

DIRECT DECOMPOSITIONS OF BASIC ALGEBRAS AND THEIR IDEMPOTENT MODIFICATIONS

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Dedicated to the 70th birthday of Alfonz Haviar

ABSTRACT. We get a necessary and sufficient condition under which a given basic algebra \mathcal{A} is isomorphic to a direct product of non-trivial basic algebras $\mathcal{A}_1, \mathcal{A}_2$ which are in fact interval subalgebras of \mathcal{A} . Further, we prove that the idempotent modification of \mathcal{A} is directly indecomposable whenever \mathcal{A} has at least one element which is not boolean.

1. INTRODUCTION

It is well-known that a bounded lattice $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ is directly decomposable into lattices $\mathcal{L}_1, \mathcal{L}_2$ isomorphic to the intervals $[a, 1], [b, 1]$ of \mathcal{L} if b is a complement of a and a, b are standard elements. Since every basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ induces a lattice $\mathcal{L}(A) = (A; \vee, \wedge)$ which is bounded by 0 and $1 = \neg 0$, we can ask if also \mathcal{A} is directly decomposable whenever there exists a complemented and standard element of $\mathcal{L}(A)$. In what follows we show that the condition concerning this element must be enlarged due to the fact that the operations \oplus and \neg cannot be derived by means of the lattice operations of $\mathcal{L}(A)$. However, we set up a natural necessary and sufficient condition for the direct decomposability of \mathcal{A} .

By a **basic algebra** (see e.g. [1, 2]) is meant an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following four axioms

$$(BA1) \quad x \oplus 0 = x;$$

$$(BA2) \quad \neg \neg x = x;$$

$$(BA3) \quad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x;$$

$$(BA4) \quad \neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0.$$

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As usual, we will write 1 instead of $\neg 0$. We say that a basic algebra \mathcal{A} is **non-trivial** if $0 \neq 1$ (i.e. $|A| > 1$).

Having a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$, one can introduce the **induced order** \leq on \mathcal{A} as follows

$$x \leq y \quad \text{if and only if} \quad \neg x \oplus y = 1.$$

It is an easy exercise to verify that \leq is really an order on A and $0 \leq x \leq 1$ for each $x \in A$. Moreover, $(A; \leq)$ is a bounded lattice in which

$$x \vee y = \neg(\neg x \oplus y) \oplus y \quad \text{and} \quad x \wedge y = \neg(\neg x \vee \neg y).$$

For some details, the reader is referred to [1]. The lattice $\mathcal{L}(A) = (A; \vee, \wedge)$ will be called the **induced lattice** of \mathcal{A} . In particular for each $a \in A$ there exists an antitone involution $x \mapsto x^a$ on the interval $[a, 1]$ (called a **section**) where $x^a = \neg x \oplus a$.

It is well-known (see e.g. [1, 3]) that also conversely, if $(A; \vee, \wedge, ({}^a)_{a \in A}, 0, 1)$ is a bounded lattice with section antitone involutions, we are able to construct a basic algebra using the operations

$$(1) \quad x \oplus y = (x^0 \vee y)^y \quad \text{and} \quad \neg x = x^0.$$

Lemma 1. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra, \leq the induced order and $a \in A$. Define the polynomial operations \neg_a and \oplus_a on the interval $[a, 1]$ as follows*

$$\neg_a x = \neg x \oplus a \quad \text{and} \quad x \oplus_a y = \neg(\neg x \oplus a) \oplus y.$$

Then $([a, 1]; \oplus_a, \neg_a, a)$ is a basic algebra.

Proof. We use the facts that $y \leq x \oplus y$, $0 \oplus x = x$ and $\neg x \oplus x = 1$ hold in each basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ (for more details see e.g. [1]). If $x, y \in [a, 1]$ then $a \leq y \leq \neg(\neg x \oplus a) \oplus y = x \oplus_a y$ thus \oplus_a is really a binary operation on $[a, 1]$. Since $a \leq \neg x \oplus a$, $\neg_a x$ is a unary operation on $[a, 1]$. Moreover, $\neg_a a = \neg a \oplus a = 1$ and $\neg_a 1 = \neg 1 \oplus a = 0 \oplus a = a$. We must check the axioms (BA1)–(BA4).

(BA1) and (BA2): For $x \in [a, 1]$ we have $x \oplus_a a = \neg(\neg x \oplus a) \oplus a = x \vee a = x$ and, analogously, $\neg_a \neg_a x = \neg(\neg x \oplus a) \oplus a = x \vee a = x$.

(BA3): Assume that $x, y \in [a, 1]$. Since $y \leq \neg x \oplus y$, then also $a \leq \neg x \oplus y$. Further, we have $\neg_a x \oplus_a y = \neg x \oplus y$. Hence, we compute

$$\neg_a(\neg_a x \oplus_a y) \oplus_a y = \neg(\neg x \oplus y) \oplus y = x \vee y$$

and, by symmetry, also

$$\neg_a(\neg_a y \oplus_a x) \oplus_a x = y \vee x = x \vee y.$$

(BA4): Let $x, y, z \in [a, 1]$. Since $a \leq x \oplus_a y$, $a \leq y \leq \neg(x \oplus_a y) \oplus y$ and $a \leq z \leq \neg(\neg(x \oplus_a y) \oplus y) \oplus z$, we obtain

$$\begin{aligned} \neg_a(\neg_a(\neg_a(x \oplus_a y) \oplus_a y) \oplus_a z) \oplus_a (x \oplus_a z) &= \\ &= \neg_a(\neg(\neg(x \oplus_a y) \oplus y) \oplus z) \oplus_a (x \oplus_a z) = \\ &= \neg(\neg(\neg(x \oplus_a y) \oplus y) \oplus z) \oplus (x \oplus_a z) = \\ &= \neg(\neg(\neg(w \oplus y) \oplus y) \oplus z) \oplus (w \oplus z) = 1, \end{aligned}$$

where $w = \neg(\neg x \oplus a)$. □

The basic algebra $([a, 1]; \oplus_a, \neg_a, a)$ where the operations \oplus_a, \neg_a are defined as in Lemma 1 will be called an **interval basic algebra**. Our motivation for introducing the operations \oplus_a and \neg_a in this way is inspired by (1), where we only replace x^0 by x^a due to the fact that a is the bottom element of the section $[a, 1]$. Since $x \in [a, 1]$, by (1) we have

$$\neg_a x = x^a = (x \vee a)^a = \neg x \oplus a$$

and then for $x, y \in [a, 1]$

$$x \oplus_a y = (\neg_a x \vee y)^y = ((\neg x \oplus a) \vee y)^y = \neg(\neg x \oplus a) \oplus y.$$

Due to the fact that every basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is in a one-to-one correspondence with the enriched lattice $\mathcal{L}(A) = (A; \vee, \wedge, ({}^a)_{a \in A}, 0, 1)$ as mentioned above (see also [1]), the interval basic algebra $([a, 1]; \oplus_a, \neg_a, a)$ is in the same correspondence with the interval enriched lattice $([a, 1]; \vee, \wedge, ({}^b)_{b \in [a, 1]}, a, 1)$ where $\vee, \wedge, {}^b$ are the same as in $\mathcal{L}(A)$. Hence, our interval basic algebra is quite a natural "cut" of the original one.

Lemma 2. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra and $a, b, c \in A$. Then*

$$(b \wedge c) \oplus a = (b \oplus a) \wedge (c \oplus a).$$

Proof. We compute by (1)

$$\begin{aligned} (b \wedge c) \oplus a &= (\neg(b \wedge c) \vee a)^a = ((\neg b \vee \neg c) \vee a)^a = ((\neg b \vee a) \vee (\neg c \vee a))^a = \\ &= (\neg b \vee a)^a \wedge (\neg c \vee a)^a = (b \oplus a) \wedge (c \oplus a) \end{aligned}$$

since $\neg b \vee a \in [a, 1]$ and $\neg c \vee a \in [a, 1]$. □

2. DIRECT DECOMPOSIBILITY OF BASIC ALGEBRAS

Now, we will set up the conditions under which a basic algebra \mathcal{A} can be directly decomposed. First, we define several concepts.

Definition 1. *An element a of a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is called **strong** if*

- (a) $x \oplus a = x \vee a$ and $x \oplus \neg a = x \vee \neg a$
for every $x \in A$.

A strong element a of \mathcal{A} is called a **decomposing element** if it moreover satisfies

- (b) $(x \oplus y) \oplus a = x \oplus (y \oplus a)$, $(x \oplus y) \oplus \neg a = x \oplus (y \oplus \neg a)$
and $x \oplus a = a \oplus x$, $x \oplus \neg a = \neg a \oplus x$
for all $x, y \in A$.

Let us note that 0 and 1 are decomposing elements for every basic algebra \mathcal{A} .

Recall (see [4]) that the element a of a lattice $(L; \vee, \wedge)$ is called **distributive** if for all $x, y \in L$

$$(x \wedge y) \vee a = (x \vee a) \wedge (y \vee a)$$

and the element a of a lattice $(L; \vee, \wedge)$ is called **standard** if for all $x, y \in L$

$$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y).$$

Further, recall that if $(L; \vee, \wedge)$ is a lattice and $a \in L$ then the following two conditions are equivalent:

- (α) a is standard
(β) a is distributive and, for $x, y \in L$,

$$a \wedge x = a \wedge y \quad \text{and} \quad a \vee x = a \vee y \quad \text{imply that} \quad x = y$$

(for more details see [4]).

Lemma 3. *Let a be a strong element of a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$. Then*

- (i) a is boolean (i.e. $a \vee \neg a = 1$, $a \wedge \neg a = 0$);
(ii) a and $\neg a$ are distributive elements.

Proof. (i) By Definition 1 we have $a \oplus a = a \vee a = a$ and $\neg a \oplus \neg a = \neg a \vee \neg a = \neg a$ thus both a and $\neg a$ are \oplus -idempotents. Then $\neg a \vee a = \neg a \oplus a = 1$ and dually (by De Morgan law) also $\neg a \wedge a = 0$.

(ii) Follows directly by Lemma 2 and the condition (a) of Definition 1. \square

Lemma 4. *Let a be a strong element of a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ and $\neg a$ be a standard element of the induced lattice $\mathcal{L}(A) = (A; \vee, \wedge)$. Then the mapping $\varphi_a(x) = (x \vee a, x \vee \neg a)$ is a lattice isomorphism of $\mathcal{L}(A)$ onto the direct product of lattices $([a, 1]; \vee, \wedge) \times ([\neg a, 1]; \vee, \wedge)$.*

Proof. The proof is only a slight modification of the proof of Theorem 1.4. (p. 200) in [4]. Namely, if $\varphi_a(x) = \varphi_a(y)$ then $x \vee a = y \vee a$ and $x \vee \neg a = y \vee \neg a$, i.e.

$$(2) \quad \neg x \wedge \neg a = \neg y \wedge \neg a$$

by the first equality and $\neg x \wedge a = \neg y \wedge a$ by the second one, i.e. also

$$(\neg x \wedge a) \vee \neg a = (\neg y \wedge a) \vee \neg a$$

thus, by Lemma 3, also

$$(3) \quad \neg x \vee \neg a = \neg y \vee \neg a.$$

Since $\neg a$ is a standard element of $\mathcal{L}(A) = (A; \vee, \wedge)$, (2) and (3) yields $\neg x = \neg y$, i.e. $x = \neg\neg x = \neg\neg y = y$. Hence, φ_a is injective. If $\langle c, d \rangle \in [a, 1] \times [\neg a, 1]$ then $a \leq c$, $\neg a \leq d$, i.e. $d \vee a \geq \neg a \vee a = 1$, $c \vee \neg a \geq a \vee \neg a = 1$ and for $c \wedge d \in A$ we have

$$\begin{aligned} \varphi_a(c \wedge d) &= ((c \wedge d) \vee a, (c \wedge d) \vee \neg a) = \\ &= ((c \vee a) \wedge (d \vee a), (c \vee \neg a) \wedge (d \vee \neg a)) = \\ &= ((c \vee a) \wedge 1, 1 \wedge (d \vee \neg a)) = (c, d) \end{aligned}$$

hence φ_a is also surjective. Therefore it is a bijection from A to $[a, 1] \times [\neg a, 1]$. Further,

$$\begin{aligned} \varphi_a(x \vee y) &= ((x \vee y) \vee a, (x \vee y) \vee \neg a) = \\ &= (x \vee a, x \vee \neg a) \vee (y \vee a, y \vee \neg a) = \varphi_a(x) \vee \varphi_a(y) \end{aligned}$$

and

$$\begin{aligned} \varphi_a(x \wedge y) &= ((x \wedge y) \vee a, (x \wedge y) \vee \neg a) = \\ &= ((x \vee a) \wedge (y \vee a), (x \vee \neg a) \wedge (y \vee \neg a)) = \varphi_a(x) \wedge \varphi_a(y) \end{aligned}$$

thus φ_a is a lattice isomorphism of $\mathcal{L}(A)$ onto $([a, 1]; \vee, \wedge) \times ([\neg a, 1]; \vee, \wedge)$. \square

Theorem 1. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra. Then \mathcal{A} is isomorphic to a direct product of non-trivial basic algebras $\mathcal{B}_1, \mathcal{B}_2$ if and only if there exists a decomposing element $a \in A$, $0 \neq a \neq 1$ such that $\neg a$ is standard in the induced lattice $\mathcal{L}(A) = (A; \vee, \wedge)$. If it is the case then \mathcal{A} is isomorphic to the direct product of interval basic algebras $([a, 1]; \oplus_a, \neg_a, a)$ and $([\neg a, 1]; \oplus_{\neg a}, \neg_{\neg a}, \neg a)$.*

Proof. Due to Lemma 4, we must only show that φ_a preserves the operations \oplus and \neg . Denote by $\hat{\oplus}$ and $\hat{\neg}$ the corresponding operations on the direct product of the interval algebras. Then, since a and $\neg a$ are strong elements, we have

$$\varphi_a(\neg x) = (\neg x \vee a, \neg x \vee \neg a) = (\neg x \oplus a, \neg x \oplus \neg a) = (\neg_a x, \neg_{\neg a} x) = \hat{\neg} \varphi_a(x).$$

Since a is a decomposing element, we derive also

$$\varphi_a(x \oplus y) = ((x \oplus y) \oplus a, (x \oplus y) \oplus \neg a)$$

and, by Lemma 2, we compute

$$\begin{aligned}
\varphi_a(x) \widehat{\oplus} \varphi_a(y) &= (x \vee a, x \vee \neg a) \widehat{\oplus} (y \vee a, y \vee \neg a) = \\
&= (x \oplus a, x \oplus \neg a) \widehat{\oplus} (y \oplus a, y \oplus \neg a) = \\
&= ((x \oplus a) \oplus_a (y \oplus a), (x \oplus \neg a) \oplus_{\neg a} (y \oplus \neg a)) = \\
&= (\neg(\neg(x \oplus a) \oplus a) \oplus (y \oplus a), \neg(\neg(x \oplus \neg a) \oplus \neg a) \oplus (y \oplus \neg a)) = \\
&= (\neg(\neg x \vee a) \oplus (y \oplus a), \neg(\neg x \vee \neg a) \oplus (y \oplus \neg a)) = \\
&= ((x \wedge \neg a) \oplus (y \oplus a), (x \wedge a) \oplus (y \oplus \neg a)) = \\
&= ((x \oplus (y \oplus a)) \wedge (\neg a \oplus (y \oplus a)), (x \oplus (y \oplus \neg a)) \wedge (a \oplus (y \oplus \neg a))) = \\
&= ((x \oplus (y \oplus a)) \wedge (\neg a \vee y \vee a), (x \oplus (y \oplus \neg a)) \wedge (a \vee y \vee \neg a)) = \\
&= ((x \oplus (y \oplus a)) \wedge 1, (x \oplus (y \oplus \neg a)) \wedge 1) = \\
&= (x \oplus (y \oplus a), x \oplus (y \oplus \neg a)).
\end{aligned}$$

Due to (b) of Definition 1, we conclude

$$\varphi_a(x \oplus y) = \varphi_a(x) \widehat{\oplus} \varphi_a(y)$$

thus φ_a preserves \oplus and \neg and hence it is an isomorphism of \mathcal{A} onto the direct product $([a, 1]; \oplus_a, \neg_a, a) \times ([-a, 1], \oplus_{\neg a}, \neg_{\neg a}, \neg a)$.

Conversely, assume that a basic algebra \mathcal{A} is isomorphic to a direct product $\mathcal{B}_1 \times \mathcal{B}_2$ of non-trivial basic algebras $\mathcal{B}_1 = (B_1; \oplus_1, \neg_1, 0_1)$ and $\mathcal{B}_2 = (B_2; \oplus_2, \neg_2, 0_2)$. It is an easy exercise to show that $(0_1, 1_2)$ (where $1_2 = \neg_2 0_2$) is a decomposing element of $\mathcal{B}_1 \times \mathcal{B}_2$ and hence $h^{-1}((0_1, 1_2))$ is a decomposing element of \mathcal{A} (where h is the isomorphism of \mathcal{A} onto $\mathcal{B}_1 \times \mathcal{B}_2$). \square

Example 1. Consider the lattice drawn in Fig. 1.

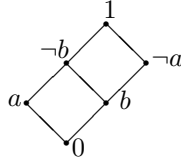


Fig. 1

We can define the operations \oplus^1 and \oplus^2 as follows

\oplus^1	0	a	b	$\neg b$	$\neg a$	1
0	0	a	b	$\neg b$	$\neg a$	1
a	a	a	$\neg a$	$\neg b$	1	1
b	b	$\neg b$	$\neg b$	1	$\neg a$	1
$\neg b$	$\neg b$	$\neg b$	1	1	1	1
$\neg a$	$\neg a$	1	$\neg b$	1	$\neg a$	1
1	1	1	1	1	1	1

\oplus^2	0	a	b	$\neg b$	$\neg a$	1
0	0	a	b	$\neg b$	$\neg a$	1
a	a	a	$\neg b$	$\neg b$	1	1
b	b	$\neg b$	$\neg a$	1	$\neg a$	1
$\neg b$	$\neg b$	$\neg b$	1	1	1	1
$\neg a$	$\neg a$	1	$\neg a$	1	$\neg a$	1
1	1	1	1	1	1	1

Then for $A = \{0, a, b, \neg b, \neg a, 1\}$ we have that $\mathcal{A}_1 = (A; \oplus^1, \neg, 0)$ and $\mathcal{A}_2 = (A; \oplus^2, \neg, 0)$ are basic algebras (where \mathcal{A}_2 is even an MV-algebra but \mathcal{A}_1 is not). In the both cases a is a strong element, but in \mathcal{A}_1 a is not a decomposing element, since for $x = b$ we have

$$a \oplus b = \neg a \neq \neg b = b \oplus a$$

which contradicts (b) of Definition 1. On the other hand, one can check by a direct computation that a is a decomposing element of \mathcal{A}_2 . \diamond

3. IDEMPOTENT MODIFICATION OF BASIC ALGEBRAS

The concept of idempotent modification of an algebra was introduced by J. Ježek [6] as follows.

Definition 2. An *idempotent modification* of an algebra $\mathcal{A} = (A; F)$ is an algebra $\mathcal{A}_I = (A; F_I)$ with the same underlying set A , where $|F| = |F_I|$ and for every $f \in F$ the corresponding operation $f_I \in F_I$ is defined as follows

- (i) if f is at most unary then $f_I = f$;
- (ii) if f is n -ary with $n > 1$ and $a_1, \dots, a_n \in A$ then

$$f_I(a_1, \dots, a_n) = \begin{cases} a_1 & \text{if } a_1 = a_2 = \dots = a_n \\ f(a_1, \dots, a_n) & \text{otherwise.} \end{cases}$$

The main result of [6] is that for any group G its idempotent modification G_I is subdirectly irreducible.

In what follows we will treat direct decomposability of an idempotent modification of a basic algebra.

For this we slightly modify our definition of basic algebra. As mentioned above, every basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ has induced lattice $\mathcal{L}(\mathcal{A}) = (A; \vee, \wedge)$ where \vee and \wedge are term operations of \mathcal{A} . Hence, inserting \vee and \wedge into the type of \mathcal{A} , we obtain an algebra with the same clone of term operations and hence term equivalent to \mathcal{A} . From now on, by a basic algebra we will understand an algebra $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ where the term operations \vee and \wedge are defined by $x \vee y = \neg(\neg x \oplus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y)$.

The reason of this insertion is that when an idempotent modification of $(A; \oplus, \neg, 0)$ is considered, the resulting algebra does not have the lattice structure. However,

if $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ is treated then the lattice structure for \mathcal{A}_I is preserved because both \vee and \wedge are idempotent operations on A .

Theorem 2. *Let $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ be a basic algebra whose at least one element is not boolean. Then its idempotent modification $\mathcal{A}_I = (A; \oplus_I, \neg, 0, \vee, \wedge)$ is not directly decomposable.*

Proof. At first we show that if $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ is a basic algebra and its idempotent modification \mathcal{A}_I is directly decomposable into non-trivial algebras $\mathcal{B}_1 = (B_1; \oplus_1, \neg_1, 0_1, \vee, \wedge)$ and $\mathcal{B}_2 = (B_2; \oplus_2, \neg_2, 0_2, \vee, \wedge)$ then also \mathcal{A} is directly decomposable. Denote by $1_1 = \neg_1 0_1$ and $1_2 = \neg_2 0_2$. Let φ be an isomorphism of \mathcal{A}_I onto $\mathcal{B}_1 \times \mathcal{B}_2$. For $x \in A$ let $\varphi(x) = (x_1, x_2)$. Define new operations \oplus^1, \oplus^2 on B_1, B_2 , respectively as follows. If $y_1, z_1 \in B_1$ and $y_1 \neq z_1$ then $y_1 \oplus^1 z_1 = y_1 \oplus_1 z_1$, if $y_2, z_2 \in B_2$ and $y_2 \neq z_2$ then $y_2 \oplus^2 z_2 = y_2 \oplus_2 z_2$. If $x_1 \in B_1$, denote by $\overline{x_1} = \varphi^{-1}((x_1, 1_2))$ and if $x_2 \in B_2$, denote by $\overline{x_2} = \varphi^{-1}((1_1, x_2))$. Now we define

$$x_1 \oplus^1 x_1 = \text{pr}_1(\varphi(\overline{x_1} \oplus \overline{x_1}))$$

and

$$x_2 \oplus^2 x_2 = \text{pr}_2(\varphi(\overline{x_2} \oplus \overline{x_2})).$$

Since $\varphi(x) = (x_1, x_2) = (x_1, 1_2) \wedge (1_1, x_2) = \varphi(\overline{x_1}) \wedge \varphi(\overline{x_2})$ and since φ and also φ^{-1} preserve the lattice operations, we have $x = \overline{x_1} \wedge \overline{x_2}$. This yields that \oplus^1, \oplus^2 are correctly defined (i.e. the result $x_1 \oplus^1 x_1$ in the first coordinate does not depend on the second coordinate and vice versa), i.e.

$$\varphi(x \oplus x) = (x_1, x_2) \overline{\oplus} (x_1, x_2) = (x_1 \oplus^1 x_1, x_2 \oplus^2 x_2),$$

where $\overline{\oplus}$ is the binary operation provided coordinatewise on the Cartesian product $B_1 \times B_2$.

It is obvious that $\mathcal{A}^1 = (B_1; \oplus^1, \neg_1, 0_1, \vee, \wedge)$ and $\mathcal{A}^2 = (B_2; \oplus^2, \neg_2, 0_2, \vee, \wedge)$ are basic algebras and φ is also an isomorphism of \mathcal{A} onto $\mathcal{A}^1 \times \mathcal{A}^2$. Moreover, we see that $\mathcal{B}_1 = \mathcal{A}_I^1$ and $\mathcal{B}_2 = \mathcal{A}_I^2$. Hence, if \mathcal{A}_I is directly decomposable then also \mathcal{A} has this property.

Assume now that $x \in A$ is not boolean and that \mathcal{A}_I is directly decomposable. We can apply the reasoning used by J. Jakubík [5]. Let $\varphi(x) = (x_1, x_2)$. Then also $\varphi(x)$ is not boolean, i.e. at least one of x_1, x_2 is not boolean. Without loss of generality, suppose that x_1 is not boolean. Then $x_1 \oplus^1 x_1 \neq x_1$. Since $|B_2| > 1$, there exists $y_2 \in B_2$ such that $x_2 \neq y_2$. Let $y = \varphi^{-1}(x_1, y_2)$. Then $x \neq y$ and $x \oplus y = x \oplus_I y$, hence $\varphi(x \oplus y) = \varphi(x \oplus_I y)$. However, $\varphi(x \oplus y) = (x_1 \oplus^1 x_1, x_2 \oplus^2 y_2)$ and $\varphi(x \oplus_I y) = (x_1 \oplus_1 x_1, x_2 \oplus_2 y_2) = (x_1, x_2 \oplus^2 y_2)$, which is a contradiction. Thus \mathcal{A}_I is not directly decomposable. \square

Call a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ **distributive** if the induced lattice $\mathcal{L}(A) = (A; \vee, \wedge)$ is distributive. For example, if \mathcal{A} is commutative then \mathcal{A} is

distributive (but not vice versa, see Example 1) see e.g. [1]. For distributive basic algebras, we can modify our result as follows

Corollary. *Let $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ be a distributive basic algebra with $|A| > 2$. Then its idempotent modification is directly indecomposable if and only if \mathcal{A} contains an element which is not boolean.*

Proof. If all elements of \mathcal{A} are boolean then, due to the fact that $\mathcal{L}(A)$ is distributive, its idempotent modification \mathcal{A}_I is in fact a Boolean algebra (where \oplus coincides with \vee). Hence, \mathcal{A}_I is directly decomposable since $|A| > 2$.

Conversely, if \mathcal{A} contains an element which is not boolean then \mathcal{A}_I is not directly decomposable by Theorem 2. \square

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