{8, 9, 10, 11\}$.
By Lemma 3.15c the graph H_5 in Figure 3.12 is a subgraph of G. It is easy to see that there are at least two vertices of C_{12} such that their eccentricities are at most 4 in G, a contradiction.

f_6) G contains C_{13} and it does not contain C_k for $k \in \{8, 9, 10, 11, 12\}$.
C_{13} is a geodesic cycle. Hence diam $G \geq 6$, a contradiction.

f_7) G contains C_{14} and it does not contain C_k for $k \in \{8, 9, 10, 11, 12, 13\}$.
We have $|E(G)| = 14$ and then diam $G = 7$, a contradiction.

□

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