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ON FORMATIONS OF LATTICES

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Dedicated to the 70th birthday of Alfonz Haviar

Abstract. A class of lattices is said to be a formation if it is closed under homomorphic images and finite subdirect products. Let us denote by $\mathbb{F}$ the collection of all formations of lattices. Then $\mathbb{F}$ can be partially ordered by the class–theoretical inclusion. We study the properties of this partially ordered class; e.g., there are described all atoms of $\mathbb{F}$.

1. Introduction

A class of algebras is said to be a formation if it is closed under homomorphic images and finite subdirect products. This concept appeared first in the 1970's in the connection with finite groups. Formations of groups were studied by several authors. Let us mention at least the monograph [3] of Shemetkov, which deals with formations of finite groups. Nevertheless, Chapter I of [3] contains a detailed presentation of basic notions of the theory without assuming the finiteness of the groups under consideration. In fact, the above definition can be used for any class of similar algebras. Formations of lattice ordered groups and GMV–algebras were investigated by Jakubík [2].

Let $\mathbb{F}$ be the collection of all formations of lattices. For $\mathcal{F}_1, \mathcal{F}_2 \in \mathbb{F}$ we write $\mathcal{F}_1 \leq \mathcal{F}_2$ if $\mathcal{F}_1$ is a subclass of $\mathcal{F}_2$. The collection $\mathbb{F}$ is large; namely, there exists an injective mapping of the class of all infinite cardinals into the collection $\mathbb{F}$. Nevertheless, with respect to the relation $\leq$ in $\mathbb{F}$, we will use the usual notions and the notation of the theory of partially ordered sets. We will prove that for any indexed system of elements of $\mathbb{F}$, both supremum and infimum exist.

For any class $\mathcal{K}$ of lattices, we will describe the least formation $\mathsf{form}(\mathcal{K})$ containing $\mathcal{K}$. Each formation of lattices, except the least one, contains subdirectly irreducible lattices. But, in contrast with varieties of lattices, different formations of lattices can have the same subclass of subdirectly irreducible lattices.

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In Section 5, we will describe all atoms in $\mathbb{F}$. They form a proper class, just like antiatoms of $\mathbb{F}$.

Finally, we will show that the class of formations of distributive lattices contains both large chains and large antichains.

2. Preliminaries

We will use the terminology and the notation as in Grätzer [1].

The direct product of an indexed system $(L_i)_{i\in I}$ of lattices is defined in the usual way; we apply the notation $\prod_{i\in I} L_i$ or $L_1 \times L_2 \times \ldots \times L_n$ if $I = \{1, 2, \ldots, n\}$. For $x = (x_i)_{i\in I}$ in $\prod_{i\in I} L_i$, x_i is the component of x in L_i; we write also $x_i = x(L_i)$. Let $K \subseteq \prod_{i\in I} L_i$ and $i_0 \in I$; we put $K(L_{i_0}) = \{x(L_{i_0}) : x \in K\}$. If K is a sublattice of $\prod_{i\in I} L_i$ and $K(L_i) = L_i$ for each $i \in I$, then K is said to be a subdirect product of the system $(L_i)_{i\in I}$. In such a case we write $K \leq \prod_{i\in I} L_i$. If the index set I is finite, K will be referred to as a finite subdirect product.

If L is a lattice, θ a congruence relation on L and $a \in L$, the symbol $[a]\theta$ will be used for the congruence class containing the element a.

3. The class $\mathsf{form}(\mathcal{K})$

Let $\mathcal{L}$ be the class of all lattices. For any class $\mathcal{K}$ of lattices we denote by

$\mathbf{H}(\mathcal{K})$– the class of all homomorphic images of elements of $\mathcal{K}$;

$\mathbf{P}_{\mathrm{FS}}(\mathcal{K})$– the class of all finite subdirect products of elements of $\mathcal{K}$.

A class $\mathcal{F}$ of lattices is said to be a *formation* if is closed with respect to the operators $\mathbf{H}$ and $\mathbf{P}_{\mathrm{FS}}$.

It is easy to see that each variety of lattices is also a formation. The converse does not hold in general; e.g., the class of all finite lattices is a formation which fails to be a variety.

Let $\mathcal{K}$ be any class of lattices. We will describe the least formation containing $\mathcal{K}$. If $\mathcal{K} = \emptyset$, then it is evidently the class of all one–element lattices. Suppose that $\mathcal{K} \neq \emptyset$. It is $\mathbf{P}_{\mathrm{FS}}\,\mathbf{H}(\mathcal{K}) \subseteq \mathbf{H}\,\mathbf{P}_{\mathrm{FS}}(\mathcal{K})$; this can be shown in the same way as the well–known inclusion $\mathbf{P}_{\mathrm{S}}\,\mathbf{H}(\mathcal{K}) \subseteq \mathbf{H}\,\mathbf{P}_{\mathrm{S}}(\mathcal{K})$, where $\mathbf{P}_{\mathrm{S}}$ stands for the operator of forming subdirect products. Using also the idempotency of the operators $\mathbf{H}$ and $\mathbf{P}_{\mathrm{FS}}$, we obtain:

Theorem 3.1. *Let $\mathcal{K}$ be any nonempty class of lattices. Then $\mathbf{H}\,\mathbf{P}_{\mathrm{FS}}(\mathcal{K})$ is a formation of lattices. Moreover, it is the least one containing $\mathcal{K}$.*

For any $\mathcal{K} \subseteq \mathcal{L}$, the least formation containing $\mathcal{K}$ will be denoted by $\mathsf{form}(\mathcal{K})$. So, if $\mathcal{K} \neq \emptyset$, then $\mathsf{form}(\mathcal{K}) = \mathbf{H}\,\mathbf{P}_{\mathrm{FS}}(\mathcal{K})$. Let us remark that $\mathbf{H}\,\mathbf{P}_{\mathrm{S}}(\mathcal{K})$ is the least variety of lattices containing $\mathcal{K}$ (cf. [1],Corollary 5.1.5).

We will show that if $\mathcal{K}$ contains only distributive lattices, then $\mathsf{form}\,\mathcal{K} = \mathbf{P}_{\mathrm{FS}}\,\mathbf{H}(\mathcal{K})$, i.e., the operators $\mathbf{H}$, $\mathbf{P}_{\mathrm{FS}}$ can be applied in arbitrary order. We will use the following assertions.

Proposition 3.2 ([1], Theorem 2.3.6). *Let K be a sublattice of a distributive lattice L. Any congruence relation θ of K can be extended to L; that is, there exists a congruence relation ϕ on L such that $x\ \phi\ y$ iff $x\ \theta\ y$ for $x, y \in K$.*

Proposition 3.3 ([1], Theorem 1.3.13). *Let L and K be lattices, let θ be a congruence relation of L, and let ϕ be a congruence relation of K. Define the relation $\theta \times \phi$ on $L \times K$ by*

$$(a, b)\ \theta \times \phi\ (c, d) \quad \text{iff} \quad a\ \theta\ c \quad \text{and} \quad b\ \phi\ d.$$

Then $\theta \times \phi$ is a congruence relation on $L \times K$. Conversely, every congruence relation of $L \times K$ is of this form.

Theorem 3.4. *Let $\mathcal{K}$ be a class containing only distributive lattices. Then $\mathsf{form}(\mathcal{K}) = \mathbf{P}_{\mathrm{FS}}\,\mathbf{H}(\mathcal{K})$.*

Proof. By Theorem 3.1, it suffices to prove the inclusion $\mathbf{H}\,\mathbf{P}_{\mathrm{FS}}(\mathcal{K}) \subseteq \mathbf{P}_{\mathrm{FS}}\,\mathbf{H}(\mathcal{K})$. Let $L \in \mathbf{H}\,\mathbf{P}_{\mathrm{FS}}(\mathcal{K})$. Then there exist lattices $A_1, \ldots, A_n \in \mathcal{K}$, $B \leq A_1 \times \ldots \times A_n$ and a homomorphism φ of B onto L. Let $\theta = \mathrm{Ker}\ \varphi$, ϕ an extension of θ to a congruence relation of $A_1 \times \ldots \times A_n$. Further, let $\phi = \phi_1 \times \ldots \times \phi_n$ with ϕ_i being a congruence relation of A_i for $i = 1, \ldots, n$.

We shall show that L is isomorphic to a subdirect product of $(A_i/\phi_i)_{i\in\{1,\ldots,n\}}$. Let us define $\psi : L \to A_1/\phi_1 \times \ldots \times A_n/\phi_n$ by

$$\psi(a) = ([b_1]\phi_1, \ldots, [b_n]\phi_n),$$

where $b = (b_1, \ldots, b_n)$ is any element of B with $\varphi(b) = a$.

It is easy to see that the definition of ψ is correct and that ψ is a one–to–one homomorphism. Moreover, if $a_i \in A_i$ and c is any element of B with $c(A_i) = a_i$, we have

$$(\psi\,(\varphi(c)))\,(A_i/\phi_i) = [a_i]\phi_i.$$

Since $A_i/\phi_i \in \mathbf{H}(\mathcal{K})$ for all $i \in \{1, \ldots, n\}$, we have proved $L \in \mathbf{P}_{\mathrm{FS}}\,\mathbf{H}(\mathcal{K})$. □

Let L be a nontrivial lattice, ω the least congruence relation of L. If ω is a completely meet–irreducible (a meet irreducible) element in the complete lattice *Con* L of all congruence relations on L, then L is said to be a subdirectly irreducible (a finitely subdirectly irreducible) lattice.

The following theorem is a slight modification of the well–known Jónsson's lemma ([1], Theorem 5.1.9).

Theorem 3.5. *Let $\mathcal{K}$ be any class of lattices. If A is a finitely subdirectly irreducible lattice, $A \in \mathsf{form}(\mathcal{K})$, then $A \in \mathbf{H}(\mathcal{K})$.*

Proof. Let A be a finitely subdirectly irreducible lattice, $A \in \mathsf{form}(\mathcal{K})$. By Theorem 3.1, there exist lattices $A_1, \ldots, A_n \in \mathcal{K}$, $B \leq A_1 \times \ldots \times A_n$, $\theta \in Con\ B$ such that $A \cong B/\theta$.

For $i \in I = \{1, \ldots, n\}$, let π_i be the projection of B onto A_i. We are going to show that there exists $i_0 \in I$ such that Ker $\pi_{i_0} \subseteq \theta$. Evidently $\bigcap_{i \in I}$ Ker $\pi_i = \omega \subseteq \theta$, so that

$$\theta = \theta \vee (\bigcap_{i \in I} \text{Ker } \pi_i) = \bigcap_{i \in I} (\theta \vee \text{Ker } \pi_i).$$

As B/θ is finitely subdirectly irreducible, we have $\theta = \theta \vee$ Ker $\pi_{i_0} \supseteq$ Ker π_{i_0} for some $i_0 \in I$.

Now using the second isomorphism theorem we obtain that B/θ is a homomorphic image of $B/$ Ker π_{i_0}. But $B/$ Ker π_{i_0} is isomorphic to A_{i_0} and thus $B/\theta \in \mathbf{H}(\{A_{i_0}\}) \subseteq \mathbf{H}(\mathcal{K})$. Consequently $A \in \mathbf{H}(\mathcal{K})$. □

Since evidently each subdirectly irreducible lattice is also finitely subdirectly irreducible, in the way described in the preceding theorem, all subdirectly irreducible lattices of $\mathsf{form}(\mathcal{K})$ are discovered. Each formation, except the least one, contains subdirectly irreducible lattices. Namely, if L is any lattice, $|L| > 1$, then L is a subdirect product of subdirectly irreducible lattices, $L \leq \prod_{i \in I} L_i$, in any variety containing L, where I need not be finite. Nevertheless, each L_i, as a homomorphic image of L, belongs to each formation containing L.

Let $\mathsf{Si}(\mathcal{F})$ denote the class of all subdirectly irreducible lattices belonging to the formation $\mathcal{F}$. Let us note that $\mathcal{F}$ is not uniquely determined by $\mathsf{Si}(\mathcal{F})$. For example, each formation of distributive lattices contains the only subdirectly irreducible lattice, the two–element chain.

4. The class of formations

Let $\mathbb{F}$ be the collection of all formations of lattices. For $\mathcal{F}_1, \mathcal{F}_2 \in \mathbb{F}$ we write $\mathcal{F}_1 \leq \mathcal{F}_2$ if $\mathcal{F}_1$ is a subclass of $\mathcal{F}_2$. The collection $\mathbb{F}$ is large; it is easy to see that for any infinite cardinal $\varkappa$, the class of all lattices of cardinality not exceeding $\varkappa$, is a formation. Nevertheless, with respect to the relation $\leq$ in $\mathbb{F}$, we can apply for $\mathbb{F}$ the usual notions and notation of the theory of partially ordered sets. Thus, for $\{\mathcal{F}_i : i \in I\} \subseteq \mathbb{F}$, the symbols $\sup\{\mathcal{F}_i : i \in I\}$ or $\bigvee_{i \in I} \mathcal{F}_i$ denote the least upper bound of $\{\mathcal{F}_i : i \in I\}$ in $\mathbb{F}$; the symbols $\inf\{\mathcal{F}_i : i \in I\}$, $\bigwedge_{i \in I} \mathcal{F}_i$ have a dual meaning.

It is easy to see that the intersection of any non–empty collection of formations is a formation. Moreover, $\mathbb{F}$ contains a least element, the class of all one–element lattices and the greatest element, the class $\mathcal{L}$ of all lattices. So we have:

Theorem 4.1. *The collection $\mathbb{F}$ of all formations of lattices is a complete lattice in the sense, that $\bigwedge_{i \in I} \mathcal{F}_i$ and $\bigvee_{i \in I} \mathcal{F}_i$ exist for any nonempty collection of formations $\{\mathcal{F}_i : i \in I\}$. Moreover,*

$$\bigwedge_{i \in I} \mathcal{F}_i = \bigcap_{i \in I} \mathcal{F}_i \ , \quad \bigvee_{i \in I} \mathcal{F}_i = \mathbf{H}\,\mathbf{P}_{\mathrm{FS}}(\bigcup_{i \in I} \mathcal{F}_i).$$

Theorem 4.2. *A formation $\mathcal{F}$ of lattices is a compact element in $\mathbb{F}$ if and only if it is generated by a single lattice.*

Proof. Let $\mathcal{F} = \mathsf{form}(\{L\})$, $L \in \mathcal{L}$. Assume $\mathcal{F} \leq \bigvee_{i\in I} \mathcal{F}_i$, where $\{\mathcal{F}_i : i \in I\} \subseteq \mathbb{F}$. Then $L \in \mathbf{H}\,\mathbf{P}_{\mathrm{FS}}(\bigcup_{i\in I} \mathcal{F}_i)$, hence there exist lattices $L_1, \dots, L_n \in \bigcup_{i\in I} \mathcal{F}_i$ with $B \leq L_1 \times \ldots \times L_n$ and a homomorphism of B onto L. If $L_1 \in \mathcal{F}_{i_1}, \dots, L_n \in \mathcal{F}_{i_n}$, we have $L \in \mathbf{H}\,\mathbf{P}_{\mathrm{FS}}(\bigcup_{j=0}^{n} \mathcal{F}_{i_j}) = \mathcal{F}_{i_1} \vee \ldots \vee \mathcal{F}_{i_n}$, which implies $\mathcal{F} \leq \mathcal{F}_{i_1} \vee \ldots \vee \mathcal{F}_{i_n}$.

Conversely, suppose that $\mathcal{F} \in \mathbb{F}$ is compact. Let $\mathcal{F} = \{L_i : i \in I\}$. As evidently $\mathcal{F} \leq \bigvee_{i\in I} \mathsf{form}(\{L_i\})$, we have $\mathcal{F} \leq \mathsf{form}(\{L_1\}) \vee \ldots \vee \mathsf{form}(\{L_n\})$ for some $L_1, \dots, L_n \in \mathcal{F}$. But then $\mathcal{F} = \mathsf{form}(\{L_1\}) \vee \ldots \vee \mathsf{form}(\{L_n\}) = \mathsf{form}(\{L_1 \times \ldots \times L_n\})$. ☐

Using the trivial fact that any formation $\mathcal{F}$ of lattices can be expressed as $\sup\{\mathsf{form}(\{L\}) : L \in \mathcal{F}\}$, we obtain:

Corollary 4.3. *The collection $\mathbb{F}$ of all formations of lattices is an algebraic lattice.*

Let $\mathbb{F}_d$ denote the collection of all formations of distributive lattices.

Theorem 4.4. *The collection $\mathbb{F}_d$ is a complete sublattice of $\mathbb{F}$; moreover, the relation*

$$\mathcal{F} \wedge \bigvee_{i\in I} \mathcal{F}_i = \bigvee_{i\in I} (\mathcal{F} \wedge \mathcal{F}_i)$$

is valid for any $\mathcal{F}, \mathcal{F}_i \in \mathbb{F}_d$.

Proof. It suffices to verify the relation

$$\mathcal{F} \wedge (\bigvee_{i\in I} \mathcal{F}_i) \subseteq \bigvee_{i\in I} (\mathcal{F} \wedge \mathcal{F}_i).$$

Using Theorem 3.4 and the fact that each $\mathcal{F}_i$ is closed under homomorphic images, we obtain that $\mathcal{F} \wedge (\bigvee_{i\in I} \mathcal{F}_i) = \mathcal{F} \cap \mathsf{form}(\bigcup_{i\in I} \mathcal{F}_i) = \mathcal{F} \cap \mathbf{P}_{\mathrm{FS}}\,\mathbf{H}(\bigcup_{i\in I} \mathcal{F}_i) = \mathcal{F} \cap \mathbf{P}_{\mathrm{FS}}(\bigcup_{i\in I} \mathbf{H}(\mathcal{F}_i)) = \mathcal{F} \cap \mathbf{P}_{\mathrm{FS}}(\bigcup_{i\in I} \mathcal{F}_i)$.

Now if $L \in \mathcal{F} \cap \mathbf{P}_{\mathrm{FS}}(\bigcup_{i\in I} \mathcal{F}_i)$, then $L \in \mathcal{F}$ and $L \leq L_1 \times \ldots \times L_k$ for some $L_1, \dots, L_k \in \bigcup_{i\in I} \mathcal{F}_i$. Each L_j, as a homomorphic image of L, belongs to $\mathcal{F}$, so each L_j belongs to $\mathcal{F} \cap (\bigcup_{i\in I} \mathcal{F}_i) = \bigcup_{i\in I} (\mathcal{F} \cap \mathcal{F}_i)$. Thus we obtain that $L \in \mathbf{P}_{\mathrm{FS}}(\bigcup_{i\in I} (\mathcal{F} \cap \mathcal{F}_i)) \subseteq \mathsf{form}(\bigcup_{i\in I} (\mathcal{F} \cap \mathcal{F}_i)) = \bigvee_{i\in I} (\mathcal{F} \wedge \mathcal{F}_i)$. ☐

The question, if this infinite distributive law or at least finite distributive law is valid in $\mathbb{F}$, is open.

Consider the following condition concerning a subclass $\mathcal{M}$ of $\mathcal{L}$:

$(*)$ $\qquad L \in \mathbf{H}(\mathcal{M}),\ L \text{ is subdirectly irreducible } \Rightarrow\ L \in \mathcal{M}.$

The following assertion is obvious.

Lemma 4.5. *Let $\mathcal{F}$ be any formation of lattices. Then* $\mathsf{Si}(\mathcal{F})$ *fulfils the condition* $(*)$.

To show that the condition $(*)$ is also sufficient for a class $\mathcal{M}$ of subdirectly irreducible lattices to be $\mathsf{Si}(\mathcal{F})$ for a formation $\mathcal{F}$, let us notice that the following holds:

Lemma 4.6. *Let* $\{\mathcal{F}_i : i \in I\}$ *be a nonempty class of formations of lattices. Then*

$$\mathsf{Si}(\bigwedge_{i\in I}\mathcal{F}_i) = \bigcap_{i\in I}\mathsf{Si}(\mathcal{F}_i)\ , \quad \mathsf{Si}(\bigvee_{i\in I}\mathcal{F}_i) = \bigcup_{i\in I}\mathsf{Si}(\mathcal{F}_i).$$

Proof. The first equality is evident, just like the inclusion $\bigcup_{i\in I}\mathsf{Si}(\mathcal{F}_i) \subseteq \mathsf{Si}(\bigvee_{i\in I}\mathcal{F}_i)$. Now let $L \in \mathsf{Si}(\bigvee_{i\in I}\mathcal{F}_i)$. Then the lattice L is subdirectly irreducible and $L \in \mathsf{form}(\bigcup_{i\in I}\mathcal{F}_i)$. By Theorem 3.5, $L \in \mathbf{H}(\bigcup_{i\in I}\mathcal{F}_i) = \bigcup_{i\in I}(\mathbf{H}(\mathcal{F}_i)) = \bigcup_{i\in I}\mathcal{F}_i$, so that $L \in \bigcup_{i\in I}\mathsf{Si}(\mathcal{F}_i)$. □

Lemma 4.7. *Let $\mathcal{M}$ be any class of subdirectly irreducible lattices satisfying the condition $(*)$. Then formations $\mathcal{F}$ with* $\mathsf{Si}(\mathcal{F}) = \mathcal{M}$ *form an interval in $\mathbb{F}$. The least element of this interval is* $\mathsf{form}(\mathcal{M})$.

Proof. First notice that $\mathsf{Si}(\mathsf{form}(\mathcal{M})) = \mathcal{M}$. The implication $\mathsf{Si}(\mathsf{form}(\mathcal{M})) \subseteq \mathcal{M}$ follows from Theorem 3.5 and from $(*)$, while the converse one is obvious. So $\mathcal{F}_0 = \mathsf{form}(\mathcal{M})$ is the least one of all formations $\mathcal{F}$ satisfying $\mathsf{Si}(\mathcal{F}) = \mathcal{M}$.

Further, let $\mathcal{F}_1$ be the least upper bound of the collection of all formations $\mathcal{F}$ with $\mathsf{Si}(\mathcal{F}) = \mathcal{M}$. By 4.6, $\mathsf{Si}(\mathcal{F}_1) = \mathcal{M}$. If $\mathcal{F}_0 \subseteq \mathcal{F} \subseteq \mathcal{F}_1$, then also $\mathsf{Si}(\mathcal{F}) = \mathcal{M}$. We have proved that $\{\mathcal{F} \in \mathbb{F} : \mathsf{Si}(\mathcal{F}) = \mathcal{M}\}$ is the interval $[\mathcal{F}_0, \mathcal{F}_1]$. □

Let C_2 be a two–element chain. Then $\mathcal{M} = \{C_2\}$ evidently satisfies $(*)$. It is easy to see that $\mathcal{F} \in \mathbb{F}$ with $\mathsf{Si}(\mathcal{F}) = \{C_2\}$ are just formations belonging to the interval $[\mathcal{F}_0, \mathcal{F}_1]$, where $\mathcal{F}_0$ is the formation containing all finite distributive lattices and $\mathcal{F}_1$ that of all distributive lattices.

Let $\mathbb{M}$ be the collection of all classes $\mathcal{M}$ of subdirectly irreducible lattices satisfying the condition $(*)$. It is easy to see that $\mathbb{M}$ is closed under arbitrary (not only finite) intersections and unions so that $(\mathbb{M}, \subseteq)$ can be considered as a complete lattice.

The following assertion is evident.

Theorem 4.8. *Let $\equiv$ be a binary relation on $\mathbb{F}$ defined by*

$$\mathcal{F} \equiv \mathcal{F}' \quad \Leftrightarrow \quad \mathsf{Si}(\mathcal{F}) = \mathsf{Si}(\mathcal{F}').$$

Then $\equiv$ is a congruence relation and the mapping $f : \mathbb{F}/\equiv\ \to \mathbb{M}$ defined by $f([\mathcal{F}]\equiv) = \mathsf{Si}(\mathcal{F})$ *is an isomorphism.*

5. Atoms and antiatoms

Let L be a lattice with a least element 0. An element $a \in L$ is said to be an atom of L if a covers 0. If $b \in L \setminus \{0\}$ and there is no atom a with $a \leq b$, then b is referred to as an antiatom. We are able to describe all atoms of $\mathbb{F}$.

Consider the following condition concerning a lattice L:

$$(**) \qquad L' \in \mathbf{H}(\{L\}),\ L' \text{ is subdirectly irreducible} \quad \Rightarrow \quad L \in \mathbf{H}(\{L'\}).$$

Theorem 5.1. *A formation $\mathcal{F}$ of lattices is an atom of $\mathbb{F}$ if and only if $\mathcal{F} = \mathsf{form}(\{L\})$ for a subdirectly irreducible lattice L satisfying the condition $(**)$.*

Proof. Let $\mathcal{F}$ be an atom. As we have remarked, $\mathcal{F}$ contains a subdirectly irreducible lattice L. Then $\mathsf{form}(\{L\}) \leq \mathcal{F}$, so that $\mathcal{F} = \mathsf{form}(\{L\})$, as $\mathcal{F}$ is an atom. We are going to show that L satisfies $(**)$. Let L' be a subdirectly irreducible lattice with $L' \in \mathbf{H}(\{L\})$. Thus it is also $\mathcal{F} = \mathsf{form}(\{L'\})$. But then $L \in \mathsf{form}(\{L'\})$ implies $L \in \mathbf{H}(\{L'\})$ by Theorem 3.5.

Conversely, let $\mathcal{F} = \mathsf{form}(\{L\})$, where L is a subdirectly irreducible lattice fulfilling $(**)$. Let $\mathcal{F}'$ be a formation of lattices different from the least one satisfying $\mathcal{F}' \leq \mathcal{F}$. Take any subdirectly irreducible lattice $L' \in \mathcal{F}'$. Then $L' \in \mathcal{F} = \mathsf{form}(\{L\})$, so that $L' \in \mathbf{H}(\{L\})$ by Theorem 3.5. Using $(**)$ we obtain $L \in \mathbf{H}(\{L'\})$, which implies $\mathcal{F} = \mathsf{form}(\{L\}) \leq \mathsf{form}(\{L'\}) \leq \mathcal{F}'$. Thus $\mathcal{F}' = \mathcal{F}$ and $\mathcal{F}$ is an atom. □

Evidently each simple lattice is a subdirectly irreducible lattice satisfying $(**)$. So each simple lattice generates an atom in $\mathbb{F}$, non–isomorphic lattices generate different atoms. Let κ be any cardinal, $\kappa \geq 3$, I any set of cardinality κ. Set $M_\kappa = \{0, 1\} \cup \{a_i : i \in I\}$ and define $\leq$ on M_κ by $0 < a_i < 1$ for all $i \in I$, a_i mutually non–comparable.

Evidently M_κ are simple lattices, mutually non-isomorphic. So we obtain:

Corollary 5.2. *Atoms of $\mathbb{F}$ form a proper class.*

If a formation $\mathcal{F}$ contains finite lattices with more than one element, then $\mathcal{F}$ contains also simple finite lattices, so that there exist atoms which lie in $\mathbb{F}$ under $\mathcal{F}$. Thus in the case that we are interested in antiatoms, we must concentrate upon formations containing, besides the one–element lattices, only infinite ones. The aim is to prove that antiatoms form a proper class, too.

Let us have an infinite ascending chain of cardinals $(\kappa_i)_{i\in\mathbb{N}}$, $\kappa_1 < \kappa_2 < \dots$, $\kappa_1 \geq 3$. Let M_{κ_i} be as above, with κ_i instead of κ. Define lattices L_i for $i \in \mathbb{N}$ by induction as follows:

$$L_1 = M_{\kappa_1}$$
$$L_{n+1} = (L_n \to M_{\kappa_{n+1}}) \quad \text{for } n \in \mathbb{N},$$

where $(L_n \to M_{\kappa_{n+1}})$ means a lattice obtained from $M_{\kappa_{n+1}}$ by interchanging one of its "middle" elements by L_n.

If we take, e.g., the sequence $3 < 4 < 5 < \dots$, we obtain a sequence of lattices, whose first three members are depicted in Fig. 1.

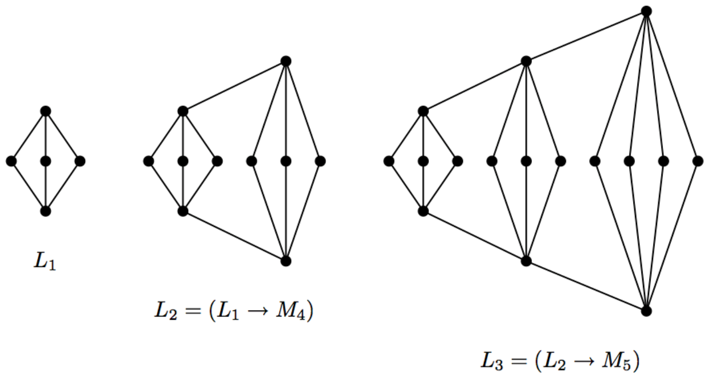

Fig. 1

It is easy to see that $(L_i)_{i\in\mathbb{N}}$, with natural embeddings $f_i : L_i \to L_{i+1}$, form a direct family of lattices. Let $L((\kappa_i)_{i\in\mathbb{N}})$ be the direct limit of this direct family. We remark, that the direct limit in this case is nothing else than a directed (set–theoretical) union. (When we consider the natural embeddings as set inclusions.)

The following assertion is easy to verify.

Lemma 5.3. *The congruence lattice of L_n is an $(n+1)$–element chain, that of $L((\kappa_i)_{i\in\mathbb{N}})$ is isomorphic to the ordinal $\omega_0 + 1$. Hence both L_n $(n \in \mathbb{N})$ and $L((\kappa_i)_{i\in\mathbb{N}})$ are subdirectly irreducible lattices.*

Lemma 5.4. *Homomorphic images of the lattice $L((\kappa_i)_{i\in\mathbb{N}})$ are just those isomorphic to $L((\kappa_{n+i})_{i\in\mathbb{N}})$ for $n \in \mathbb{N}_0$, and one–element lattices.*

Proof. Let $\theta_0 \subset \theta_1 \subset \dots$ be the sequence of all congruence relations of $L = L((\kappa_i)_{i\in\mathbb{N}})$ different from the greatest one. Then L/θ_0 is isomorphic to L; L/θ_1 is isomorphic to $L((\kappa_{1+i})_{i\in\mathbb{N}})$, and so on. In particular, for all $n \in \mathbb{N}_0$, L/θ_n is isomorphic to $L((\kappa_{n+i})_{i\in\mathbb{N}})$. □

Theorem 5.5. *Let $L = L((\kappa_i)_{i\in\mathbb{N}})$, Then the formation $\mathsf{form}(\{L\})$ is an antiatom in $\mathbb{F}$.*

Proof. By way of contradiction, let $\mathcal{F}$ be an atom in $\mathbb{F}$ with $\mathcal{F} \leq \mathsf{form}(\{L\})$. Then $\mathcal{F} = \mathsf{form}(\{M\})$ for a subdirectly irreducible lattice M satisfying the condition

$(**)$, $M \in \mathsf{form}(\{L\})$. By Theorem 3.5, $M \in \mathbf{H}(\{L\})$, so that M is isomorphic to $L((\kappa_{n+i})_{i\in\mathbb{N}})$ for some $n \in \mathbb{N}_0$. As $L((\kappa_{n+1+i})_{i\in\mathbb{N}}) \in \mathbf{H}(L((\kappa_{n+i})_{i\in\mathbb{N}})) = \mathbf{H}(\{M\})$, using $(**)$ we obtain $M \in \mathbf{H}(\{L((\kappa_{n+1+i})_{i\in\mathbb{N}})\})$. This contradicts Lemma 5.4. □

Theorem 5.6. *There exists a proper class of mutually non–comparable antiatoms in* $\mathbb{F}$.

Proof. Let $(\kappa_i)_{i\in\mathbb{N}}$ and $(\varkappa_i)_{i\in\mathbb{N}}$ be infinite ascending sequences of cardinals with $\kappa_i < \varkappa_j$ for all $i, j \in \mathbb{N}$. Denote $\mathcal{F}_\kappa$, $\mathcal{F}_\varkappa$ the formation generated by the lattice $L((\kappa_i)_{i\in\mathbb{N}})$, $L((\varkappa_i)_{i\in\mathbb{N}})$, respectively.

Suppose that $\mathcal{F}_\kappa \le \mathcal{F}_\varkappa$. Then $L((\kappa_i)_{i\in\mathbb{N}}) \in \mathsf{form}(L((\varkappa_i)_{i\in\mathbb{N}}))$ and using Theorem 3.5 we obtain $L((\kappa_i)_{i\in\mathbb{N}}) \in \mathbf{H}(L((\varkappa_i)_{i\in\mathbb{N}}))$. By Lemma 5.4, $L((\kappa_i)_{i\in\mathbb{N}})$ is isomorphic to $L((\varkappa_{n+i})_{i\in\mathbb{N}})$ for some $n \in \mathbb{N}_0$, which implies $\kappa_1 = \varkappa_{n+1}$, a contradiction. Similarly, $\mathcal{F}_\varkappa \le \mathcal{F}_\kappa$ implies $\varkappa_1 = \kappa_{m+1}$, for some $m \in \mathbb{N}_0$, again a contradiction.

In order to complete the proof, it is sufficient to find a proper class of such sequences of cardinals. Obviously, $\{(\aleph_{\alpha+i})_{i\in\mathbb{N}} : \alpha \text{ limit ordinal }\}$ forms a proper class and for limit ordinals α, β with $\alpha < \beta$ and $i, j \in \mathbb{N}$, we have $\aleph_{\alpha+i} < \aleph_{\beta+j}$. □

6. Formations of distributive lattices

In Section 4, we have introduced the denotation $\mathbb{F}_d$ for the collection of all formations of distributive lattices. This collection is a proper class. For any infinite cardinal κ, let $\mathcal{F}_d(\kappa)$ be the class of all distributive lattices with cardinalities not exceeding κ. Then $\mathcal{F}_d(\kappa)$, for various infinite cardinals κ, form a large chain. We are going to show that $\mathbb{F}_d$ contains also large antichains.

Lemma 6.1. *Let α, β be any limit ordinals. Then $\beta \in \mathbf{H}(\{\alpha\})$ if and only if α contains a cofinal subset of the type β.*

Proof. Let f be a homomorphism of α onto β. For $y \in \beta$, let $x(y)$ be the least element of $f^{-1}(y)$. It is easy to see that $\{x(y) : y \in \beta\}$ is a cofinal subset of α isomorphic to β.

Conversely, let $X \subseteq \alpha$ be a cofinal subset of α, g an isomorphism of X onto β. For any $a \in \alpha$, let x_a be the least element of the set $\{x \in X : x \ge a\}$. Set $f(a) = g(x_a)$. Then f is a homomorphism of α onto β. □

For any limit ordinal $\alpha > 0$, let $\mathrm{cf}(\alpha)$ denote the cofinality of α. If $\mathrm{cf}(\alpha) = \alpha$, then α is said to be a regular ordinal. In fact, each regular ordinal is an initial ordinal, i.e., cardinal. In the sequel we will denote the initial ordinals as usual by ω_α, $\alpha \in \mathit{Ord}$. The Axiom of Choice guarantees the existence of a proper class of regular ordinals, in particular for each $\alpha \in \mathit{Ord}$, $\omega_{\alpha+1}$ is a regular ordinal.

If ω_α is regular and $L \in \mathbf{H}(\{\omega_\alpha\})$, then L is isomorphic to ω_α or to a successor ordinal less then ω_α, by Lemma 6.1.

Theorem 6.2. *Let $\omega_\alpha, \omega_\beta$ be any different regular ordinals. Then formations generated by ω_α and ω_β are non–comparable.*

Proof. Let us suppose that $\omega_\alpha < \omega_\beta$. Since $\mathsf{form}(\{\omega_\alpha\})$ contains only lattices L with $|L| \leq \aleph_\alpha$, ω_β does not belong to $\mathsf{form}(\{\omega_\alpha\})$.

Further, we will show that $\omega_\alpha \notin \mathsf{form}(\{\omega_\beta\})$. By way of contradiction, let $\omega_\alpha \in \mathsf{form}(\{\omega_\beta\}) = \mathbf{P}_{\mathrm{FS}}\,\mathbf{H}(\{\omega_\beta\})$, due to Theorem 3.4. Then ω_α is a subdirect product of some L_i, $(i = 1, \ldots, n)$, $L_i \in \mathbf{H}(\{\omega_\beta\})$. Each L_i is a homomorphic image of ω_α, too. A homomorphic image of ω_α must be a well ordered chain, which (for the cardinality reason) cannot be isomorphic to a cofinal subset of ω_β. By 6.1, L_i cannot be a limit ordinal, which means that L_i has a greatest element. Hence, the same holds for ω_α, a contradiction. □

Corollary 6.3. *Formations generated by regular ordinals form an antichain which is a proper class.*

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