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MINIMAL REPRESENTATIVES OF $\mathcal{G}$-CLASSES OF 3-MANIFOLDS OF GENUS TWO

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Abstract. One of the central problems for 3-manifolds is the isomorphism problem. Since 70's several methods to attack it were developed. The method introduced in a paper of Ferri and Gagliardi is not easy to use, since no bound for the number of steps in a computer representation is known. Some approximations were introduced in the paper of Grasselli, Mulazzani and Nedela. The present method based on these approximations leads to a simple algorithm finding representatives of a given equivalence classes of 3-manifolds of genus two. We have applied the algorithm to reduce a known list of representatives of 3-manifolds of genus 2 and to derive some new results as well.

Introduction

A n-manifold ($n \geq 1$) is the topological space, in which every point has a neighbourhood $O(x)$ homeomorphic to the n-dimensional Euclidean space. Next, every compact connected n-manifold, $n \leq 3$, can be expressed as a simplicial complex containing a finite set of simplices of dimension n. For instance, a compact connected surface can be triangulated. However, we can form a triangulation of a surface by infinitely many ways. For example, we can choose a point in a triangle of a given triangulation, connect it with the vertices of that triangle and form a new triangulation of the same surface. A general problem is to decide, whether two different simplicial complexes represent the same n-manifold. In what follows we shall only consider compact connected piecewise-linear 3-manifolds. Each such a 3-manifold can be triangulated as was already mentioned.

There is a well defined equivalence relation on the set of n-dimensional complexes representing n-manifolds based on *wave moves* [4]. This equivalence allows us to decide, which simplicial complexes represent the same n-manifold. Unfortunately, the straight use of wave moves to solve the above isomorphism problem seems to be intractable. In fact, if $n > 2$ no limit for the number of steps (moves) needed to decide whether two complexes determine the same compact connected manifold is known.

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Let $\mathcal{M}$ be an n-manifold. It is well known, that a simplicial complex representing $\mathcal{M}$ can be represented by a graph $\Gamma(\mathcal{M})$, which vertices represents simplices of dimension n of the complex and edges represent a "gluing" of maximal simplices of the complex in subsimplices of dimension $n-1$. Given graph $\Gamma(\mathcal{M})$ can be "drawn" on an orientable surface. The *genus* of $\mathcal{M}$ is the minimal genus of an orientable surface into which $\Gamma(\mathcal{M})$ embeds in a particular way described in the next section.

Following [3] we represent a 3-manifold of genus two as a vector of integers of length six and consider certain equivalence relations defined on 6-tuples and preserving the associated 3-manifold of genus two. There is an equivalence on the set of 6-tuples introduced in [2] called $\mathcal{H}$-equivalence. In [5] other equivalence on the set of 6-tuples is defined. This equivalence is called $\mathcal{G}$-equivalence and it extends $\mathcal{H}$-equivalence. If f and g are $\mathcal{G}$-equivalent 6-tuples then they represent isomorphic 3-manifolds of genus two. Hence the $\mathcal{G}$-equivalence provides an approximation of the "isomorphism problem".

Main aim of this paper is to investigate the $\mathcal{G}$-equivalence in details. As an application a list of representations of "small" 3-manifolds of genus two is produced.

Preliminaries

Each 3-dimensional simplicial complex can be represented by a bipartite 4-edge-coloured graph. Let T be any simplicial triangulation and T' be its first barycentric subdivision. Each vertex $\hat{\omega}$, which is the barycenter of the simplex ω of T is labelled by the dimension of ω. Take the dual graph Γ of T' and if uv is an edge and $\{i, j, k\}$ are the colours of respective triangle in T use the colour complementary to $\{i, j, k\}$ to colour the edge uv. The labelling of vertices of T induces a decomposition of the tetrahedrons of T into two classes, where adjacent tetrahedrons belong to different classes. Thus Γ is bipartite. The dual graph Γ of T', together with the edge-colouring ν, is a 4-coloured graph, representing T.

Definition 1. Let $\Gamma = (V(\Gamma, E(\Gamma)))$ be a bipartite graph and let there exist a mapping $\nu : E(\Gamma) \to \Delta_4 = \{1, 2, 3, 4\}$ such that for all incident edges $f, g \in E(\Gamma)$: $\nu(f) \neq \nu(g)$. This mapping called a *graph colouring* and the graph Γ_{Δ_4} *4-coloured graph.*

It is proved [4] that the above mentioned simplicial complexes can be represented by a 4-coloured bipartite and connected graph Γ_{Δ_4} (next, the graph). The colouring is regular, i.e. two incident edges share distinct colours. Since the colouring is regular a factor induced by two colours is a disjoint union of bicoloured cycles. Let $\mathcal{I}$ denotes the set of 2-cell embeddings of Γ_{Δ_4} into a closed orientable surface such that the local rotation of colours induced by the embedding in "black" vertices is the same, say ρ, while the local rotation of colours in "white" vertices is ρ^{-1}. Note that there are six possibilities for choosing ρ. It follows that faces of such embedding are bounded by bicoloured cycles. Out of these six possibilities for ρ we choose such ρ that the genus of the underlying surface is minimal in $\mathcal{I}$. The integer g is an invariant of a 3-manifold $\mathcal{M}$ represented by Γ_{Δ_4} and it is called the *regular genus* of $\mathcal{M}$ (or shortly the genus of $\mathcal{M}$). It is known that the regular genus of $\mathcal{M}$ is equal to the Heegaard genus of $\mathcal{M}$ [1].

Let Γ_{Δ_4} is a 4-coloured graph and let Θ is subgraph of Γ_{Δ_4} contains of vertices X, Y joined by h edges $(1 \leq h \leq 3)$ coloured by colours $c_1, \ldots, c_h$. If X and Y are in two different components of graph $\Gamma_{\Delta_4 - \{c_1,\ldots,c_h\}}$ induced by the set of complementary colours $\Delta_4 - \{c_1, \ldots, c_h\}$ then the subgraph Θ will be called a *dipole of type* h.

There is a well defined operator over the set of 4-coloured graphs [4] called *wave move*. Note that a wave move can be defined for $(n+1)$-coloured, connected and bipartite graphs (n-manifolds) generally.

Definition 2. If Θ is a dipole of type h in Γ_{Δ_4} coloured by colours $\{c_1, \ldots, c_h\}$ we define a wave move as follows (see Fig. 1):

(a) Cutting of Θ

- remove edges and vertices of Θ
- glue "hanging" edges of graph Γ_{Δ_4} of same colour

(b) Adding of Θ as inverse to cutting

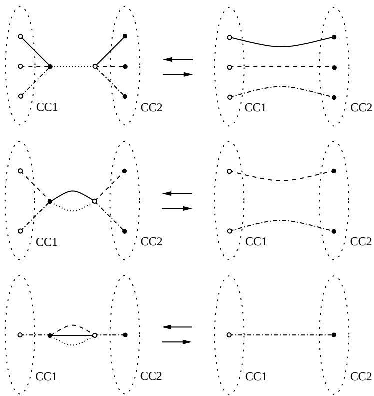

FIG. 1. Wave moves

The main result of [4] states that graphs Γ_{Δ_4} and Γ'_{Δ_4} represent isomorphic 3-manifolds if and only if there is a finite sequence of wave-moves transforming Γ_{Δ_4} to Γ'_{Δ_4}. Hence the "isomorphism problem" reduces to the problem to decide whether two 4-coloured graphs are "wave-move equivalent".

It follows from [3] that each (closed) genus two 3-manifold can be represented by a graph Γ_{Δ_4} which structure can be coded by a 6-tuple of integers satisfying certain conditions. Let $\widetilde{\mathcal{F}}_2$ is set of 6-tuples:

$$f = (h_0, h_1, h_2; q_0, q_1, q_2), h_i, q_i \in \mathbb{N}.$$

The set of 6-tuples representing genus two 3-manifolds satisfy the following axioms:

(i) $\forall i \in \mathbb{Z}_3 : h_i > 0$,
(ii) all h_i has the same parity,
(iii) $\forall i \in \mathbb{Z}_3 : 0 \le q_i < h_{i-1} + h_i = 2l_i$,
(iv) all q_i has the same parity.

Remark 1. From here all operations with numbers q_i will be considered modulo $2l_i$, and according to (iii), q_i will be always the least non-negative integer of the class.

Now let us define the set $V(f)$ for a 6-tuple $f \in \widetilde{\mathcal{F}}_2$:

$$V(f) = \bigcup_{i \in \mathbb{Z}_3} \{i\} \times \mathbb{Z}_{2l_i}$$

and the following permutations on $V(f)$:

$$\begin{aligned}
\alpha_0(i,j) &= (i, j + (-1)^j), \\
\alpha_1(i,j) &= (i, j - (-1)^j), \\
\alpha_2(i,j) &= \begin{cases} (i+1, 2l_k - j - 1)\,;\ k = i+1\,; & 0 \le j < h_i \\ (i-1, 2l_k - j - 1)\,;\ k = i\,; & h_i \le j < 2l_i \end{cases}, \\
\alpha_3(i,j) &= \rho \circ \alpha_2 \circ \rho^{-1},
\end{aligned}$$

where $\rho : V(f) \to V(f)$ is a bijection defined by rule

$$\rho(i,j) = (i, j + q_i).$$

Now let $f \in \mathcal{F}_2$ satisfy the following conditions:

(v) $\forall i \in \mathbb{Z}_3 : h_i + q_i$ is odd, h_i and q_i have different parity,

(vi) the group $\langle \alpha_2, \alpha_3 \rangle$ has exactly three orbits.

Given 6-tuple f we define the associated graph $\Gamma_{\Delta_4}(f)$ as follows. Let $V = V(f)$ be the set of vertices of $\Gamma_{\Delta_4}(f)$. Then the permutations $\alpha_0, \alpha_1, \alpha_2$ and α_3 define the decomposition of the edge set into four colours, the orbits of α_i form the edges of Γ_{Δ_i} coloured by i, for $i = 0, 1, 2, 3$. Observe that the subgraphs $\Gamma_{\{0,1,2\}}$, $\Gamma_{\{0,1,3\}}$ induced by the respective sets of colours are isomorphic planar graphs.

Vice-versa let Γ_{Δ_4} be a 4-coloured graph with a bicoloured 2-factor containing three circles of even length C_0, C_1, C_2 coloured by colours 0 and 1. Other edges coloured by colours 2 and 3 join vertices of Γ_{Δ_4} such that the induced subgraphs $\Gamma_{\{0,1,2\}}$ and $\Gamma_{\{0,1,3\}}$ are planar and isomorphic. Now, let us code the graph by the 6-tuple $f = (h_0, h_1, h_2; q_0, q_1, q_2)$ [2]. The first three items code the numbers of edges coloured by 2 (3) joining the circles C_{i-1} and C_i, $(i = 0, 1, 2)$ of Γ_{Δ_4}. Clearly, the planar subgraphs $\Gamma_{\{0,1,2\}} \simeq \Gamma_{\{0,1,3\}}$ of Γ_{Δ_4} are uniquely determined by the integers h_0, h_1 and h_2. Then Γ_{Δ_4} arises by gluing $\Gamma_{\{0,1,2\}}$ with $\Gamma_{\{0,1,3\}}$ in the three cycles C_0, C_1 and C_2 coloured by 0 and 1. The integers q_0, q_1 and q_2 determine the rotations of cycles C_0, C_1, C_2 in $\Gamma_{\{0,1,3\}}$ before the gluing is done. In this way we get an embedding of Γ_{Δ_4} into bitorus (see Fig. 2)

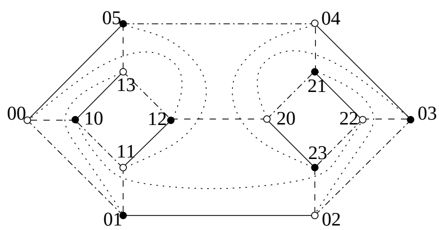

FIG. 2. The graph represented by 6-tuple $(3, 1, 3; 2, 2, 0)$

The conditions (i) – (vi) come in part from the interpretation while in part they are forced by the requirement that Γ represents a compact connected 3-manifold of genus 2.

Theorem 1 [3]**.** For every compact connected 3-manifold $\mathcal{M}$ of genus ≤ 2 there exists $f \in \mathcal{F}_2$ such that $\Gamma(f)$ represents $\mathcal{M}$.

Definition 3. The elements of the set $\mathcal{F}_2 \subset \widetilde{\mathcal{F}}_2$ satisfying conditions (i) – (vi) will be called *admissible 6-tuples.*

Definition 4. Let $f \in \mathcal{F}_2$. The number $z(f) = h_0+h_1+h_2$ is called the *complexity of the 6-tuple f.*

It is easy to design an algorithm to verify the conditions (i) – (vi) for a given integer vector with six items. The most complicated seems to be to verify the condition (vi), but the complexity of this algorithm is polynomial, depending on complexity of given 6-tuple f. Therefore we can construct the set $\mathcal{F}_2$ up to a fixed complexity in an effective way.

Now we introduce the equivalence relations on $\mathcal{F}_2$ defined in [2] and [5]. If $f = (h_0, h_1, h_2; q_0, q_1, q_2)$ is an admissible 6-tuple define the permutations ψ_1, ψ_2, ψ_3 acting on $\mathcal{F}_2$ as follows [2]:

$$\begin{aligned}
\psi_1(h_0, h_1, h_2; q_0, q_1, q_2) &= (h_1, h_2, h_0; q_1, q_2, q_0)\\
\psi_2(h_0, h_1, h_2; q_0, q_1, q_2) &= (h_2, h_1, h_0; q_0, q_2, q_1)\\
\psi_3(h_0, h_1, h_2; q_0, q_1, q_2) &= (h_0, h_1, h_2; 2l_0 - q_0, 2l_1 - q_1, 2l_2 - q_2)
\end{aligned}$$

The above described permutations represents some recolourings of the graph $\Gamma_{\Delta_4}(f)$.

Definition 5. Let $f, g \in \mathcal{F}_2$. Let us define the relation

$$f \overset{\mathcal{H}}{\approx} g : \exists \eta \in \langle \psi_1, \psi_2, \psi_3 \rangle, \eta(f) = g$$

This relation is an equivalence and we will call it *$\mathcal{H}$-equivalence on $\mathcal{F}_2$.* The equivalence classes will be called *$\mathcal{H}$-orbits.*

Lemma 1. [2, Prop. 16] $\mathcal{H}$-equivalence preserves the admissibility of the 6-tuple.

Lemma 2. The group $\mathcal{H} = \langle \psi_1, \psi_2, \psi_3 \rangle$ is isomorphic to $\mathcal{D}_{12}$, where $\mathcal{D}_{12}$ is the group of symmetries of a regular hexagon. In particular, each $\mathcal{H}$-orbit has at most 12 elements.

Proof. It follows from the definition of ψ_1, ψ_2, ψ_3 that $\psi_1^3 = \psi_2^2 = \psi_3^2 = 1$. The group $\langle \psi_1, \psi_2 \rangle$ is isomorphic to the group $\mathcal{S}_3$ of symmetries of a regular triangle because

$$\psi_2 \psi_1 \psi_2 = \psi_1^{-1}.$$

Also ψ_3 commutes with the members of $\langle \psi_1, \psi_2 \rangle$. Hence the group $\mathcal{H}$ satisfies the relations of dihedral group $\mathcal{D}_{12}$. Thus $\mathcal{H}$ is an epimorphic image of $\mathcal{D}_{12}$. To prove that the epimorphism is an isomorphism it is sufficient to find at least one admissible 6-tuple such that the respective $\mathcal{H}$-orbit has 12 different 6-tuples. The 6-tuple $(1, 3, 5; 2, 2, 2)$ is the such a 6-tuple. $\square$

Following [5], let us define mapping $\sigma : \mathcal{F}_2 \to \mathcal{F}_2$:

$$\sigma(h_0, h_1, h_2; q_0, q_1, q_2) = \begin{cases} (h_0, h_1, h_2; q_0, q_1, q_2); & \text{if } q_0 = 0 \\ (h'_0, h'_1, h'_2; q'_0, q'_1, q'_2); & \text{if } q_0 \neq 0 \end{cases}$$

where $f = (h'_0, h'_1, h'_2; q'_0, q'_1, q'_2)$ is a 6-tuple defined by the next rules

$$\left.\begin{array}{ll} h'_0 = h_0 + h_1 - q_0 & q'_0 = h_0 + h_1 + h_2 - 2q_0 \\ h'_1 = q_0 & q'_1 = q_0 + q_1 + h_1 \\ h'_2 = h_2 + h_1 - q_0 & q'_2 = q_0 + q_2 + h_1 \end{array}\right\} \quad \text{iff } 0 < q_0 < h_0, h_2$$

$$\left.\begin{array}{ll} h'_0 = q_0 + h_1 - h_2 & q'_0 = h_1 \\ h'_1 = h_0 + h_2 - q_0 & q'_1 = q_0 + q_1 - h_2 \\ h'_2 = q_0 + h_1 - h_0 & q'_2 = q_0 + q_2 - h_0 \end{array}\right\} \quad \text{iff } q_0 > h_0, h_2$$

$$\left.\begin{array}{ll} h'_0 = h_1 & q'_0 = h_1 + h_2 - q_0 \\ h'_1 = h_0 & q'_1 = q_1 \\ h'_2 = h_1 + h_2 - h_0 & q'_2 = 2q_0 + q_2 + h_1 - h_0 \end{array}\right\} \quad \text{iff } h_0 < q_0 < h_2$$

$$\left.\begin{array}{ll} h'_0 = h_1 + h_0 - h_2 & q'_0 = h_1 + h_0 - q_0 \\ h'_1 = h_2 & q'_1 = 2q_0 + q_1 + h_1 - h_2 \\ h'_2 = h_1 & q'_2 = q_2 \end{array}\right\} \quad \text{iff } h_2 < q_0 < h_0$$

The above described operation represents a sequence of wave moves such that applying it to the graph represented by an admissible 6-tuple we get the new graph, which can be represented by an admissible 6-tuple too.

Definition 6. Let $f, g \in \mathcal{F}_2$. We define a relation:

$$f \overset{\mathcal{G}}{\approx} g : \exists \gamma \in \langle \psi_1, \psi_2, \psi_3, \sigma \rangle, \gamma(f) = g$$

This relation will be called *$\mathcal{G}$-equivalence on $\mathcal{F}_2$*. The equivalence classes will be called *$\mathcal{G}$-orbits* and will be marked as usual $[f]_{\mathcal{G}}$.

Agreement. Denote by $[f]_{\mathcal{H}}$ a $\mathcal{H}$-orbit containing f. Similar, denote by $[f]_{\mathcal{G}}$ a $\mathcal{G}$-orbit containing f.

Lemma 3. [5, Th. 5.1] $\mathcal{G}$-equivalence preserves the admissibility of the given 6-tuple.

Obviously, any $\mathcal{G}$-orbit is a union of some $\mathcal{H}$-orbits.

Definition 7. Let $\mathcal{H}, \mathcal{H}'$ be two different $\mathcal{H}$-orbits. Let $f \in \mathcal{H} \wedge g \in \mathcal{H}' : g = \sigma(f)$. Then we define a *derivation of* f as the difference $\delta(f) = z(g) - z(f)$ [5].

Straightforward from Definitions 6 and 7 we get the following lemma.

Lemma 4 [5]**.** With the above notation

$$\delta(f) = \begin{cases} 0 & \text{iff } q_0 = 0 & \text{(a)} \\ h_1 - q_0 & \text{iff } 0 < q_0 < h_0, h_2 & \text{(b)} \\ h_1 - h_0 & \text{iff } h_0 < q_0 < h_2 & \text{(c)} \\ h_1 - h_2 & \text{iff } h_2 < q_0 < h_0 & \text{(d)} \\ q_0 + h_1 - h_0 - h_2 & \text{iff } q_0 > h_0, h_2 & \text{(e)} \end{cases}$$

Note. We denote by $f_i (i = 1, 2, \ldots, 6)$ the i-th item of the vector representing an admissible 6-tuple.

Definition 8. Let $f, g \in \mathcal{F}_2$ be two 6-tuples. Let $I = \{1, 2, 3, 4, 5, 6\}$ be the set of indexes of components of these vectors. We define the lexical order $\prec$ as follows:

$$f \prec g \Leftrightarrow \text{for } j = \inf\{i | (i \in I) \wedge f(i) \neq g(i)\}, f(j) < g(j)$$

Definition 9. Using the lexical order we derive an order on $\mathcal{F}_2$ in the following way

$$f \ll g \Leftrightarrow (z(f) < z(g)) \vee ((z(f) = z(g)) \wedge (f \prec g))$$

We call this order the *natural order of* $\mathcal{F}_2$.

Finally, we define representatives of $\mathcal{H}$-orbits.

Definition 10. Let $F \subseteq \mathcal{F}_2$.

The member f of F satisfying

$$f \in F : \neg(\exists g \in F), g \ll f$$

is called a *minimal representative of* F.

The member f of F satisfying

$$f \in F : \forall g \in F, f \ll g$$

is called the *least representative of* F.

Since $\mathcal{F}_2$ with respect to $\ll$ is a well-ordered set, for each F there exists a unique minimal representative which is in the same time the least representative of F.

Agreement. In the notation $[f]_{\mathcal{H}}$ denoting an orbit of $\mathcal{H}$-equivalence we shall always assume that 6-tuple f is minimal unless otherwise follows from the context.

To create a $\mathcal{H}$-orbit from a given f is a trivial problem, which can be represented by simple algorithm following i.e. from [2, Prop. 16]. By Lemma 2 the members of $[f]_{\mathcal{H}}$ are

$$\begin{aligned}
f &= (h_0, h_1, h_2; q_0, q_1, q_2) \\
\psi_1 f &= (h_1, h_2, h_0; q_1, q_2, q_0) \\
\psi_2 f &= (h_2, h_1, h_0; q_0, q_2, q_1) \\
\psi_3 f &= (h_0, h_1, h_2; 2l_0 - q_0, 2l_1 - q_1, 2l_2 - q_2) \\
\psi_1^2 f &= (h_2, h_0, h_1; q_2, q_0, q_1) \\
\psi_2 \psi_1 f &= (h_0, h_2, h_1; q_1, q_0, q_2) \\
\psi_3 \psi_1 f &= (h_1, h_2, h_0; 2l_1 - q_1, 2l_2 - q_2, 2l_0 - q_0) \\
\psi_2 \psi_1^2 f &= (h_1, h_0, h_2; q_2, q_1, q_0) \\
\psi_3 \psi_1^2 f &= (h_2, h_0, h_1; 2l_2 - q_2, 2l_0 - q_0, 2l_1 - q_1) \\
\psi_3 \psi_2 f &= (h_2, h_1, h_0; 2l_0 - q_0, 2l_2 - q_2, 2l_1 - q_1) \\
\psi_3 \psi_2 \psi_1 f &= (h_0, h_2, h_1; 2l_1 - q_1, 2l_0 - q_0, 2l_2 - q_2) \\
\psi_3 \psi_2 \psi_1^2 f &= (h_1, h_0, h_2; 2l_2 - q_2, 2l_1 - q_1, 2l_0 - q_0)
\end{aligned}$$

Similarly, it is not complicated to compute an image $\sigma(f)$ for any $f \in \mathcal{F}_2$. On the other hand, a $\mathcal{G}$-orbit may be infinite. In what follows we give a simple method for deciding whether $g \approx_{\mathcal{G}} h$.

Example. The 6-tuple $(1, 3, k; 2, 2, k-1)$ $k \geq 3$ belongs to an infinite $\mathcal{G}$-orbit. Since $\sigma((1, 3, k; 2, 2, k-1)) = (3, 1, k+2; k+1, 2, 2)$ and this 6-tuple is $\mathcal{H}$-equivalent to the 6-tuple $(1, 3, k+2; 2, 2, k+1)$. Hence $(1, 3, k+2; 2, 2, k+1) \approx_{\mathcal{G}} (1, 3, k; 2, 2, k-1)$, and $z(\sigma^{j+1}(f)) > z(\sigma^j(f))$ for every positive integer j. Thus $[(1, 3, k; 2, 2, k-1)]_{\mathcal{G}}$ is infinite.

Representatives of $\mathcal{G}$-orbits

Next lemma appears in [5, Prop. 6.1] without proof.

Lemma 5. Let $f \in \mathcal{F}_2$. Then ψ_2, ψ_3 and σ satisfy the following relations:

$$\begin{aligned} \text{a) } \sigma^2 &= 1 \\ \text{b) } \psi_2\sigma &= \sigma\psi_2 \\ \text{c) } \psi_3\sigma &= \sigma\psi_3 \end{aligned}$$

Proof. The proof is done by direct computation. We have to deal with four cases related with the action of σ-operator. Recall that all the computations with q_i will be done modulo $h_i + h_{i+1}, i \in \mathbb{Z}_3$.

a) Let $f' = \sigma(f)$ and $f'' = \sigma(f')$. We prove $f = f''$.
The case $q_0 = 0$ implies the identity by definition.

$$\text{(I)} \qquad \begin{aligned} 0 < q_0 < h_0 \Rightarrow h_0 + h_1 - q_0 < h_0 + h_1 + h_2 - 2q_0 \Rightarrow h_0' < q_0' \\ 0 < q_0 < h_2 \Rightarrow h_2 + h_1 - q_0 < h_0 + h_1 + h_2 - 2q_0 \Rightarrow h_2' < q_0' \end{aligned}$$

Hence we have to apply Case II in the definition of σ to compute $\sigma(f'')$

$$\begin{aligned}
h_0'' &= (h_0 + h_1 + h_2 - 2q_0) + q_0 - (h_2 + h_1 - q_0) = h_0 \\
h_1'' &= (h_0 + h_1 - q_0) + (h_2 + h_1 - q_0) - (h_0 + h_1 + h_2 - 2q_0) = h_1 \\
h_2'' &= (h_0 + h_1 + h_2 - 2q_0) + q_0 - (h_0 + h_1 - q_0) = h_2 \\
q_0'' &= q_0 \bmod (h_0 + h_2) \\
q_1'' &= (h_0 + h_1 + h_2 - 2q_0) + (q_0 + q_1 + h_1) - (h_2 + h_1 - q_0) \equiv q_1 \bmod (h_0 + h_1) \\
q_2'' &= (h_0 + h_1 + h_2 - 2q_0) + (q_0 + q_2 + h_1) - (h_0 + h_1 - q_0) \equiv q_2 \bmod (h_1 + h_2)
\end{aligned}$$

$$\text{(II)} \qquad \begin{aligned} q_0 > h_0, h_2 \Rightarrow q_0 - h_2 > 0 \Rightarrow h_1 < h_1 + (q_0 - h_2) \Rightarrow 0 < q_0' < h_0' \\ q_0 > h_0, h_2 \Rightarrow q_0 - h_0 > 0 \Rightarrow h_1 < h_1 + (q_0 - h_0) \Rightarrow 0 < q_0' < h_2' \end{aligned}$$

Hence we have to apply Case I in the definition of σ to compute $\sigma(f'')$

$$\begin{aligned}
h_0'' &= (q_0 + h_1 - h_2) + (h_0 + h_2 - q_0) - h_1 = h_0 \\
h_1'' &= h_1 \\
h_2'' &= (q_0 + h_1 - h_0) + (h_0 + h_2 - q_0) - h_1 = h_2 \\
q_0'' &= (q_0 + h_1 - h_2) + (h_0 + h_2 - q_0) + (q_0 + h_1 - h_0) - 2h_1 = q_0 \bmod (h_0 + h_2) \\
q_1'' &= h_1 + (q_0 + q_1 - h_2) + (h_0 + h_2 - q_0) \equiv q_1 \bmod (h_0 + h_1) \\
q_2'' &= h_1 + (q_0 + q_2 - h_0) + (h_0 + h_2 - q_0) \equiv q_2 \bmod (h_1 + h_2)
\end{aligned}$$

$$\text{(III)}\qquad \begin{array}{l} q_0 < h_2 \Rightarrow h_1 < (h_2 - q_0) + h_1 \Rightarrow h_0' < q_0' \\ q_0 > h_0 \Rightarrow (h_1 + h_2) - q_0 < (h_1 + h_2) - h_0 \Rightarrow q_0' < h_2' \end{array}$$

Hence we have to apply Case III in the definition of σ to compute $\sigma(f'')$

$$\begin{aligned}
&h_0'' = h_0 \\
&h_1'' = h_1 \\
&h_2'' = h_0 + (h_1 + h_2 - h_0) - h_1 = h_2 \\
&q_0'' = h_0 + (h_1 + h_2 - h_0) - (h_1 + h_2 - q_0) = q_0 \bmod (h_0 + h_2) \\
&q_1'' = q_1 \bmod (h_0 + h_1) \\
&q_2'' = 2(h_1 + h_2 - q_0) + (2q_0 + q_2 + h_1 - h_0) + h_0 - h_1 \equiv q_2 \bmod (h_1 + h_2)
\end{aligned}$$

$$\text{(IV)}\qquad \begin{array}{l} q_0 > h_2 \Rightarrow (h_1 + h_0) - h_2 > (h_1 + h_0) - q_0 \Rightarrow h_0' > q_0' \\ h_0 > q_0 \Rightarrow (h_0 - q_0) + h_1 > h_1 \Rightarrow q_0' > h_2' \end{array}$$

Hence we have to apply Case IV in the definition of σ to compute $\sigma(f'')$

$$\begin{aligned}
&h_0'' = h_2 + (h_1 + h_0 - h_2) - h_1 = h_0 \\
&h_1'' = h_1 \\
&h_2'' = h_2 \\
&q_0'' = h_2 + (h_1 + h_0 - h_2) - (h_1 + h_0 - q_0) = q_0 \bmod (h_0 + h_2) \\
&q_1'' = 2(h_1 + h_0 - q_0) + (2q_0 + q_1 + h_1 - h_2) + h_2 - h_1 \equiv q_1 \bmod (h_0 + h_1) \\
&q_2'' = q_2 \bmod (h_1 + h_2)
\end{aligned}$$

b) In the following calculations the usage of the respective Case in computation of images under σ is signed as follows $\ldots \overset{I.}{=} \ldots, \ldots \overset{IV.}{=} \ldots$. Let $f' = \sigma(f)$ and $f'' = \psi_2(f)$

$$\begin{aligned}
&\text{(I)}\quad 0 < q_0 < h_0, h_2 \Rightarrow \psi_2(f') \overset{I.}{=} \\
&\overset{I.}{=} \psi_2(h_0 + h_1 - q_0, q_0, h_2 + h_1 - q_0; h_0 + h_1 + h_2 - 2q_0, q_0 + q_1 + h_1, q_0 + q_1 + h_1) = \\
&= (h_2 + h_1 - q_0, q_0, h_0 + h_1 - q_0; h_2 + h_1 + h_0 - 2q_0, q_0 + q_2 + h_1, q_0 + q_1 + h_1) \overset{I.}{=} \\
&\overset{I.}{=} \sigma(h_2, h_1, h_0; q_0, q_2, q_1) = \sigma(f'') \\
&\text{(II)}\quad q_0 > h_0, h_2 \Rightarrow \psi_2(f') \overset{II.}{=} \\
&\overset{II.}{=} \psi_2(q_0 + h_1 - h_2, h_0 + h_2 - q_0, q_0 + h_1 - h_0; h_1, q_0 + q_1 - h_2, q_0 + q_2 - h_0) = \\
&= (q_0 + h_1 - h_0, h_2 + h_0 - q_0, q_0 + h_1 - h_2; h_1, q_0 + q_2 - h_0, q_0 + q_1 - h_2) \overset{II.}{=} \\
&\overset{II.}{=} \sigma(h_2, h_1, h_0; q_0, q_2, q_1) = \sigma(f'')
\end{aligned}$$

Note, that using ψ_2 in Cases (III) and (IV) of σ swaps the input conditions. However, we need to prove the following equalities:

Using Case (IV) in the definition of σ we get:

(III) $\quad h_0 < q_0 < h_2 \Rightarrow \psi_2(f') \overset{III.}{=}$

$\overset{III.}{=} \psi_2(h_1, h_0, h_1 + h_2 - h_0; h_1 + h_2 - q_0, q_1, 2q_0 + q_2 + q_2 + h_1 - h_0) =$

$= (h_1 + h_2 - h_0, h_0, h_1; h_1 + h_0 - q_0, 2q_0 + q_2 + h_1 - h_0, q_1) \overset{IV.}{=}$

$\overset{IV.}{=} \sigma(h_2, h_1, h_0; q_0, q_2, q_1) = \sigma(f'')$

Using Case (III) in the definition of σ we get:

(IV) $\quad h_2 < q_0 < h_0 \Rightarrow \psi_2(f') \overset{IV.}{=}$

$\overset{IV.}{=} \psi_2(h_1 + h_0 - h_2, h_2, h_1; h_1 + h_0 - q_0, q_2, 2q_0 + q_1 + h_1 - h_2) =$

$= (h_1, h_2, h_1 + h_0 - h_2; h_1 + h_0 - q_0, q_2, 2q_0 + q_1 + h_1 - h_2) \overset{III.}{=}$

$\overset{III.}{=} \sigma(h_2, h_1, h_0; q_0, q_2, q_1) = \sigma(f'')$

c) To prove commutativity of ψ_3 and σ note that for the minimum non-negative representatives q_0, q_1, q_2 of the repective residual classes the following relations hold (see Remark 1):

$$-q_i = (h_i + h_{i-1}) - q_i \bmod (h_i + h_{i-1}); i \in \mathbb{Z}_3$$

$q_i < h_{i-1} \Rightarrow (h_i + h_{i-1}) - q_i > (h_i + h_{i-1}) - h_{i-1} \Rightarrow$

$\Rightarrow -q_i > h_i \bmod (h_i + h_{i-1})$

$q_i < h_i \Rightarrow (h_i + h_{i-1}) - q_i > (h_i + h_{i-1}) - h_i \Rightarrow$

$\Rightarrow -q_i > h_{i-1} \bmod (h_i + h_{i-1})$

Let $f' = \psi_3\sigma f$ and $f'' = \sigma\psi_3 f$. We have to prove $f' = f''$.

I) $0 < q_0 < h_0, h_2$

$h'_0 = h_0 + h_1 - q_0$

$h'_1 = q_0$

$h'_2 = h_2 + h_1 - q_0$

$q'_0 = (h_0+h_1-q_0)+(h_2+h_1-q_0)-(h_0+h_1+h_2-2q_0) \equiv h_1 \bmod (2h_1+h_0+h_2-2q_0)$

$q'_1 = (h_0 + h_1 - q_0) + q_0 - (q_0 + q_1 + h_1) = h_0 - q_0 - q_1 \bmod (h_0 + h_1)$

$q'_2 = q_0 + (h_2 + h_1 - q_0) - (q_0 + q_2 + h_1) \equiv h_2 - q_0 - q_2 \bmod (h_1 + h_2)$

Since $0 < q_0 < h_0, h_2 \Rightarrow -q_0 > h_0, h_2 \bmod (h_0 + h_2)$ Case II in calculation of f'' applies.

$h''_0 = (h_0 + h_2) - q_0 + h_1 - h_2 = h_0 + h_1 - q_0$

$h''_1 = h_0 + h_2 - ((h_0 + h_2) - q_0) = q_0$

$h''_2 = (h_0 + h_2) - q_0 + h_1 - h_0 = h_2 + h_1 - q_0$

$q''_0 = h_1 \bmod (2h_1 + h_0 + h_2 - 2q_0)$

$q''_1 = (h_0 + h_2 - q_0) + (h_0 + h_1 - q_1) - h_2 \equiv h_0 - q_0 - q_1 \bmod (h_0 + h_1)$

$q''_2 = (h_0 + h_2 - q_0) + (h_1 + h_2 - q_2) \equiv h_2 - q_0 - q_2 \bmod (h_1 + h_2)$

II) $h_0, h_2 < q_0$

$h'_0 = h_1 - h_2 + q_0$

$h'_1 = h_0 + h_2 - q_0$

$h'_2 = h_1 - h_0 + q_0$

$q'_0 = (h_1 - h_2 + q_0) + (h_1 - h_0 + q_0) - h_1 =$

$= h_1 - h_0 - h_2 + 2q_0 \equiv -h_1 \bmod (2h_1 - h_0 - h_2 + 2q_0)$

$q'_1 = (h_1 + h_2 + q_0) + (h_1 - h_0 + q_0) - (q_0 + q_1 - h_2) =$

$= h_0 + h_1 + h_2 - q_0 - q_1 \equiv h_2 - q_0 - q_1 \bmod (h_1 + h_0)$

$q'_2 = (h_0 + h_2 - q_0) + (h_1 - h_0 + q_0) - q_0 + q_2 - h_0 =$

$= h_0 + h_1 + h_2 - q_0 - q_2 \equiv h_0 - q_0 - q_0 \bmod (h_1 + h_2)$

Since $h_0, h_2 < q_0 \Rightarrow 0 < -q_0 < h_0, h_2 \bmod (h_0 + h_2)$ Case I in calculation of f'' applies.

$h''_0 = h_0 + h_1 - (h_0 + h_2 - q_0) = h_1 - h_2 + q_0$

$h''_1 = h_0 + h_2 - q_0$

$h''_2 = h_2 + h_1 - (h_0 + h_2 - q_0) = h_1 - h_0 + q_0$

$q''_0 = h_0 + h_1 + h_2 - 2(h_0 + h_2 - q_0) =$

$= h_1 - h_0 - h_2 + 2q_0 \equiv -h_1 \bmod (2h_1 - h_0 - h_2 + 2q_0)$

$q''_1 = (h_0 + h_2 - q_0) + (h_0 + h_1 - q_1) + h_1 =$

$= h_0 + h_1 + h_2 - q_0 - q_1 \equiv h_2 - q_0 - q_1 \bmod (h_0 + h_1)$

$q''_2 = (h_0 + h_2 - q_0) + (h_1 + h_2 - q_2) + h_1 =$

$= h_0 + h_1 + h_2 - q_0 - q_2 \equiv h_0 - q_0 - q_2 \bmod (h_1 + h_2)$

III) $h_0 < q_0 < h_2$

$h'_0 = h_1$

$h'_1 = h_0$

$h'_2 = h_1 + h_2 - h_0$

$q'_0 = [h_1 + (h_1 + h_2 - h_0)] - (h_1 + h_2 - q_0) = h_1 - h_0 + q_0 \bmod (2h_1 + h_2 - h_0)$

$q'_1 = [h_1 + h_0 - q_1] \equiv -q_1 \bmod (h_1 + h_0)$

$q'_2 = [h_0 + (h_1 + h_2 - h_0)] - (2q_0 + q_2 + h_1 - h_0) = h_2 + h_0 - 2q_0 - q_2 \bmod (h_1 + h_2)$

Since $h_0 < q_0 < h_2 \Rightarrow h_0 < -q_0 < h_2 \bmod (h_0 + h_2)$ Case III in calculation of f'' applies.

$h''_0 = h_1$

$h''_1 = h_0$

$h''_2 = h_1 + h_2 - h_0$

$q''_0 = h_1 + h_2 - (h_0 + h_2 - q_0) = h_1 - h_0 + q_0 \bmod (2h_1 + h_2 - h_0)$

$q''_1 = h_0 + h_1 - q_1 = h_0 + h_1 - q_1 \equiv -q_1 \bmod (h_0 + h_1)$

$q''_2 = 2(h_0 + h_2 - q_0) + (h_1 + h_2 - q_2) + h_1 - h_0 = h_2 + h_0 - 2q_0 - q_2 \bmod (h_1 + h_2)$

IV) $h_2 < q_0 < h_0 \Rightarrow h_2 < -q_0 < h_0 \bmod (h_0 + h_2)$

$h'_0 = h_1 + h_0 - h_2$

$h'_1 = h_2$

$h'_2 = h_1$

$q'_0 = [(h_1 + h_0 - h_2) + h_1] - (h_1 + h_0 - q_0) = h_1 - h_2 + q_0 \bmod (2h_1 + h_0 - h_2)$

$q'_1 = [(h_1 + h_0 - h_2) + h_2] - (2q_0 + q_1 + h_1 - h_2) = h_0 + h_2 - 2q_0 - q_1 \bmod (h_1 + h_0)$

$q'_2 = [h_1 + h_2] - q_2 \equiv -q_2 \bmod (h_1 + h_2)$

Since $h_2 < q_0 < h_0 \Rightarrow h_2 < -q_0 < h_0 \bmod (h_0 + h_2)$ Case IV in calculation of f'' applies.

$h''_0 = h_1 + h_0 - h_2$

$h''_1 = h_2$

$h''_2 = h_1$

$q''_0 = h_1 + h_0 - (h_0 + h_2 - q_0) = h_1 - h_2 + q_0 \bmod (2h_1 + h_0 - h_2)$

$q''_1 = 2(h_0 + h_2 - q_0) + (h_0 + h_1 - q_1) + h_1 - h_2 = h_0 + h_2 - 2q_0 - q_1 \bmod (h_0 + h_1)$

$q''_2 = h_1 + h_2 - q_2 \equiv -q_2 \bmod (h_1 + h_2)$

□

The application of σ is now easier. It follows that to calculate the action of σ it is sufficient to consider the images of the three members σf, $\sigma\psi_1 f$ and $\sigma\psi_1^2 f$ of an $\mathcal{H}$-orbit.

Definition 11. Let $\mathcal{S} = \{V, E\}$ be a graph which vertices are $\mathcal{H}$-orbits and the adjacency relation is given by:

$$[f]_{\mathcal{H}} \sim [g]_{\mathcal{H}} \Leftrightarrow \exists g' \in [g]_{\mathcal{H}} \wedge \exists f' \in [f]_{\mathcal{H}} : g' = \sigma f'.$$

Since $\sigma^2 = 1$, the graph $\mathcal{S}$ is undirected. Note that $\mathcal{S}$ contains loops.

The connectivity components of $\mathcal{S}$ are in a correspondence with the $\mathcal{G}$-orbits. Therefore we call the connectivity components of $\mathcal{S}$, $\mathcal{G}$-orbits too. By the definition, a $\mathcal{G}$-orbit is a class of equivalence. We can describe its minimal representatives.

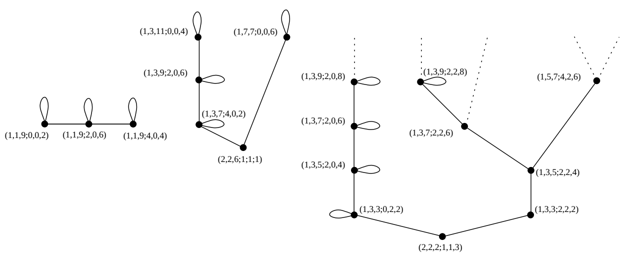

FIG. 3. Some components of connectivity of $\mathcal{S}$.

Agreement. Since the members of each $\mathcal{H}$-orbit have the same complexity, we define the complexity of a $\mathcal{H}$-orbit to be the complexity of its members. Since each $\mathcal{H}$-orbit corresponds to a vertex in $\mathcal{S}$, we can speak about complexity of a vertex. Moreover, we say that $\mathbf{u} \ll \mathbf{v}$ for $\mathbf{u} = [f]_{\mathcal{H}}$ and $\mathbf{v} = [g]_{\mathcal{H}}$, if $f \ll g$.

Lemma 6. The set of neighbours of a vertex $\mathbf{u} = [f]_{\mathcal{H}}$ in the graph $\mathcal{S}$ is $N = \{[\sigma f]_{\mathcal{H}}, [\sigma\psi_1 f]_{\mathcal{H}}, [\sigma\psi_1^2 f]_{\mathcal{H}}\}$. In particular, a vertex in $\mathcal{S}$ has at most 3 neighbours.

Proof. Let $A = \langle\psi_2, \psi_3\rangle$. Each $\mathcal{H}$-orbit decomposes into the orbits induced by the action of A. Since σ commutes with the elements of A (see Lemma 5), it follows that for $g = \phi f, \phi \in A$ we have $\sigma g = \sigma\phi f = \phi\sigma f$, hence $[\sigma g]_{\mathcal{H}} = [\sigma f]_{\mathcal{H}}$. Hence, the set of neighbours of vertex $\mathbf{u}$ is N. □

Theorem 2. Let $\mathbf{v}, \mathbf{u}, \mathbf{w}$ be three pairwise distinct vertices in $\mathcal{S}$. Let $\mathbf{u}$ and $\mathbf{w}$ be neighbours of $\mathbf{v}$. Then

$$z(\mathbf{u}) < z(\mathbf{v}) \Rightarrow z(\mathbf{w}) > z(\mathbf{v}), \tag{1}$$
$$z(\mathbf{u}) = z(\mathbf{v}) \Rightarrow z(\mathbf{w}) \geq z(\mathbf{v}). \tag{2}$$

Proof. Let us analyse the derivation of complexity $\delta(f)$ for a vertex $\mathbf{v}, f \in [f]_{\mathcal{H}} = \mathbf{v}$. Recall that $f = (h_0, h_1, h_2; q_0; q_1, q_2)$ is the minimal representative of $[f]_{\mathcal{H}}$. It follows that $h_0 \leq h_1 \leq h_2$. By Lemma 6 $\mathbf{u}, \mathbf{w} \in \{[\sigma f]_{\mathcal{H}}, [\sigma\psi_1 f]_{\mathcal{H}}, [\sigma\psi_1^2 f]_{\mathcal{H}}\}$. Hence we need to analyse the three derivations: $\delta(f)$, $\delta(\psi_1 f)$ and $\delta(\psi_1^2 f)$. In the following discussion we refer to Lemma 4.

I. For $\delta(f)$ we get:

(a) $q_0 = 0 \Rightarrow \delta(f) = 0,$

(b) $0 < q_0 < h_0, h_2 \Rightarrow \delta(f) > 0,$ therefore $h_1 - q_0 \geq h_0 - q_0 > 0,$

(c) $h_0 < q_0 < h_2,$ we consider subcases:

$h_0 < q_0 < h_2 \Rightarrow \delta(f) > 0$

or

$h_0 = h_1 < q_0 < h_2 \Rightarrow \delta(f) = 0,$

(d) $h_2 < h_0$ is in a contradiction with the minimality of f,

(e) $q_0 > h_0, h_2 \Rightarrow \delta(f) > 0,$ therefore $q_0 + h_1 - h_0 - h_2 > h_1 - h_0 \geq 0.$

II. For $\delta(\psi_1 f)$ we get:

(a) $q_1 = 0 \Rightarrow \delta(\psi_1 f) = 0,$

(b) $0 < q_1 < h_1, h_0 \Rightarrow \delta(\psi_1 f) > 0,$ therefore $h_2 - q_1 \geq h_1 - q_1 > 0,$

(c) $h_1 < h_0$ is in a contradiction with the minimality of f,

(d) $h_0 < q_1 < h_1 \Rightarrow \delta(\psi_1 f) > 0,$

(e) $q_1 > h_1, h_0 \Rightarrow \delta(\psi_1 f) > 0,$ therefore $q_1 + h_2 - h_1 - h_0 > h_2 - h_0 \geq 0.$

III. For $\delta(\psi_1^2 f)$ we get:

(a) $q_2 = 0 \Rightarrow \delta(\psi_1^2 f) = 0,$

(b) $0 < q_2 < h_1, h_2$ we must consider following cases:

$q_2 < h_0 \leq h_1, h_2 \Rightarrow \delta(\psi_1^2 f) > 0$

or

$h_0 < q_2 < h_1 \Rightarrow \delta(\psi_1^2 f) < 0,$

(c) $h_2 < h_1$, is in a contradiction with the minimality of f,

(d) $h_1 < q_2 < h_2$ we consider subcases:

$h_0 < h_1 < q_2 < h_2 \Rightarrow \delta(\psi_1^2 f) < 0$

or

$h_0 = h_1 < q_2 < h_2 \Rightarrow \delta(\psi_1^2 f) = 0,$

(e) $q_2 > h_1, h_2$ we consider subcases:

$h_0 \leq h_1 \leq h_2 < q_2 < h_1 + h_2 - h_0 \Rightarrow \delta(\psi_1^2 f) < 0$

or

$h_0 \leq h_1 \leq h_2 < q_2 = h_1 + h_2 - h_0 \Rightarrow \delta(\psi_1^2 f) = 0$

or

$h_1, h_2, h_1 + h_2 - h_0 \leq q_2 \Rightarrow \delta(\psi_1^2 f) > 0.$

The previous discussion describes the derivation of each neighbour of the vertex $\mathbf{v}$. The subcases are pairwise eliminative and they cover all the possibilities. Since only one from the three possible neighbours of a given vertex $\mathbf{x} = [f]_{\mathcal{H}}$ can have a smaller complexity as $\mathbf{x}$, it follows that two edges incident to vertices $\mathbf{u}$, $\mathbf{w}$ with given complexity never enter the vertex $\mathbf{v}$ with higher complexity. This neighbour is $\mathbf{y} = [\sigma\psi_1^2 f]_{\mathcal{H}}$ and $z(\mathbf{y}) < z(\mathbf{x})$ in some subcases of Case **III**. The complexity of a neighbour of $\mathbf{v}$ can be smaller only in Case **III**.

Now assume $z(\mathbf{u}) < z(\mathbf{v})$. We have already observed $z(\mathbf{w}) \geq z(\mathbf{v})$. Assume $z(\mathbf{w}) = z(\mathbf{v})$. Analysing Cases **I**, **II** and **III** we see that $\mathbf{u}$ satisfies one of the conditions **III-b**, **III-d**, **III-e**. Moreover, $\mathbf{w}$ satisfies the condition **I-c**. Combining **I-c** with **III-b**, or **III-d**, or **III-e** we derive the following contradictions:

$$h_0 = h_1 < q_0 < h_2 \wedge h_0 = q_2 < h_1, h_2 \Rightarrow h_1 < q_0 \leq q_2 < h_1$$
$$h_0 = h_1 < q_0 < h_2 \wedge h_0 \leq h_1 \leq h_2 < q_2 = h_1 + h_2 - h_0 \Rightarrow h_2 < q_2 < h_2$$
$$h_0 = h_1 < q_0 < h_2 \wedge h_0 < h_1 < q_2 < h_2.$$

Hence $z(\mathbf{w}) > z(\mathbf{v})$ and we are done. □

Definition 12. A vertex $\mathbf{v}$ is in an *horisontal branch* $\mathcal{B} = \mathcal{B}(\mathbf{v})$ of a $\mathcal{G}$-orbit if the following holds:

$$\mathbf{v} \in \mathcal{B}(\mathbf{v}) \Leftrightarrow \forall \mathbf{u} \in N(\mathbf{v}) : z(\mathbf{u}) \geq z(\mathbf{v})$$

Lemma 7. In every $\mathcal{G}$-orbit there is precisely one horisontal branch $\mathcal{B}$ and $\mathcal{B}$ contains the minimum element $\mathbf{m}$ of the $\mathcal{G}$-orbit with respect to the order $\ll$. The complexity of all elements of $\mathcal{B}$ is equal to $z(\mathbf{m})$.

Proof. By the definition and by Theorem 2 a horisontal branch $\mathcal{B}$ consists of the 6-tuples with a fixed complexity. A minimal representative $\mathbf{m}$ of a $\mathcal{G}$-orbit containing $\mathcal{B}$ belongs to $\mathcal{B}$ as well. Moreover, Theorem 2 implies that the complexity of the 6-tuples in $\mathcal{B}(\mathbf{m})$ is equal to $z(\mathbf{m})$. □

Notice that a horisontal branch may contain only one vertex of $\mathcal{S}$.

Theorem 3. There exists a polynomial-time algorithm to decide whether two 6-tuples in $\mathcal{F}_2$ are $\mathcal{G}$-equivalent.

Proof. Let f and g be 6-tuples in $\mathcal{F}_2$ such that $z(f) \geq z(g)$. Using Theorem 2 we find $f_1 \in N(f)$ and $g_1 \in N(g)$ so that $z(f_1) \leq z(f)$ and $z(g_1) \leq z(g)$. Note that if the complexity of f_1 (g_1) is less than $z(f)$ $(z(g))$, f_1 (g_1) is uniquely determined. By proceeding at most $z(f) = n$ iterations we reach the horisontal branches of the respective $\mathcal{G}$-orbits containing f and g. If $z(f_n) \neq z(g_n)$, the 6-tuples are not $\mathcal{G}$-equivalent. The complexity of this procedure is $O(n)$. If $z(f_n) = z(g_n)$ the algorithm continues. We choose the minimal representatives of horisontal branches $\mathcal{B}_1(f_n)$ and $\mathcal{B}_2(g_n)$ containing f_n and g_n. If the minimal representatives are equal then $f \approx_{\mathcal{G}} g$. The complexity of this part of algorithm can be rougly estimated by $O(z_n(f)^3)$. By Lemma 7 the 6-tuples f and g are not $\mathcal{G}$-equivalent in the other case. □

List of $\mathcal{G}$-minimal representatives of 6-tuples

By using [2], [5] and previous results we have generated two catalogues of minimal representatives of $\mathcal{G}$-classes of 3-manifolds of genus two.

The first one is a reduction of the catalogue introduced in [2]. We have applied the $\mathcal{G}$-equivalence on it. The new version includes only minimal representatives of $\mathcal{G}$-orbits. It is created by a simple algorithm which computes $\sigma(f)$, $\sigma(\psi_1 f)$ and $\sigma(\psi_1^2 f)$ for every 6-tuple. Only 6-tuples satisfying $f \leq \sigma(f) \wedge f \leq \sigma(\psi_1 f) \wedge f \leq \sigma(\psi_1^2 f)$ (see Theorem 2) are listed in this catalogue. The new catalogue contains 309 of 6-tuples with complexity $z \leq 21$ instead of 695 6-tuples of the original.

The second catalogue is formed by computing of all admissible 6-tuples with complexity $z \leq 21$. After creating, a 6-tuple is processed by a similar way as described above and the minimal representatives of horisontal branches were listed. Since we do not use any further criteria to reduce it, this catalogue is more rich as the first one. It contains 433 of minimal 6-tuples. We have excluded the representatives of *traps* defined by a condition introduced in [5].

The catalogue up to complexity $z = 50$ was created and reduced in thirty minutes. Using the same program minimal prepresentatives of $\mathcal{G}$-orbits up to complexity $z = 100$ and higher can be generated in a real time. The program can be parallelised.

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E-mail: karabas@savbb.sk, nedela@savbb.sk

Appendix A: Reduced catalogue introuced in [2]

```
+++ z =  7 +++

( 1, 3, 3; 2, 2, 0)

+++ z =  9 +++

( 1, 3, 5; 2, 2, 0)
( 3, 3, 3; 2, 2, 2)

+++ z = 10 +++


+++ z = 11 +++

( 1, 3, 7; 2, 2, 0)
( 1, 5, 5; 2, 2, 0)
( 1, 5, 5; 2, 4, 0)
( 3, 3, 5; 0, 2, 4)
( 3, 3, 5; 2, 2, 4)

+++ z = 12 +++

( 4, 4, 4; 1, 1, 1)
( 4, 4, 4; 1, 1, 5)
( 4, 4, 4; 3, 3, 3)

+++ z = 13 +++

( 1, 3, 9; 2, 2, 0)
( 1, 3, 9; 4, 2, 0)
( 1, 5, 7; 2, 2, 0)
( 1, 5, 7; 2, 4, 0)
( 3, 3, 7; 2, 2, 2)
( 3, 3, 7; 2, 2, 6)
( 3, 5, 5; 2, 4, 0)
( 3, 5, 5; 4, 4, 2)

+++ z = 14 +++

( 4, 4, 6; 1, 1, 1)
( 4, 4, 6; 1, 1, 3)
( 4, 4, 6; 1, 1, 5)
( 4, 4, 6; 1, 1, 7)
( 4, 4, 6; 1, 5, 1)
( 4, 4, 6; 1, 5, 5)
( 4, 4, 6; 3, 1, 5)
( 4, 4, 6; 3, 3, 5)

+++ z = 15 +++

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( 1, 5, 9; 2, 4, 0)
( 1, 5, 9; 4, 2, 0)
( 1, 5, 9; 4, 4, 0)
( 1, 7, 7; 2, 2, 0)
( 1, 7, 7; 2, 6, 0)
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( 3, 3, 9; 2, 0, 2)
( 3, 3, 9; 2, 0, 6)
( 3, 3, 9; 2, 2, 8)
( 3, 5, 7; 2, 4, 0)
( 3, 5, 7; 2, 4, 2)
( 3, 5, 7; 4, 4, 0)
( 3, 5, 7; 4, 4,10)
( 5, 5, 5; 0, 4, 4)
( 5, 5, 5; 2, 2, 2)
( 5, 5, 5; 4, 4, 4)

+++ z = 16 +++

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( 4, 4, 8; 1, 1, 9)
( 4, 4, 8; 1, 5, 1)
( 4, 4, 8; 3, 1, 7)
( 4, 4, 8; 3, 3, 7)
( 4, 4, 8; 3, 7, 3)
( 4, 6, 6; 1, 1, 1)
( 4, 6, 6; 1, 1, 9)
( 4, 6, 6; 1, 7, 1)
( 4, 6, 6; 3, 5,11)
( 4, 6, 6; 5, 5, 3)

+++ z = 17 +++

( 1, 3,13; 2, 2, 0)
( 1, 3,13; 4, 2, 0)
( 1, 3,13; 6, 2, 0)
( 1, 5,11; 2, 2, 0)
( 1, 5,11; 2, 4, 0)
( 1, 7, 9; 2, 2, 0)
( 1, 7, 9; 2, 6, 0)
( 1, 7, 9; 4, 2, 0)
( 1, 7, 9; 4, 6, 0)
( 3, 3,11; 2, 2, 2)
( 3, 3,11; 2, 2, 4)
( 3, 3,11; 2, 2,10)
( 3, 5, 9; 2, 0, 2)
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( 3, 5, 9; 4, 2, 0)
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( 3, 5, 9; 4, 6, 0)
( 3, 7, 7; 2, 2, 2)
( 3, 7, 7; 2, 6, 2)
( 3, 7, 7; 4, 4,12)
( 5, 5, 7; 0, 2, 6)
( 5, 5, 7; 0, 4, 6)
```

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( 5, 5, 7; 2, 4, 6)
( 5, 5, 7; 2, 6, 2)
( 5, 5, 7; 2, 6, 6)
( 5, 5, 7; 4, 0, 6)
( 5, 5, 7; 4, 2, 4)
( 5, 5, 7; 4, 4, 4)
( 5, 5, 7; 4, 4, 6)

+++ z = 18 +++

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( 2, 8, 8; 3, 5, 1)
( 2, 8, 8; 5, 5, 1)
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( 4, 4,10; 1, 1, 3)
( 4, 4,10; 1, 1, 5)
( 4, 4,10; 1, 1, 7)
( 4, 4,10; 1, 1, 9)
( 4, 4,10; 1, 1,11)
( 4, 4,10; 1, 5, 1)
( 4, 4,10; 1, 5, 7)
( 4, 4,10; 1, 5, 9)
( 4, 4,10; 3, 1, 9)
( 4, 4,10; 3, 3, 3)
( 4, 4,10; 3, 3, 9)
( 4, 6, 8; 1, 1, 1)
( 4, 6, 8; 1, 1, 3)
( 4, 6, 8; 1, 1,11)
( 4, 6, 8; 1, 5, 3)
( 4, 6, 8; 1, 7, 1)
( 4, 6, 8; 3, 5,13)
( 4, 6, 8; 3, 9, 3)
( 4, 6, 8; 3, 9,13)
( 4, 6, 8; 5, 1, 3)
( 4, 6, 8; 5, 5, 3)
( 4, 6, 8; 5, 5,11)
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( 6, 6, 6; 1, 1, 9)
( 6, 6, 6; 1, 3, 3)
( 6, 6, 6; 1, 3, 7)
( 6, 6, 6; 1, 5, 9)
( 6, 6, 6; 1, 7, 7)
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( 6, 6, 6; 3, 5, 5)
( 6, 6, 6; 5, 5, 5)

+++ z = 19 +++

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( 1, 5,13; 4, 4, 0)
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( 1, 7,11; 2, 6, 0)
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( 1, 9, 9; 2, 8, 0)
( 1, 9, 9; 4, 4, 0)
( 1, 9, 9; 4, 6, 0)
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( 3, 3,13; 2, 2, 8)
( 3, 3,13; 2, 2,12)
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( 3, 5,11; 2, 4, 2)
( 3, 5,11; 4, 4, 0)
( 3, 5,11; 4, 4, 2)
( 3, 5,11; 4, 4,14)
( 3, 5,11; 6, 4, 0)
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( 3, 7, 9; 2, 4, 2)
( 3, 7, 9; 4, 2, 0)
( 3, 7, 9; 4, 4, 0)
( 3, 7, 9; 4, 4,14)
( 3, 7, 9; 4, 6, 0)
( 3, 7, 9; 4, 8, 0)
( 3, 7, 9; 4, 8,14)
( 5, 5, 9; 0, 4, 4)
( 5, 5, 9; 0, 4, 6)
( 5, 5, 9; 0, 4,12)
( 5, 5, 9; 2, 0, 2)
( 5, 5, 9; 2, 0,10)
( 5, 5, 9; 2, 2, 2)
( 5, 5, 9; 4, 0, 4)
( 5, 5, 9; 4, 0, 8)
( 5, 5, 9; 4, 4, 8)
( 5, 5, 9; 4, 8, 4)
( 5, 7, 7; 0, 4,12)
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( 5, 7, 7; 2, 6, 0)
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( 5, 7, 7; 4, 4, 2)
( 5, 7, 7; 4, 6, 4)
( 5, 7, 7; 4, 6,12)
( 5, 7, 7; 6, 6, 4)

+++ z = 20 +++

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( 4, 4,12; 1, 1, 9)
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( 4, 4,12; 1, 5, 5)
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( 4, 4,12; 3, 3,11)
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( 4, 6,10; 5, 1, 1)
( 4, 6,10; 5, 3,15)
( 4, 6,10; 5, 5, 1)
( 4, 6,10; 5, 5,13)
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( 4, 6,10; 7, 1, 1)
( 4, 6,10; 7, 3,15)
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( 4, 8, 8; 1, 1,13)
( 4, 8, 8; 1, 7, 3)
( 4, 8, 8; 1, 9, 1)
( 4, 8, 8; 1, 9,13)
( 4, 8, 8; 3, 3, 1)
( 4, 8, 8; 5, 5,13)
( 4, 8, 8; 5, 7, 3)
( 6, 6, 8; 1, 1, 1)
( 6, 6, 8; 1, 1, 3)
( 6, 6, 8; 1, 1, 5)
( 6, 6, 8; 1, 1, 7)
( 6, 6, 8; 1, 1, 9)
( 6, 6, 8; 1, 1,11)
( 6, 6, 8; 1, 5, 5)
( 6, 6, 8; 1, 5,11)
( 6, 6, 8; 1, 9, 1)
( 6, 6, 8; 1, 9, 7)
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( 6, 6, 8; 3, 7, 7)
( 6, 6, 8; 3,11, 3)
( 6, 6, 8; 3,11, 9)
( 6, 6, 8; 5, 1, 7)
( 6, 6, 8; 5, 3, 7)
( 6, 6, 8; 5, 5, 7)
( 6, 6, 8; 5,11, 7)

+++ z = 21 +++

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( 1, 3,17; 8, 2, 0)
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( 1, 5,15; 2, 4, 0)
( 1, 5,15; 6, 2, 0)
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( 1, 7,13; 4, 6, 0)
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( 1, 7,13; 6, 6, 0)
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( 1, 9,11; 2, 4, 0)
( 1, 9,11; 2, 6, 0)
( 1, 9,11; 2, 8, 0)
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( 3, 3,15; 2, 2, 6)
( 3, 3,15; 2, 2,14)
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( 3, 5,13; 2, 4, 0)
( 3, 5,13; 4, 2, 0)
( 3, 5,13; 4, 4,16)
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( 3, 5,13; 6, 4, 0)
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( 3, 7,11; 2, 6, 2)
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( 3, 7,11; 4, 4,16)
( 3, 7,11; 4, 6, 2)
( 3, 7,11; 4, 8,16)
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( 3, 9, 9; 2, 4, 0)
( 3, 9, 9; 2, 8, 0)

( 3, 9, 9; 4, 4,16)
( 5, 5,11; 0, 4,10)
( 5, 5,11; 0, 4,14)
( 5, 5,11; 2, 0, 8)
( 5, 5,11; 2, 0,12)
( 5, 5,11; 2, 2, 8)
( 5, 5,11; 2, 4, 2)
( 5, 5,11; 2, 4,10)
( 5, 5,11; 2, 6, 2)
( 5, 5,11; 2, 6,10)
( 5, 5,11; 4, 0,10)
( 5, 5,11; 4, 2, 4)
( 5, 5,11; 4, 2, 8)
( 5, 5,11; 4, 4, 8)
( 5, 5,11; 4, 4,10)
( 5, 5,11; 4, 8, 4)
( 5, 7, 9; 0, 2, 4)
( 5, 7, 9; 0, 2,12)
( 5, 7, 9; 0, 4,14)
( 5, 7, 9; 0, 6, 4)
( 5, 7, 9; 2, 0, 2)
( 5, 7, 9; 2, 2, 4)
( 5, 7, 9; 2, 4, 4)
( 5, 7, 9; 2, 6, 0)
( 5, 7, 9; 4, 0, 2)
( 5, 7, 9; 4, 0,14)
( 5, 7, 9; 4, 2, 2)
( 5, 7, 9; 4, 6, 0)
( 5, 7, 9; 4, 6, 2)
( 5, 7, 9; 4, 6,14)
( 5, 7, 9; 4,10, 4)
( 5, 7, 9; 4,10,14)
( 5, 7, 9; 6, 4, 4)
( 5, 7, 9; 6, 6, 0)
( 5, 7, 9; 6, 6,12)
( 5, 7, 9; 6, 6,14)
( 5, 7, 9; 6, 8, 2)
( 5, 7, 9; 6, 8,12)
( 5, 7, 9; 6,10,14)
```

```
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( 7, 7, 7; 2, 2, 6)
( 7, 7, 7; 2, 2,10)
( 7, 7, 7; 2, 6,10)
( 7, 7, 7; 2, 8, 8)
( 7, 7, 7; 4, 4, 4)
( 7, 7, 7; 4, 4, 8)
( 7, 7, 7; 6, 6, 6)
```

Appendix B: Our version of catalogue

```
+++ z =  7 +++

( 1, 3, 3; 2, 2, 0)

+++ z =  9 +++

( 1, 1, 7; 2, 0, 2)
( 1, 3, 5; 2, 2, 0)
( 3, 3, 3; 2, 2, 2)

+++ z = 11 +++

( 1, 3, 7; 2, 2, 0)
( 1, 5, 5; 2, 2, 0)
( 1, 5, 5; 2, 4, 0)
( 3, 3, 5; 0, 2, 4)
( 3, 3, 5; 2, 2, 4)

+++ z = 12 +++

( 2, 4, 6; 3, 3, 1)
( 4, 4, 4; 1, 1, 1)
( 4, 4, 4; 1, 1, 5)
( 4, 4, 4; 3, 3, 3)

+++ z = 13 +++

( 1, 1,11; 2, 0, 2)
( 1, 3, 9; 2, 2, 0)
( 1, 3, 9; 4, 2, 0)
( 1, 5, 7; 2, 2, 0)
( 1, 5, 7; 2, 4, 0)
( 3, 3, 7; 2, 2, 2)
( 3, 3, 7; 2, 2, 6)
( 3, 5, 5; 2, 4, 0)
( 3, 5, 5; 4, 4, 2)

+++ z = 14 +++

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( 2, 6, 6; 3, 5, 1)
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( 4, 4, 6; 1, 1, 3)
( 4, 4, 6; 1, 1, 5)
( 4, 4, 6; 1, 1, 7)
( 4, 4, 6; 1, 5, 1)
( 4, 4, 6; 1, 5, 5)
( 4, 4, 6; 3, 1, 5)
( 4, 4, 6; 3, 3, 3)
( 4, 4, 6; 3, 3, 5)

+++ z = 15 +++

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( 1, 5, 9; 2, 4, 0)
( 1, 5, 9; 4, 2, 0)
( 1, 5, 9; 4, 4, 0)
( 1, 7, 7; 2, 2, 0)
( 1, 7, 7; 2, 6, 0)
( 3, 3, 9; 0, 2, 4)
( 3, 3, 9; 0, 2, 8)
( 3, 3, 9; 2, 0, 2)
( 3, 3, 9; 2, 0, 4)
( 3, 3, 9; 2, 0, 6)
( 3, 3, 9; 2, 2, 8)
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( 3, 5, 7; 2, 4, 2)
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( 5, 5, 5; 0, 4, 4)
( 5, 5, 5; 2, 2, 2)
( 5, 5, 5; 2, 2, 6)
( 5, 5, 5; 4, 4, 4)

+++ z = 16 +++

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( 2, 4,10; 5, 3,13)
( 2, 6, 8; 3, 3, 1)
( 2, 6, 8; 3, 5,13)
( 2, 6, 8; 5, 3, 1)
( 4, 4, 8; 1, 1, 1)
( 4, 4, 8; 1, 1, 9)
( 4, 4, 8; 1, 3, 5)
( 4, 4, 8; 1, 5, 1)
( 4, 4, 8; 3, 1, 7)
( 4, 4, 8; 3, 3, 7)
( 4, 4, 8; 3, 7, 3)
( 4, 6, 6; 1, 1, 1)
( 4, 6, 6; 1, 1, 9)
( 4, 6, 6; 1, 7, 1)
( 4, 6, 6; 3, 5, 3)
( 4, 6, 6; 3, 5,11)
( 4, 6, 6; 5, 5, 3)

+++ z = 17 +++
```

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( 1, 5,11; 2, 4, 0)
( 1, 7, 9; 2, 2, 0)
( 1, 7, 9; 2, 6, 0)
( 1, 7, 9; 4, 2, 0)
( 1, 7, 9; 4, 6, 0)
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( 3, 3,11; 2, 2, 4)
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( 3, 5, 9; 4, 2, 0)
( 3, 5, 9; 4, 4, 2)
( 3, 5, 9; 4, 4,12)
( 3, 5, 9; 4, 6, 0)
( 3, 7, 7; 2, 2, 2)
( 3, 7, 7; 2, 6, 2)
( 3, 7, 7; 4, 4, 0)
( 3, 7, 7; 4, 4,12)
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( 5, 5, 7; 0, 4, 6)
( 5, 5, 7; 0, 4,10)
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( 5, 5, 7; 2, 0, 6)
( 5, 5, 7; 2, 0, 8)
( 5, 5, 7; 2, 2, 8)
( 5, 5, 7; 2, 4, 2)
( 5, 5, 7; 2, 4, 6)
( 5, 5, 7; 2, 6, 2)
( 5, 5, 7; 2, 6, 6)
( 5, 5, 7; 4, 0, 6)
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( 5, 5, 7; 4, 4, 4)
( 5, 5, 7; 4, 4, 6)

+++ z = 18 +++

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( 2, 6,10; 5, 5,15)
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( 2, 8, 8; 5, 5, 1)
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( 4, 4,10; 1, 1, 3)
( 4, 4,10; 1, 1, 5)
( 4, 4,10; 1, 1, 7)
( 4, 4,10; 1, 1, 9)
( 4, 4,10; 1, 1,11)
( 4, 4,10; 1, 5, 1)
( 4, 4,10; 1, 5, 7)
( 4, 4,10; 1, 5, 9)
( 4, 4,10; 3, 1, 9)
( 4, 4,10; 3, 3, 3)
( 4, 4,10; 3, 3, 7)
( 4, 4,10; 3, 3, 9)
( 4, 4,10; 3, 7, 7)
( 4, 6, 8; 1, 1, 1)
( 4, 6, 8; 1, 1, 3)
( 4, 6, 8; 1, 1,11)
( 4, 6, 8; 1, 5, 3)
( 4, 6, 8; 1, 7, 1)
( 4, 6, 8; 3, 5,13)
( 4, 6, 8; 3, 9, 3)
( 4, 6, 8; 3, 9,13)
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( 4, 6, 8; 5, 5, 3)
( 4, 6, 8; 5, 5,11)
( 4, 6, 8; 5, 7, 1)
( 4, 6, 8; 5, 7,11)
( 4, 6, 8; 5, 7,13)
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( 6, 6, 6; 1, 1, 9)
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( 6, 6, 6; 1, 3, 7)
( 6, 6, 6; 1, 5, 9)
( 6, 6, 6; 1, 7, 7)
( 6, 6, 6; 3, 3, 5)
( 6, 6, 6; 3, 3, 7)
( 6, 6, 6; 3, 5, 5)
( 6, 6, 6; 5, 5, 5)

+++ z = 19 +++

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( 1, 5,13; 2, 4, 0)
( 1, 5,13; 4, 2, 0)
( 1, 5,13; 4, 4, 0)
( 1, 5,13; 6, 2, 0)
( 1, 5,13; 6, 4, 0)
( 1, 7,11; 2, 2, 0)
( 1, 7,11; 2, 6, 0)
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( 1, 9, 9; 4, 6, 0)
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```

```
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( 3, 7, 9; 4, 8,14)
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( 5, 5, 9; 0, 4,12)
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