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A NEW PROOF FOR CHORDAL GRAPHS

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Abstract. Chordal graphs are those without induced cycles longer than three. It is a classical fact that every nonempty chordal graph contains a vertex, the neighbourhood of which induces a clique. This was first proved by Lekkerkerker and Boland in 1962. Few more proofs are known nowadays. We present yet another, very short and elementary proof.

Chordal Graphs

We consider nonempty finite simple graphs. A *clique* is a complete (sub)graph. A *chordal graph* is a graph containing no induced cycle of length greater than three. For a graph G and vertex $v \in V(G)$, we denote by $N_G(v)$ the subgraph induced on the neighbours of v in G (not including v itself – an "open" neighbourhood). A vertex v of G is called *simplicial* if $N_G(v)$ is a clique.

Simplicial vertices in chordal graphs were, perhaps, first considered by Lekkerkerker and Boland in an old paper [2] describing interval graphs. One of their results – a key fact in a characterization of chordal graphs via a simplicial decomposition, reads:

Theorem 1. (Lekkerkerker and Boland, 1962) If G is a chordal graph, then G has a simplicial vertex.

We say that a graph G is *bisimplicial* if G is either a clique, or G has two nonadjacent simplicial vertices. The "folklore" short proof of Theorem 1 establishes the following claim by induction.

Lemma 2. Every chordal graph is bisimplicial.

The inductive step in this proof finds a minimal vertex cut X which separates the remaining vertices in G into sets Y_1, Y_2. It is proved that X induces a clique in G, and then the claim is applied to the smaller graphs $G - Y_1$ and $G - Y_2$.

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New Proof

The above sketched proof of Lemma 2 is short and elegant, but it is not elementary — the proof that a minimal vertex cut X in a chordal graph induces a clique needs the Menger theorem. This had shown to be a major obstacle when we tried to extend the notion of chordality to represented matroids (cf. [1]). That is why we have looked for another, short and elementary proof of Theorem 1.

Our alternative proof of Lemma 2 proceeds in a sequence of three simple (and elementary) claims, which lead to a contradiction showing that no minimal counterexample to Lemma 2 exists.

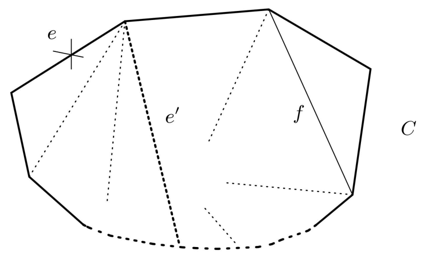

FIG. 1. An illustration to a chordal cycle C

Lemma 3. Let G be a chordal graph. If $C \subseteq G$ is a cycle, and e is an edge of C, then there is an edge $f \in E(G)$ such that f forms a triangle with two edges of $C - \{e\}$. (Fig 1.)

Proof. We proceed by induction on $|C|$. If C is a triangle, then the claim holds for $f = e$. So suppose a cycle $C \subseteq G$ of length greater than three. Since C is not induced in G, there is an edge $e' \in E(G - C)$ having both ends on C (a "chord" of C). The graph $(C - \{e\}) \cup \{e'\}$ contains a unique cycle C' which is shorter than C. We find the edge f inductively for C' and e'. $\square$

Lemma 4. Let G be a graph, and let u, v be adjacent vertices in G such that the neighbourhood subgraph $N_G(v)$ is bisimplicial. If v is simplicial in the neighbourhood subgraph $N_G(u)$, but v is not simplicial in the whole graph G, then there is a vertex w which is adjacent to v but not to u, and w is simplicial in $N_G(v)$.

Proof. By the assumption, there are two nonadjacent simplicial vertices w, w' in the neighbourhood subgraph $N_G(v)$. At least one of them, say w, does not belong to the clique $N_G(u) \cap N_G(v)$. (See also Fig. 2.) $\square$

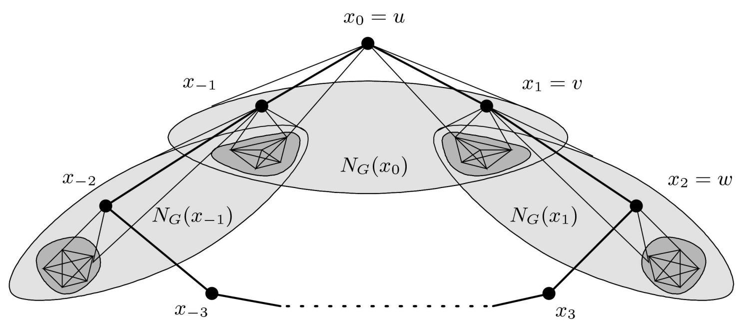

Fig 2. An illustration to the proofs

Lemma 5. Let G be a non-bisimplicial graph such that the neighbourhood subgraph $N_G(v)$ is bisimplicial for each vertex $v \in V(G)$. Then there is a cycle $C \subseteq G$ and an edge e of C, such that no triangle of G has two edges on $C - \{e\}$.

Proof. By the assumptions, G contains a non-simplicial vertex x_0. Let x_1 and x_{-1} be two nonadjacent neighbours of x_0 that are simplicial in $N_G(x_0)$. For $i = 1, 2, 3, \ldots$, we apply Lemma 4 to $u = x_{i-1}$ and $v = x_i$, and we set $x_{i+1} = w$. Notice that x_{i-1}, x_i, x_{i+1} induce a 2-path but *not a triangle*. (See Fig. 2.) The sequence proceeds inductively until x_i is simplicial. In the other direction we analogously get x_{i-1} from x_i, x_{i+1} for $i = -1, -2, -3, \ldots$.

Since G is finite, we eventually find indices $i, j \in \mathbb{Z}$, $i - j > 2$; such that both x_i, x_j are simplicial or that $x_{i+1} = x_j$. Then, unless G is bisimplicial in the first case, the vertices x_i, x_j are adjacent. Denote by C the cycle on $x_j, x_{j+1}, \ldots, x_i$ in G, and by $e = x_j x_i$. We have found the required objects C, e. $\square$

Suppose that G_0 is a counterexample to Lemma 2 on the smallest possible number of vertices. Then G_0 is a non-bisimplicial chordal graph on more than one vertex. Since an induced subgraph of a chordal graph is chordal by definition, the (smaller) neighbourhood subgraphs $N_{G_0}(v)$ are bisimplicial for each $v \in V(G_0)$. We have a contradiction between Lemmas 3 and 5 for $G = G_0$.

References

[1] P. Hliněný and G. Whittle, *Tree-Width and Superchordal Matroids*, in preparation.

[2] C.B. Lekkerkerker and J.C. Boland, *Representation of finite graphs by a set of intervals on the real line*, Fund. Math. **51** (1962), 45–64.

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