Acta Univ. M. Belii
Math. no.4 (1996), pp. 73-78

SURELY COMPLETE MATRICES

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Abstract. The "sure completness" of the 3x3 matrices over the set $\{0, 1, 2, *\}$ is defined and it is found an example of such matrix with only 4 numbers. (No such matrix with less than 4 numbers can be surely complete.) Using surely complete matrices, a lot of functionally complete algebras can be generated.

Studying the functional completness and other properties of the algebras of the type (2) on the set $\{0, 1, 2, \}$, we will use the matrix denotation by [3]. Similarly, the unary functions we will write in the vector form.

Definition. Let G be a 3x3 matrix over the set $\{0, 1, 2, *\}$ and let H be a 3x3 matrix over the set $\{0, 1, 2\}$. The matrix H will be called a **specification** of the matrix G iff the following implication is satisfied:

$$G(i, j) \in \{0, 1, 2\} \Rightarrow H(i, j) = G(i, j)$$

Example 1. The matrix

$$H = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

is a specification of the matrix

$$G = \begin{pmatrix} 1 & * & 0 \\ * & 0 & 2 \\ * & * & * \end{pmatrix}.$$

Definition. Let G be a 3x3 matrix over the set $\{0, 1, 2, *\}$. The matrix G will be called **surely complete** iff the following condition is satisfied: For every specification H of the matrix G, the algebra $(\{0, 1, 2\}, H)$ of the type (2) is funcionally complete.

1991 *Mathematics Subject Classification.* 20M35.
Key words and phrases. Polynomial function, functionally complete algebra.

Example 2. Put

$$G = \begin{pmatrix} 1 & 0 & 2 \\ * & 2 & * \\ * & * & 0 \end{pmatrix}, \; H = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

The matrix G is not surely complete. In fact, its specification H can be described by the formula

$$H(x, y) = 2x + 2y + 1 \; modulo \; 3.$$

and every polynomial of the algebra $(\{0, 1, 2, \}, H)$ is a polynomial of the algebra $(\{0, 1, 2, \}, +)$, too. On the other hand, the last algebra is not functionally complete.

In [3], the following theorem is proved.

Theorem 1. *Assume that $A = \{0, 1, 2\}$ and that the algebra (A, F) has the following properties:*

1) *among unary polynomial functions there exist at least one transposition, at least one 3-cycle and at least one function with exactly 2 values,*
2) *among binary polynomial functions there exists a function G and there exist $a, b, c, d \in A$ such that*

$$\{G(a, c), \; G(a, d), \; G(b, c), \; G(b, d)\} = A.$$

Then (A, F) is functionally complete.

Example 3. Put

$$G = \begin{pmatrix} 1 & 0 & 2 \\ * & 2 & 0 \\ * & * & 1 \end{pmatrix}.$$

This matrix G is surely complete. In fact, the polynomial fuction $G(0, x) = (1, 0, 2)$ is a transposition, $G(x, 2) = (2, 0, 1)$ is a 3-cycle, $G(x, x) = (1, 2, 1)$ has exactly 2 values and

$$\{G(0, 0), \; G(0, 1), \; G(1, 0), \; G(1, 1)\} = \{0, 1, 2\}.$$

Lemma 1. *The matrix*

$$G = \begin{pmatrix} 1 & * & 0 \\ * & 0 & 2 \\ * & * & 0 \end{pmatrix}$$

is surely complete.

Proof. Following unary functions are polynomial:

$$p_1(x) = G(x, x) = (1, 0, 0) \quad (it \; has \; 2 \; values),$$
$$p_2(x) = G(x, 2) = (0, 2, 0),$$
$$p_3(x) = p_1(p_1(x)) = (0, 1, 1),$$
$$p_4(x) = G(p_3(x), x) = (1, 0, 2) \quad (a \; transposition),$$
$$p_5(x) = p_2(p_3(x)) = (0, 2, 2),$$
$$p_6(x) = G(x, p_5(x)) = (1, 2, 0) \quad (a \; 3 - cycle).$$

Moreover, $\{G(0, 0), \; G(0, 2), \; G(1, 0), \; G(1, 2)\} = \{0, 1, 2\}$. Now apply Theorem 1.

Lemma 2. *The matrix*

$$G = \begin{pmatrix} 1 & * & 0 \\ * & 0 & 2 \\ * & * & 1 \end{pmatrix}$$

is surely complete.

Proof. Following unary functions are polynomial:

$$p_1(x) = G(x,x) = (1,0,1) \quad (\textit{it has } 2 \textit{ values}),$$

$$p_2(x) = G(x,2) = (0,2,1) \quad (\textit{a transposition}),$$

$$p_3(x) = p_1(p_1(x)) = (0,1,0),$$

$$p_4(x) = G(p_3(x),x) = (1,0,0),$$

$$p_5(x) = p_1(p_4(x)) = (0,1,1),$$

$$p_6(x) = G(p_5(x),x) = (1,0,2),$$

$$p_7(x) = p_2(p_6(x)) = (2,0,1) \quad (a\ 3-cycle).$$

Moreover, $\{G(0,0),\ G(0,2),\ G(1,0),\ G(1,2)\} = \{0,1,2\}$. Now apply Theorem 1.

Lemma 3. *The matrix*

$$G = \begin{pmatrix} 1 & 0 & 0 \\ * & 0 & 2 \\ * & * & 2 \end{pmatrix}$$

is surely complete.

Proof. Following unary functions are polynomial:

$$p_1(x) = G(x,2) = (0,2,2) \quad (\textit{it has } 2 \textit{ values}),$$

$$p_2(x) = G(x,x) = (1,0,2) \quad (\textit{a transposition}),$$

$$p_3(x) = p_1(p_2(x)) = (2,0,2),$$

$$p_4(x) = G(0,p_3(x)) = (0,1,0),$$

$$p_5(x) = G(p_4(x),p_1(x)) = (1,2,0) \quad (a\ 3-cycle).$$

Moreover, $\{G(0,0),\ G(0,2),\ G(1,0),\ G(1,2)\} = \{0,1,2\}$. Now apply Theorem 1.

Lemma 4. *The matrix*

$$G = \begin{pmatrix} 1 & 1 & 0 \\ * & 0 & 2 \\ * & * & 2 \end{pmatrix}$$

is surely complete.

Proof. Following unary functions are polynomial:

$$p_1(x) = G(0, x) = (1, 1, 0) \quad (\textit{it has } 2 \textit{ values}),$$

$$p_2(x) = G(x, x) = (1, 0, 2) \quad (a\ transposition),$$

$$p_3(x) = G(x, 2) = (0, 2, 2),$$

$$p_4(x) = p_3(p_1(x)) = (2, 2, 0),$$

$$p_5(x) = p_1(p_3(x)) = (1, 0, 0),$$

$$p_6(x) = G(p_5(x), p_4(x)) = (2, 0, 1) \quad (a\ 3 - cycle).$$

Moreover, $\{G(0,0),\ G(0,2),\ G(1,0),\ G(1,2)\} = \{0,1,2\}$. Now apply Theorem 1.

Lemma 5. *The matrix*

$$G = \begin{pmatrix} 1 & * & 0 \\ * & 0 & 2 \\ * & * & 2 \end{pmatrix}$$

is surely complete.

Proof. For the value $G(0,1)$ we have only 3 possibilities. In the case $G(0,1) = 0$, it suffices to apply Lemma 3. In the case $G(0,1) = 1$, it suffices to apply Lemma 4. In the case $G(0,1) = 2$, the following unary functions are polynomial:

$$G(x, 2) = (0, 2, 2) \quad (\textit{it has } 2 \textit{ values}),$$

$$G(x, x) = (1, 0, 2) \quad (a\ transposition),$$

$$G(0, x) = (1, 2, 0) \quad (a\ 3 - cycle).$$

Moreover, $\{G(0,0),\ G(0,2),\ G(1,0),\ G(1,2)\} = \{0,1,2\}$. Now apply Theorem 1.

Theorem 2. *The matrix*

$$G = \begin{pmatrix} 1 & * & 0 \\ * & 0 & 2 \\ * & * & * \end{pmatrix}$$

is surely complete.

Proof. Apply Lemma 1, Lemma 2 and Lemma 5.

Theorem 3. *The algebra $(\{0,1,2\}, F)$ is functionally complete iff there exists a binary polynomial function G such that*

$$G(0,0) = 1,\ G(0,2) = 0,\ G(1,1) = 0,\ G(1,2) = 2.$$

Lemma 6. *Let G be a surely complete 3x3 matrix. Then at least one row of the matrix G contains at least 2 different numbers.*

Proof. Assume that no row of the matrix G contains different numbers. Then the matrix G has a specification H of the form

$$H = \begin{pmatrix} a & a & a \\ b & b & b \\ c & c & c \end{pmatrix}.$$

Trivially, the algebra $(\{0,1,2\}, H)$ is not functionally complete.

Lemma 7. *Let G be a surely complete 3x3 matrix. Then at least one column of the matrix G contains at least 2 different numbers.*

Lemma 8. *Let G be a 3x3 matrix over the set $\{0,1,2,*\}$. Assume that at least one row and at least one column of the matrix G are "number-free". Then the matrix G is not surely complete.*

The idea of the proof. For example, the matrix

$$G = \begin{pmatrix} * & a & b \\ * & c & d \\ * & * & * \end{pmatrix}$$

has a specification

$$H = \begin{pmatrix} b & a & b \\ d & c & d \\ b & a & b \end{pmatrix}.$$

The algebra $(\{0,1,2\}, H)$ is not functionally complete. In fact, it has a non-trivial congruence $cg(0,2)$.

Theorem 4. *Let G be a surely complete 3x3 matrix over the set $\{0,1,2\}$. Then G contains at least 4 numbers.*

Proof. Apply Lemma 6, Lemma 7 and Lemma 8.

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(Received September 5, 1996)

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