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RELATIVELY COMPLEMENTED λ-LATTICES

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Abstract. In this paper, we discuss relatively complemented λ-lattices.

1 λ-posets and λ-lattices

For terminology and notation throughout the paper see [1] and [2].

Definition 1.1. *Let $P = (P, \leq)$ be an ordered set. For $A \subseteq P$, we denote*

$$L(A) = \{x \in P;\ x \leq a \text{ for all } a \in A\}$$
$$U(A) = \{x \in P;\ x \geq a \text{ for all } a \in A\}.$$

Call $L(A)$ the lower and $U(A)$ the upper cone of A, respectively. If B is a finite family of elements of P, say $A = \{a_1, a_2, \ldots, a_n\}$, we write briefly $L(a_1, a_2, \ldots, a_n)$ or $U(a_1, a_2, \ldots, a_n)$ for $L(A)$ or $U(A)$, respectively.

A λ-poset is a poset $(P, \leq)$, where $L(a, b) \neq \emptyset \neq U(a, b)$ for every two elements $a, b \in P$, with a choice function λ which chooses a single element from both $L(a, b)$ and $U(a, b)$; moreover λ satisfies the following conditions:

$$\lambda(L(a, b)) = \lambda(L(b, a)) \text{ and } \lambda(U(a, b)) = \lambda(U(b, a)) \tag{1}$$

$$\text{If } a \leq b \text{ then } \lambda(L(a, b)) = a \text{ and } \lambda(U(a, b)) = b. \tag{2}$$

In the rest of this paper, we will use $\lambda_L(a, b)$ instead of $\lambda(L(a, b))$ and $\lambda_U(a, b)$ instead of $\lambda(U(a, b))$.

The chosen elements $\lambda_L(a, b)$ and $\lambda_U(a, b)$ are denoted by $a \cdot b$ and by $a + b$ respectively. After the choice of λ, the elements $a \cdot b$ and $a + b$ are fixed. Because of (1), the choice of λ is independent on the order of the elements a and b.

On the other hand, the choice is not assumed to be consequential, i.e. if $L(a, b) = L(c, d)$ for some elements $a, b, c, d \in P$, $(a, b) \neq (c, d)$, $a \cdot b$ and $c \cdot d$ may not be equal; the same holds for $a + b$ and $c + d$. Thus choice of λ depends only on the elements a and b.

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Definition 1.2. *A λ-lattice is an algebra $P = (P, \cdot, +)$ where $+$ and $\cdot$ are two binary operations on P, satisfying the following laws for all $a, b, c \in P$:*

$i.)$ $a \cdot a = a$	$i_+)$ $a + a = a$
$c.)$ $a \cdot b = b \cdot a$	$c_+)$ $a + b = b + a$
$t.)$ $a \cdot ((a \cdot b) \cdot c) = (a \cdot b) \cdot c$	$t_+)$ $a + ((a + b) + c) = (a + b) + c$
$a.)$ $a \cdot (a + b) = a$	$a_+)$ $a + (a \cdot b) = a$

Lemma 1.3. From the laws $i.), t.)$ we obtain $a \cdot (a \cdot b) = a \cdot b$ and similarly from $i_+), t_+)$ we obtain $a + (a + b) = a + b$.

Proof. We will provide proof for $\cdot$, the second part can be proved dually:

$a \cdot ((a \cdot a) \cdot b) = (a \cdot a) \cdot b$	according to $t.)$
$(a \cdot a) \cdot b = a \cdot b$	according to $i.)$
$a \cdot ((a \cdot a) \cdot b) = a \cdot (a \cdot b)$	according to $i.)$

Hence $a \cdot (a \cdot b) = a \cdot b$ □

Theorem 1.4. *Let $(P, \leq, \lambda)$ be a λ-poset. Then the induced algebra $P = (P, \cdot, +)$ with binary operations $\cdot$ and $+$, where $a \cdot b = \lambda_L(a, b)$ and $a + b = \lambda_U(a, b)$, is a λ-lattice.*

Proof. We prove that the induced algebra satisfies all laws to be a λ-lattice:

- $i.)$ $a \cdot a = \lambda_L(a, a) = a.$;
- $c.)$ This is true because the choice of λ is independent on the order of elements a and b;
- $t.)$ Because $a \cdot b$ is from $L(a, b)$, $a \cdot b \leq a$ in $(P, \leq)$; analogously $(a \cdot b) \cdot c \leq a \cdot b$, and from transitivity $(a \cdot b) \cdot c \leq a$. From equation ((2)), we obtain $a \cdot ((a \cdot b) \cdot c) = \lambda_L((a \cdot b) \cdot c, a) = (a \cdot b) \cdot c.$;
- $a.)$ Because $a + b \in U(a, b)$, $a + b \leq a$. From equation ((2)), we obtain $\lambda_L(a + b, a) = a..$

The proof of $i_+)$, $c_+)$, $t_+)$ and $a_+)$ can be provided analogously. □

Theorem 1.5. *Let $P = (P, \cdot, +)$ be a λ-lattice. Then the relation $\leq$ denoted by $a + b = b \implies a \leq b$ is an ordering and the partially ordered set $(P, \leq)$ with a choice function λ, where $\lambda_L(a, b) = \lambda(L(a, b)) = a \cdot b$ and $\lambda_U(a, b) = \lambda(U(a, b)) = a + b$ is a λ-poset $(P, \leq, \lambda)$. Moreover, the relation $\leq'$ denoted by $a \cdot b = b \implies a \leq' b$ is also an ordering and $\leq = \leq'$.*

Proof. First, we will prove that the relation $\leq$ is an ordering:

- $RE)$ $\leq$ is reflexive, because $a + a = a$;
- $AN)$ $\leq$ is anti-symmetric, because if $a \leq b$ and $b \leq a$ then $b = a + b = b + a = a$;
- $TR)$ $\leq$ is transitive, because if $a \leq b$ and $b \leq c$ then $a + b = b$ and $b + c = c$, hence $a + ((a + b) + c) = a + (b + c) = a + c$. From $t_+)$ we obtain $a + ((a + b) + c) = (a + b) + c = b + c = c$, hence $c = a + c$, which means that $a \leq c$.

According to Lemma 1.3 the cones $L(a, b)$ and $U(a, b)$ are non-empty for every $a, b \in P$. Moreover, we must verify, that the properties (1), (2) hold:

(1) holds because the operations $\cdot, +$ are commutative,
(2) is a consequence of Lemma 1.3.

Finally, it must be verified that $\leq = \leq'$:
If $a \leq b$ then $a + b = b$. So $a \cdot b = a \cdot (a + b) = a$. From $a.)$ we see, that $a \leq' b$ holds. Conversely we can show that from $a \leq' b$ implies $a \leq b$. □

Definition 1.6. *A subset $I \subseteq P$ of a λ-lattice $(P, \cdot, +)$ is called an ideal of P, if*

(i) $i \in I$ *and* $a \in P \Rightarrow i \cdot a \in I$ *and*

(ii) $i, j \in I \Rightarrow i + j \in I$.

Lemma 1.7. Let Θ be a congruence on a λ-lattice P. Then $[a]_\Theta$ is a convex sub-λ-lattice for every $a \in P$.

Proof. First we prove that $[a]_\Theta$ is a sub-λ-lattice. From $x \equiv a(\Theta)$ and $y \equiv a(\Theta)$ it follows that $x + y \equiv a(\Theta)$ and $x \cdot y \equiv a(\Theta)$ and we have that $[a]_\Theta$ is a sub-λ-lattice. Further we prove that $[a]_\Theta$ is convex. If $x \leq t \leq y$, $x, y \in [a]_\Theta$ and $t \in P$ then $t = t \cdot y \equiv t \cdot a(\Theta)$, $t = t + x \equiv (t \cdot a) + x \equiv (t \cdot a) + a = a(\Theta)$, so we have $t \in [a]_\Theta$. □

Theorem 1.8. *Let P be a λ-lattice. A reflexive binary relation on P is a congruence on P iff the following three properties are satisfied for any $x, y, z, t \in P$.*

(i) $x \equiv y(\Theta) \iff x \cdot y \equiv x + y(\Theta)$;

(ii) $x \leq y \leq z, x \equiv y(\Theta), y \equiv z(\Theta) \Rightarrow x \equiv z(\Theta)$;

(iii) $x \leq y$ *and* $x \equiv y(\Theta) \Rightarrow x \cdot t \equiv y \cdot t(\Theta), x + t \equiv y + t(\Theta)$.

Proof. If Θ is a congruence on P, then it obviously satisfies the conditions (i), (ii) and (iii). Hence we will prove the converse condition only. At first we prove that if $b, c \in [a, d] = \{x;\ a \leq x \leq d\}$ and if $a \equiv d(\Theta)$ then $b \equiv c(\Theta)$. According to (iii), we obtain $b \equiv d(\Theta), a \equiv b(\Theta)$. By using of (iii) again we obtain $b \cdot c \equiv c(\Theta)$, $c \equiv c + b(\Theta)$. Because $b \cdot c \leq c \leq b + c$, (ii) implies $b \cdot c \equiv b + c(\Theta)$, and by (i) also $b \equiv c(\Theta)$. According to (i) Θ is symmetric. To prove transitivity of Θ, we assume that $x \equiv y(\Theta), y \equiv z(\Theta)$. Then by (i) $x \cdot y \equiv x + y(\Theta)$, $y \cdot z \equiv y + z(\Theta)$, and by (iii)

$$y + z = (y + z) + (y \cdot x) \equiv (y + z) + (y + x)(\Theta),$$

$$y \cdot z = (y \cdot z) \cdot (y + x) \equiv (y \cdot z) \cdot (y \cdot x)(\Theta).$$

Because $y + z \equiv (y + z) + (y + x)(\Theta)$, $y \cdot z \equiv (y \cdot z) \cdot (y \cdot x)(\Theta)$ and

$$(y \cdot z) \cdot (y \cdot x) \leq y \cdot z \leq y + z \leq (y + z) + (y + x),$$

we apply (ii) twice to obtain

$$(y \cdot z) \cdot (y \cdot x) \equiv (y + z) + (y + x)(\Theta).$$

Because

$$x, z \in [(y \cdot z) \cdot (y \cdot x), (y + z) + (y + x)],$$

the proof in the preceding paragraph implies that $x \equiv z(\Theta)$.
Next we prove the assertion: if $x \equiv y(\Theta)$, then $x + t \equiv y + t(\Theta)$.

Since $x, y \in [x \cdot y, x + y]$, (i) and the proof of the first paragraph implies that $x \equiv x + y(\Theta), y \equiv x + y(\Theta)$. Now, according to (iii), $x + t \equiv (x + y) + t(\Theta), y + t \equiv (x + y) + t(\Theta)$, and by applying transitivity proved above, we obtain $x + t \equiv y + t(\Theta)$.

We are now able to prove the substitution property of Θ for $+$ operation:
Let $x_0 \equiv y_0(\Theta), x_1 \equiv y_1(\Theta)$. Then $x_0 + x_1 \equiv x_0 + y_1(\Theta), x_0 + y_1 \equiv y_0 + y_1(\Theta)$, and according to the transitivity, also $x_0 + x_1 \equiv y_0 + y_1(\Theta)$. The substitution property for $\cdot$ can be proved similarly. □

2 Relatively complemented λ-lattices

Relatively complemented lattices play an important role in the lattice theory (see e.g. [7,5]). In this chapter, we will deal with relatively complemented λ-lattices. We use Skala's [6] relatively complemented λ-lattice definition.

Definition 2.1. *Let P be λ-lattice. We say that P is relatively complemented if for each $x, y, z \in P$ such that $x \leq y \leq z$ there exists $v \in P$,so that $x \leq v \leq z$ and $y \cdot v = x$ a $y + v = z$. We call element v the relative complement of element y in the $[x, z]$ interval.*

Example 2.2 Let $P = \{0, x, 1, a, b, c, d\} \bigcup \{\alpha_i\}_{i=1}^{\infty} \bigcup \{\beta_i\}_{i=1}^{\infty}$, $i \in N$ where we define

$$a + b = c \text{ and } c \cdot d = a$$

and remaining elements of + resp. $\cdot$ operation are supremum resp. infimum. This λ-lattice, the diagram of which is provided in figure 1, is not a lattice. We shall verify, that P is relatively complemented.

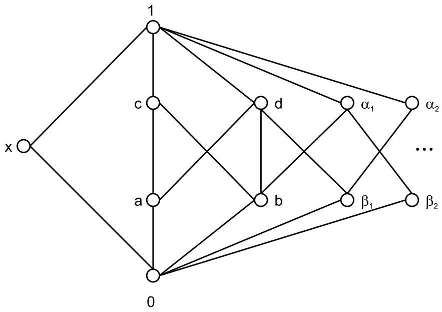

Fig. 1 A λ-lattice which is not a lattice

The element x is a complement to all elements in $[0, 1]$ interval except for 0, 1, x. Intervals $[0, c] = \{0, a, b, c\}$ and $[0, d] = \{0, a, b, \beta_1, d\}$ are evidently complementary. The proposition is again clear for $[0, \alpha_1]$, $[a, 1]$, $[b, 1]$ and $[\beta_1, 1]$ intervals.

Theorem 2.3. *If the λ-lattice P is relatively complemented, it has regular congruences.*

Proof. Let θ and ψ be two different congruences with a shared class K. First, we will show that when $x, y \geq a$ where $x, y \in P$, $a \in K$, then $x \equiv y(\theta)$ implies $x \equiv y(\psi)$.

Let us take z, which is a complement of the $(x \cdot y)$ element in $[a \cdot (x \cdot y), x + y]$ interval. Then we have $z \leq x + y$, so that $z = z \cdot (x + y) \equiv z \cdot (x \cdot y) = a \cdot (x \cdot y) \equiv a \cdot (x \cdot x) = a(\theta)$ and consequently $z \in K$. Moreover it holds, that $x + y = z + (x \cdot y) \equiv a + (x \cdot y)(\psi)$. Because $a \leq x, y$, we can see that $a = a \cdot (x + y) \equiv a \cdot (x \cdot y)(\theta)$ and $a \cdot (x \cdot y) \in K$. Consequently we get $x + y \equiv a + (x \cdot y) \equiv (a \cdot (x \cdot y)) + (x \cdot y) = x \cdot y(\psi)$. According to Lemma 1.7 this means, that $x \equiv y(\psi)$.

Similarly we can prove that if $x, y \leq a$ for some $a \in K$ then from $x \equiv y(\theta)$ follows $x \equiv y(\psi)$.

We claim that if $a \in K$ and $L = [b]_\theta$, where $b \leq a$, then $L = [b]_\psi$. If $x, y \in [b]_\theta$, then $a \cdot x \equiv a \cdot b = b(\theta)$ and it holds that $a \cdot b \leq a, b \leq a$, $a + x \equiv a + b = a(\theta)$ and $a + b \geq a, b \geq a$ as well.

Using frst part of the proof with $(a \cdot x) \cdot y$ and $[(a \cdot x) \cdot y]_\theta$ playing the roles of a and K respectively, we get $a \cdot x \equiv b(\psi)$, $a + x \equiv a(\psi)$, hence $x = (a+x) \cdot x \equiv a \cdot x \equiv b(\psi)$. Similarly we can show that $y \equiv b(\psi)$, hence $x \equiv y(\psi)$. In the last step we will show that $x \equiv y(\theta)$ implies $x \equiv y(\psi)$. We will choose the element $a \in K$ and in following reasoning we will use the class $L = [(a \cdot x) \cdot y]_\theta$. So we have $(a \cdot x) \cdot y \leq a$, a and according to previous item $x \equiv y(\psi)$. Hence we proved, that $\theta \subseteq \psi$. The opposite inclusion can be proved analogically. So we proved that $\theta = \psi$. □

Theorem 2.4. *Let P be relatively complemented λ-lattice. Then P has permutable congruences.*

Proof. Let θ and ψ be congruences on P. Be $a, b, x \in P$, $a \equiv x(\theta)$, $x \equiv b(\psi)$. First, we will show that if $x \in [a, b]$, then there exists $y \in [a, b]$ such that $a \equiv y(\psi)$ and $y \equiv b(\theta)$ hold. Let us choose y to be a complement of element x in $[a, b]$ interval. Then it holds that $y = b \cdot y \equiv x \cdot y = a(\psi)$, $y = a + y \equiv x + y = b(\theta)$.

Second we have $a = a + a \equiv a + x = (a + x) + x(\theta)$, $(a + x) + x \equiv (a + b) + x(\psi)$, $(a+b)+x \equiv (x+b)+x = b+x(\theta)$, $b+x \equiv b+b = b(\psi)$. We will use first part of the proof which confirms the existence of elements $u \in [a, (a+b)+x]$, $v \in [b, (a+b)+x]$ such that $a \equiv u(\psi), u \equiv (a + b) + x(\theta)$, $(a + b) + x \equiv v(\psi), v \equiv b(\theta)$. Hence we have $u = u(\cdot(a + b) + x) \equiv u \cdot v(\psi)$, $v = v \cdot ((a + b) + x) \equiv u \cdot v(\theta)$, and further $a \equiv u \cdot v(\psi), u \cdot v \equiv b(\theta)$. □

Theorem 2.5. *In every relatively complemented λ-lattice P with zero element every congruence θ is determined by the ideal $I_\theta = \{x \in P, x \equiv 0(\theta)\}$.*

Proof. This theorem is a trivial consequence of Theorem 2.3.

3 Conclusion

In this paper, we discussed one class of λ-lattices, namely relatively complemented λ-lattices. In future, we would like to use λ-lattices in XML indexing (for a detailed application see [8]) in cases, when the XML document's structure prevents the usage of classic lattices.

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