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ON INTEGRAL BALANCED ROOTED TREES OF DIAMETER 10

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Abstract. A graph G is called integral if all the roots of the characteristic polynomial $P(G;x)$ are integers. A tree T is called balanced if the vertices at the same distance from the centre of T have the same degree. In the present paper the infinite class of integral balanced rooted trees of diameter 10, which has not been known so far, is given. The problem of the existence of integral balanced rooted trees of arbitrarily large diameter remains open.

1.Introduction

Let $G = (V, E)$ be a graph. The characteristic polynomial $P(G;x)$ of the graph G is defined to be characteristic polynomial of the adjacency matrix of G. The spectrum of the adjacency matrix is also called the spectrum of G, and is denoted by $Sp(G)$. We assume that the eigenvalues of G are given in non-increasing order

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G).$$

$\lambda_1(G)$ is called the index of G, where $n = |V|$. Given $v \in V(G), G - v$ denotes the subgraph of G obtained by deleting the vertex v. For all other facts on graph spectra (or terminology), see [1] (or [2]).

We say that G has an integral spectrum if all the roots of $P(G;x)$ are integers. A graph G is called integral if it has an integral spectrum. In general, the problem of characterizing integral graphs seems to be difficult. In this paper we restrict our investigations to integral balanced rooted trees, which present one interesting family of graphs. It is known that there are infinitely many integral trees. In 1998 Híc and Nedela (see[4]) published the problem if there are integral balanced trees of arbitrarily large diameter. There exist integral balanced trees with diameter 2, 3, 4, 6, 8. Integral balanced trees with diameter 5, 7, and 9 do not exist, as well as integral balanced trees with diameter $4k + 1$ (k is an arbitrary integer). Hence, the first unsolved case of above problem is diameter 10. The main concern of this paper is to investigate integral balanced rooted trees of diameter 6, 8, and 10. The infinite class of integral balanced rooted trees with diameter 10 is given here (see Proposition 6). In general, the problem remains still open.

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The structure of a balanced tree (without vertices of degree 2) is determined by the parity of its diameter and the sequence $(n_k, n_{k-1}, ..., n_1)$, where k is the radius of T and n_j $(1 \leq j \leq k)$ denotes the number of successors of the vertex at the distance $k-j$ from the centre $Z(T)$. In what follows, n_i $(i = 1, 2, ...)$ always stands for an integer ≥ 2. The balanced trees of diameter $2k$ are encoded by the sequence $(n_k, n_{k-1}, ...n_1)$ and denoted by $T_k = T(n_k, n_{k-1}, ...n_1)$. These trees are called balanced rooted trees. (see [4])

Sequence $(n_k, n_{k-1}, ..., n_1)$ is called integral if the corresponding balanced rooted tree $T(n_k, n_{k-1}, ..., n_1)$ is integral. In 1974 Harary and Schwenk (see [3]) proved that (n_1) is integral if and only if n_1 is a square. Later, Schwenk and Watanabe (see [6]) proved that the sequence (n_2, n_1) is integral if and only if both n_1 and $n_1 + n_2$ are squares.

Híc and Nedela (see [4], [5]) published the following results:

(1) If $(n_k, n_{k-1}, ..., n_1)$ is integral, then $(n_j, n_{j-1}, ..., n_1)$ is integral for $1 \leq j \leq k-1$.

(2) The sequence $(n_k, n_{k-1}, ..., n_1)$ of positive integers is integral if and only if for every $q \in \mathrm{N}$ the sequence $(q^2 n_k, q^2 n_{k-1}, ..., q^2 n_1)$ is integral.

(3) All roots of the characteristic polynomial of the balanced tree with the sequence $(n_k, n_{k-1}, ..., n_1)$ are roots of the following recursively defined polynomial $P_k(x)$:

$P_0(x) = x$
$P_1(x) = x^2 - n_1$
$P_j(x) = x.P_{j-1}(x) - n_j P_{j-2}(x)$ where $j = 2, ..., k$.

The integral sequence $(n_k, n_{k-1}, ..., n_1)$ such that the g.c.d. $(n_k, n_{k-1}, ..., n_1)$ is square-free is called the primitive integral sequence and the corresponding integral tree $T(n_k, n_{k-1}, ..., n_1)$ is called the primitive integral tree.

2.Preliminaries

Here, we give some useful results from spectral graph theory, and then deduce some elementary facts about the spectrum of the integral balanced trees of even diameter.

Theorem 1. (see [1, Theorem 0.6, p. 19]) Let G be a connected graph. Then $\lambda_1(G) > \lambda_1(G - v)$ for any vertex $v \in V(G)$.

Lemma 2. Let $\{P_i(x)\}$ be the sequence of polynomials (3) and $\{T_i\}$ be the sequence of corresponding balanced trees. Then the sequence $\{\lambda_1(T_i)\}$ is increasing.

Proof. Let i be a positive integer, and let v be the central vertex of T_i. Now, since $T_i - v = n_i T_{i-1}$, where $n_i T_{i-1}$ are n_i disjoint copies of T_{i-1}, the proof follows from Theorem 1. □

Now, let μ_i be the smallest positive root of the polynomial $P_i(x)$, $i = 1, 2, ...$ Denote by $\{\mu_k\}$ the sequence of the smallest positive roots corresponding to the sequence $\{P_k(x)\}$. The following theorem is given in [5].

Theorem 3. (see [5, Theorem 1]) For every $i \geq 1$, there exists a positive root of the polynomial $P_i(x)$. Moreover, the following statements hold:

a. $\{\mu_{2k+1}\}$ is decreasing;
b. $\{\mu_{2k}\}$ is decreasing;

c. $\mu_{2k+2} > \mu_{2k+1}$ for $k = 0, 1, ...$

Lemma 4. Let $f_i(x) = \frac{xP_i(x)}{P_{i-1}(x)}$ for $i = 1, 2, ...,$ where $P_i(x)$ is the polynomial of (3). Then $f_i(x)$ is increasing and positive for $x \in (\lambda_1(T_i), \infty)$.

Proof. We proceed by mathematical induction.

1. If $i = 1$, then $f_1(x) = \frac{xP_1(x)}{P_0(x)} = x^2 - n_1$. Notice that $\sqrt{n_1} = \lambda_1(T_1)$. Clearly, the function $f_1(x)$ is increasing and positive for any $x \in (\lambda_1(T_1), \infty)$.

2. Now, let $f_{i-1}(x) = \frac{xP_{i-1}(x)}{P_{i-2}(x)}$ be increasing and positive for any $x \in (\lambda_1(T_{i-1}), \infty)$. Using (3), after some routine calculations from the formula

$f_i(x) = \frac{xP_i(x)}{P_{i-1}(x)}$

we get

$f_i(x) = x^2(1 - \frac{n_i}{f_{i-1}(x)})$.

By the induction hypothesis, the function $f_{i-1}(x)$ is increasing and positive for $x \in (\lambda_1(T_{i-1}), \infty)$. Using lemma 2, the function $f_{i-1}(x)$ is increasing and positive also for $x \in (\lambda_1(T_i), \infty)$. Hence, the function $\frac{1}{f_{i-1}(x)}$ is decreasing and positive for $x \in (\lambda_1(T_i), \infty)$. Since n_i is a positive integer, the function $\frac{n_i}{f_{i-1}(x)}$ is decreasing and positive for $x \in (\lambda_1(T_i), \infty)$, too. We will prove that $\frac{n_i}{f_{i-1}(x)} < 1$ for every $x \in (\lambda_1(T_i), \infty)$.

Using substitution $x = \lambda_1(T_i)$ in the equation (3) we have

$P_i(\lambda_1(T_i)) = \lambda_1(T_i).P_{i-1}(\lambda_1(T_i)) - n_i P_{i-2}(\lambda_1(T_i))$. Since $\lambda_1(T_i)$ is the root of $P_i(x)$, we have

$n_i = \frac{\lambda_1(T_i)P_{i-1}(\lambda_1(T_i))}{P_{i-2}(\lambda_1(T_i))} = f_{i-1}(\lambda_1(T_i))$.

Notice that the function $f_{i-1}(x)$ is increasing for $x \in (\lambda_1(T_{i-1}), \infty)$. Using the previous formula, we have $f_{i-1}(x) > n_i$ for $x \in (\lambda_1(T_i), \infty)$. Therefore, $\frac{n_i}{f_{i-1}(x)} < 1$ for $x \in (\lambda_1(T_i), \infty)$. Now, the function $1 - \frac{n_i}{f_{i-1}(x)}$ is increasing and positive for $x \in (\lambda_1(T_i), \infty)$. Since the function x^2 is also increasing and positive for $x \in (\lambda_1(T_i), \infty)$, the function $f_i(x) = x^2(1 - \frac{n_i}{f_{i-1}(x)})$ is increasing and positive for $x \in (\lambda_1(T_i), \infty)$, too. The proof is complete. □

Theorem 5. (see [6, Theorem 1 and Theorem 2])

a) The balanced rooted tree $T(n_1)$ is integral if and only if $n_1 = k^2$ for some $k \in N$.

b) The balanced rooted tree $T(n_2, n_1)$ is integral if and only if $n_1 = k^2$ and $n_2 = n^2 + 2nk$ for some $k, n \in N$.

3. Results

3.1 Integral balanced rooted trees of diameter 6

Theorem 5 enables us to construct all integral balanced rooted trees $T(n^2 + 2nk, k^2)$ of diameter 4 for given $k, n \in N$. It follows from (1) that every integral balanced rooted tree of diameter 6 can be expressed as $T(n_3, n^2 + 2nk, k^2)$ for $k, n \in N$. For the roots of its characteristic polynomial the following equations hold:

(4) $$P_0(x) = x$$

$$P_1(x) = x^2 - k^2$$

$$P_2(x) = x.P_1(x) - n_2 P_0(x) = x(x^2 - (k+n)^2)$$

$$P_3(x) = x.P_2(x) - n_3 P_1(x)$$

The roots of the first three equations are $0, \pm k, \pm(n+k)$. Now, it is easy to see that every root of equation $x.P_2(x) - n_3 P_1(x) = 0$ is also the root of the equation

$$n_3 = \frac{x.P_2(x)}{P_1(x)}. \quad (5)$$

The equation $P_3(x) = 0$ is the polynomial equation of degree 4 with the roots $\pm a, \pm b$ (because the spectrum of every tree is symmetric). Using (4) and (5), we have $0 < a < k, n+k < b$.

The problem of finding all integral balanced rooted trees $T(n_3, n^2+2nk, k^2)$ for given $k, n \in N$ is identical to the problem of finding all integers n_3, for which the equation (5) has only integer roots. This problem can be solved by the following algorithm:

readln(k);
readln(n);
for a:=1 to $k-1$ do
if $\frac{a.P_2(a)}{P_1(a)}$ is integer then
begin
$b := n + k + 1$;
while $\frac{b.P_2(b)}{P_1(b)} \leq \frac{a.P_2(a)}{P_1(a)}$ do begin { end condition follows from lemma 4 }
if $\frac{b.P_2(b)}{P_1(b)} = \frac{a.P_2(a)}{P_1(a)}$ then write $(T(\frac{b.P_2(b)}{P_1(b)}, n^2+2nk, k^2))$;
$b := b + 1$;
end;
end;

Using the programme based on this algorithm we have found all integral balanced rooted trees $T(n_3, n^2+2nk, k^2)$ of diameter 6 for $n = 1...10000$, $k = 1...2000$. The number of them is 270 814, but only 96 720 of them are primitive. That means that we have found 96 720 infinite classes of integral balanced rooted trees of diameter 6. Their list you can get by e-mail from authors.

3.2 Integral balanced rooted trees of diameter 8

To find integral balanced rooted trees of diameter 8, we will use similar method as we did in the section 3.1. It follows from (1) that every integral balanced rooted tree of diameter 8 can be expressed as $T(n_4, n_3, n^2+2nk, k^2)$, where $T(n_3, n^2+2nk, k^2)$ is the integral balanced rooted tree of diameter 6. The roots of its characteristic polynomial have to satisfy the equations (4) and the equation

$$P_4(x) = x.P_3(x) - n_4 P_2(x) = 0.$$

The roots of the equations (4) are $0, \pm a, \pm k, \pm(n+k), \pm b$. Every root of the equation $x.P_3(x) - n_4 P_2(x) = 0$ is the root of the equation

$$n_4 = \frac{x.P_3(x)}{P_2(x)} \quad (6)$$

The equation $P_4(x) = 0$ is the polynomial equation of degree 5 with the roots $0, \pm c, \pm d$. The problem of finding all integral balanced rooted trees $T(n_4, n_3, n^2 + 2nk, k^2)$ for given integral balanced rooted tree $T(n_3, n^2 + 2nk, k^2)$ of diameter 6 is identical to the problem of finding all integers n_4, for which the equation (6) has only integer roots. This problem can be solved by the following algorithm (the values of variables k, n and n_3 are taken from the results of the algorithm, which is described in the section 3.1):

readln(k);
readln(n);
readln(n_3);
for c:=$a + 1$ to $n + k - 1$ do
if $\frac{c.P_3(c)}{P_2(c)}$ is integer then
begin
$d := b + 1$;
while $\frac{d.P_3(d)}{P_2(d)} \leq \frac{c.P_3(c)}{P_2(c)}$ do begin { end condition follows from lemma 4 }
if $\frac{d.P_3(d)}{P_2(d)} = \frac{c.P_3(c)}{P_2(c)}$then write $(T(\frac{d.P_3(d)}{P_2(d)}, n_3, n^2 + 2nk, k^2))$;
$d := d + 1$;
end;
end;

Using the programme based on this algorithm we have found all integral balanced rooted trees $T(n_4, n_3, n^2 + 2nk, k^2)$ of diameter 8 for $n = 1...10000$, $k = 1...2000$. The number of them is 31 558, but only 5 784 of them are primitive. That means that we have found 5 784 infinite classes of integral balanced rooted trees of diameter 8. Their list you can get by e-mail from authors.

3.3 Integral balanced rooted trees of diameter 10

Every integral balanced rooted tree of diameter 10 can be expressed as $T(n_5, n_4, n_3, n^2 + 2nk, k^2)$, where $T(n_4, n_3, n^2 + 2nk, k^2)$ is the integral balanced rooted tree of diameter 8. The roots of its characteristic polynomial have to satisfy the equations (4), (6), and the equation $P_5(x) = x.P_4(x) - n_5P_3(x)$. The roots of the equations (4) and (6) are $0, \pm a, \pm c, \pm k, \pm(n+k), \pm b, \pm d$. If x is the root of the equation

$$x.P_4(x) - n_5P_3(x) = 0,$$

then x is also the root of the equation

$$n_5 = \frac{x.P_4(x)}{P_3(x).} \qquad (7)$$

The equation $P_5(x) = 0$ is polynomial equation of degree 6 with the roots $\pm e, \pm f, \pm g$. The problem of finding all integral balanced rooted trees $T(n_5, n_4, n_3, n^2 + 2nk, k^2)$ for given integral balanced rooted tree $T(n_4, n_3, n^2 + 2nk, k^2)$ of diameter 8 is identical to the problem of finding all integers n_5, for which the equation (7) has only integer roots. This problem can be solved by the following algorithm (the values of variables k, n, n_3 and n_4 are taken from the results of the algorithm, which is described in the section 3.2):

readln(k);
readln(n);

readln(n_3);
readln(n_4);
for e:=1 to $a-1$ do
if $\frac{e.P_4(e)}{P_3(e)}$ is integer then
for $f := c+1$ to $b-1$ do
if $\frac{f.P_4(f)}{P_3(f)} = \frac{e.P_4(e)}{P_3(e)}$ then
begin
$g := d+1$;
while $\frac{g.P_4(g)}{P_3(g)} \leq \frac{e.P_4(e)}{P_3(e)}$ do begin { end condition follows from lemma 4 }
if $\frac{g.P_4(g)}{P_3(g)} = \frac{e.P_4(e)}{P_3(e)}$ then write ($T(\frac{g.P_4(g)}{P_3(g)}, n_4, n_3, n^2+2nk, k^2)$);
$g := g+1$;
end;
end;

Using the programme based on this algorithm we have found all integral balanced rooted trees $T(n_5, n_4, n_3, n^2+2nk, k^2)$ of diameter 10 for $n = 1...10000, k = 1...2000$. Their list is in the table below. Only the tree in the first column after the heading is primitive, the other trees can be construct from it using (2).

n_5	3006756	12027024	27060804	48108096	75 168 900
n_4	1051960	4207840	9467640	16831360	26 299 000
n_3	751689	3006756	6765201	12027024	18 792 225
n_2	283360	1133440	2550240	4533760	7 084 000
n_1	133956	535824	1205604	2143296	3 348 900
k	366	732	1098	1464	1 830
n	280	560	840	1120	1 400
a	306	612	918	1224	1 530
b	1037	2074	3111	4148	5 185
c	527	1054	1581	2108	2 635
d	1394	2788	4182	5576	6 970
e	289	578	867	1156	1 445
f	918	1836	2754	3672	4 590
g	2074	4148	6222	8296	10 370

Proposition 6.
For any $q \in N$, the tree $T(q^2 3006756, q^2 1051960, q^2 751689, q^2 283360, q^2 133956)$ is integral balanced rooted tree of diameter 10, and its spectrum
$Sp = \{0, \pm 280q, \pm 289q, \pm 306q, \pm 366q, \pm 527q, \pm 918q, \pm 1037q, \pm 1394q, \pm 2074q\}$.

Conclusion

The method which is written in this paper can be used for finding integral balanced rooted trees with arbitrarily large even diameter. The disadvantage of this method is that we have not been able to find any integral balanced rooted tree with diameter larger than 10 yet because we have to work with extremely huge numbers and investigate a huge number of possibilities. This problem can be solved by development of computers only partially.

We have used 10 computers with Pentium II processors and with the operating system Windows 98. The computation has lasted for two months, but only at weekends. The programmes were made using Delphi Pascal.

About the problems connected with finding integral balanced rooted tree of diameter larger than 10 tells the fact that from 20 million integral balanced rooted trees of diameter 4 only 270 814 can be expanded to integral balanced rooted trees of diameter 6, from them only 31 558 can be expanded to integral balanced rooted trees of diameter 8, and from them only 5 can be expanded to integral balanced rooted trees of diameter 10.

We can suppose that if we wanted to find integral balanced rooted tree of diameter 12, we would have to investigate much more integral balanced rooted trees of diameter 4, 6, 8, and 10, and work with much larger numbers than we have worked so far.

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References

[1] Cvetkovič, D.M. - Doob, M. - Sachs, H., *Spectra of graphs - Theory and Application*, Deutscher Verlag der Wissenschaften - Academic Press, Berlin-New York, 1980.

[2] Harary, F., *Graph Theory*, Addison - Wesley Publishing Company, Reading, MA, 1969.

[3] Harary, F. - Schwenk, A. J., *Which graphs have integral spectra? In: Graphs and Combinatorics. Lecture Notes in Math* **406** (1974), Springer-Verlag, Berlin, pp. 45-51.

[4] Híc, P. - Nedela, R., *Balanced integral trees. Math. Slovaca* **48** (1998), No. 5, pp. 429-445.

[5] Híc, P. - Nedela, R., *Note on zeros of the characteristic polynomial of balanced trees, Acta Univ. M. Bellii, Math.* (1996), No. 3, pp. 31-35.

[6] Schwenk, A. J. - Watanabe, M., *Integral starlike trees, J. Austral. Math. Soc.* **28** (1979), Ser. A, pp. 120-128.

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