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ORBITAL STRUCTURE OF THE DERIVATION OPERATOR ON A CERTAIN SEMIRING OF NONSINGULAR COMPLEX MATRICES

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Abstract. We consider a special semiring of complex matrices with real and imaginary parts formed by positive nonsingular matrices taken from sets of pairwise commuting matrices. We describe an orbital structure of acertain transformation of the mentioned semiring satisfying rules of derivation for sum and product.

Preliminaries

Matrices with nonnegative or positive elements [2, 4, 10], the systematic study of which dates back to G. Frobenius (1912), belong to important mathematical tools of modelling especially in the theory of stochastic processes (modelling of Markoff processes), in mathematical economy and elsewhere. In the theory of ordinary differential equations the types of equations solutions of which and systems of differential equations are expressed by linear combinations of products of elementary functions and exponential functions or vector functions and exponential functions of functional matrices. Considering more details, let us remined the Floquet theorem [8] concerning systems of ordinary linear differential equations with periodical coefficients, which says that if U is the standard matrix of the Cauchy problem $x' = A(t)x, x'(0) = \xi$ then there exists a continuously differentiable nonsingular $n \times n$ matrix function P and an $n \times n$ constant matrix R, such that function $U(t)$ can be represented by the ordered pair of matrix functions $[P(t), R(t)]$. Calculations rules considered in this paper are motivated by calculations of these ordered pairs, e. g. $[P_1(t), R_1(t)], [P_2(t), R_2(t)]$ can be added whenever $R_1(t) = R_2(t)$; in the opposite case the sum of corresponding pair is substituted by an ideal element, whereas the usual product of the above pairs is the pair $[P_1(t)P_2(t), R_1(t) + R_2(t)]$. In connection with papers [5, 10] and monography [18, Chap. I.] we will analyze the orbital structure of a special differential operator defined on a certain semiring of complex matrices $M = A + iB$, where A, B are square nonsingular matrices of order $n \geqq 2$ with entirely positive elements (*i.e.* positive matrices). The considered

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semiring structure can be defined on more general carrier set formed by all complex matrices with square positive real and imaginary parts.

A mono-unary algebra (called also an unar) is a pair (L, f), where L is e set and $f : L \longrightarrow L$ is a mapping [6, 7, 11-14]. If $S \subseteq L$ such that $f(S) \subseteq S$ then $(S, f \mid S)$ is called a subalgebra of the algebra (L, f). An algebra (L, f) is said to be connected whenever for any pair $a, b \in L$ of its elements there exists a pair of non-negative integers $m, n \in \mathbb{N}_0$ such that $f^m(a) = f^n(b)$; where f^m is the m-th iteration of the map f. A maximal (with respect to the set inclusion) subalgebra (S, φ) of a mono-unary algebra (L, f) is termed as a component of (L, f). If $\{(S_\gamma, \varphi_\gamma); \gamma \in \Gamma)\}$ is e collection of all components of the unar (L, f), we write

$$(L, f) = \sum_{\gamma\in\Gamma} (S_\gamma, \varphi_\gamma).$$

In fact φ_γ is the restriction of $f : L \longrightarrow L$ onto the subset $S \subseteq L$.

By a semiring we mean an algebra $(S, +, \cdot)$, where $(S, +)$ is a commutative semigroup, $(S, \cdot)$ is a semigroup and $a \cdot (b + c) = a \cdot b + a \cdot c, (a + b) \cdot c = a \cdot c + b \cdot c$ for any triad $a, b, c \in S$, *i. e.* both distributive laws are satisfied [5].

Denote by $Matr_n(\mathbb{R}^+)$ the semiring (with respect to usual operations of addition and matrix multiplication) of all square matrices over $\mathbb{R}^+$ (the set of all positive real numbers) of order $n \geqq 2$

It is convenient to deal with complex matrices instead of pairs of real matrices. Algebraic differential structures form an important transfer between algebraic structures and structures of mathematical analysis.We denote by

$$\mathcal{M} = \{M; M = A + iB; A, B \in Matr_n(\mathbb{R}^+\} \cup \{O, iI\},$$

where O is the zero matrix and I the unit matrix, both of the order n.Let us denote $\mathcal{M}^+ = \mathcal{M} - \{O, iI\}$.

We define two binary operations $\oplus, \odot$ on the set $\mathcal{M}$, which are derived from the calculus of solution sets of linear ordinary homogeneous differential equations or from properties of exponential function of matrices. For any pair of matrices $M_1, M_2 \in \mathcal{M}^+$ where $M_k = A_k + iB_k;\ k = 1, 2, \dots$

$$M_1 \oplus M_2 = A_1 + A_2 + iB \quad if \;\; B_1 = B_2 = B, \tag{1}$$

$$M_1 \oplus M_2 = iI \quad if \;\; B_1 \neq B_2. \tag{2}$$

For $M \in \mathcal{M}$ arbitrary we set

$$M \oplus O = O \oplus M = M, \tag{3}$$

$$M \oplus iI = iI \oplus M = iI. \tag{4}$$

Further we define for any pair $M_1, M_2 \in \mathcal{M}^+$

$$M_1 \odot M_2 = A_1A_2 + i(B_1 + B_2), \tag{5}$$

$$M \odot O = O \odot M = O \tag{6}$$

for any $M \in \mathcal{M}$ and

$$M \odot iI = iI \odot M = iI \tag{7}$$

for any matrix $M \in \mathcal{M} - \{O\}$.

Lemma 1. The pair $(\mathcal{M}, \oplus)$ is a commutative monoid.

Proof. We prove first, that the operation $\oplus$ is commutative in the set $\mathcal{M}^+$ and also in the set $\mathcal{M}$. Let M_1, M_2 be matrices in M^+, such that

a) $B_1 = B_2 = B$,
then $M_1 \oplus M_2 = A_1 + A_2 + iB = A_2 + A_1 + iB = (A_2 + iB_2) + (A_1 + iB_2) = M_2 \oplus M_1$.

b) $B_1 \neq B_2$,
then $M_1 \oplus M_2 = (A_1 + iB_1) + (A_2 + iB_2) = iI = (A_2 + iB_2)(A_1 + iB_1) = M_2 \oplus M_1$.

c) If any of the matrices M_1, M_2 is equal to the zero matrix or to the matrix iI then commutativity follows from (3) a (4).

The operation $\oplus$ is thus commutative and we will use this property in the proof of associativity.

a) Let $M_k = A_k + iB_k$ for $k \in \{1, 2, 3\}$ are matrices from $\mathcal{M}^+$, *i. e.* :
  α) if $B_1 = B_2 = B_3 = B$, then $(M_1 \oplus M_2) \oplus M_3 = (A_1 + A_2 + iB) \oplus (A_3 + iB_3) = A_1 + A_2 + A_3 + iB = (A_1 + iB) \oplus (A_2 + A_3 + iB) = M_1 \oplus (M_2 \oplus M_3)$,
  β) if $B_1 = B_2 = B \neq B_3$, then $(M_1 \oplus M_2) \oplus M_3 = (A_1 + A_2 + iB) \oplus (A_3 + iB_3) = iI = (A_1 + iB) \oplus iI = M_1 \oplus (M_2 \oplus M_3)$,
  γ) if $B_1 \neq B_2 \neq B_3 \neq B_1$, then $(M_1 \oplus M_2) \oplus M_3 = ((A_1 + iB_1) \oplus (A_2 + iB_2)) \oplus (A_3 + iB_3) = iI \oplus (A_3 + iB_3) = iI = (A_1 + iB_1) \oplus iI = (A_1 + iB_1) \oplus ((A_2 + iB_2) \oplus (A_3 + iB_3)) = M_1 \oplus (M_2 \oplus M_3)$.

b) Let M_1, M_2 be any matrices from $\mathcal{M}$ and
  α) if matrix M_3 is zero matrix, then $(M_1 \oplus M_2) \oplus M_3 = (M_1 \oplus M_2) \oplus O = M_1 \oplus M_2 = M_1 \oplus (M_2 \oplus O) = M_1 \oplus (M_2 \oplus M_3)$,
  β) z if $M_3 = iI$, then $(M_1 \oplus M_2) \oplus M_3 = (M_1 \oplus M_2) \oplus iI = iI = M_2 \oplus iI = M_1 \oplus (M_2 \oplus iI) = M_1 \oplus (M_2 \oplus M_3)$.

The neutral element is matrix O by (3). □

Lemma 2. The pair $(\mathcal{M}, \odot)$ is a (noncommutative) semigroup with zero O.

Proof. The operation $\odot$ is noncommutative, because multiplication of square matrices $n \times n$ for $n \geq 2$ is noncommutative.

Let us prove that the operation $\odot$ is associative.

a) Let M_1, M_2, M_3 be square matrices from $\mathcal{M}^+$. Then $(M_1 \odot M_2) \odot M_3 = (A_1A_2 + i(B_1 + B_2)) \odot M_3 = (A_1A_2 + i(B_1 + B_2)) \odot (A_3 + iB_3) = (A_1A_2)A_3 + i((B_1 + B_2) + B_3) = A_1(A_2A_3) + i(B_1 + (B_2 + B_3)) = M_1 \odot ((A_2A_3) + i(B_2 + B_3)) = M_1 \odot (M_2 \odot M_3)$.

b) If at least one of matrices $M_1, M_2, M_3 \in \mathcal{M}$ is a zero matrix, then: $(M_1 \odot M_2) \odot M_3 = O = M_1 \odot (M_2 \odot M_3)$.

c) Let $M_1, M_2, M_3 \in (M)$, $M_1, M_2, M_3 \neq O$ and at least one of matrices M_1, M_2 equal to iI. Then $(M_1 \odot M_2) \odot M_3 = iI \odot M_3 = iI = M_1 \odot (M_2 \odot M_3)$.

d) Let $M_1, M_2 \neq O$, $M_1, M_2 \in \mathcal{M}$ and $M_3 = iI$. Then $(M_1 \odot M_2) \odot M_3 = (M_1 \odot M_2) \odot iI = iI = (M_1 \odot iI = M_1 \odot (M_2 \odot iI) = M_1 \odot (M_2 \odot M_3)$.

According to (6) is O zero. □

Theorem 3. The triad $(\mathcal{M}, \oplus, \odot)$ is a noncommutative semiring.

Proof. By Lemma 1 the structure $(\mathcal{M}, \oplus)$ is a commutative monoid and according to Lemma 2 $(\mathcal{M}, \odot)$ is a noncommutative semigroup, so we need to prove the validity of distributive laws:

$$M_1 \odot (M_2 \oplus M_3) = (M_1 \odot M_2) \oplus (M_2 \odot M_3),$$

$$(M_1 \oplus M_2) \odot M_3 = (M_1 \odot M_3) \oplus (M_2 \odot M_3).$$

a) Suppose $M_2, M_3 \in \mathcal{M}$, $B_2 = B_3$ and

α) M_1 is a matrix from the set $\mathcal{M}^+$. First we prove the left distributive law. $M_1 \odot (M_2 \oplus M_3) = (A_1 + iB_1) \odot ((A_2 + iB) + (A_3 + iB)) = (A_1 + iB_1) \odot (A_2 + A_3 + iB) = A_1(A_2 + A_3) + i(B_1 + B) = A_1A_2 + A_1A_3 + i(B_1 + B) = (A_1A_2 + i(B_1 + B_2)) \oplus ((A_1A_3 + i(B_1 + B)) = ((A_1 + iB_1) \odot (A_2 + iB)) \oplus ((A_1 + iB_1) \odot (A_3 + iB)) = (M_1 \odot M_2) \oplus (M_1 \odot M_3)$.

Then we prove the right distributive law: $(M_2 \oplus M_3) \odot M_1 = ((A_2 + iB) \oplus (A_3 + iB)) \odot (A_1 + iB_1) = (A_2 + A_3 + iB) \odot (A_1 + iB_1) = (A_2 + A_3)A_1 + i(B + B_1) = A_2A_1 + A_3A_1 + i(B + B_1) = (A_2A_1 + i(B + B_1)) \oplus (A_3A_1 + i(B + B_1)) = ((A_2 + iB) \odot (A_1 + iB)) \oplus ((A_3 + iB) \odot (A_1 + iB)) = (M_2 \odot M_1) \oplus (M_3 \odot M_1)$.

β) M_1 is a zero matrix.

Then: $M_1 \odot (M_2 \oplus M_3) = O \odot (M_2 \oplus M_3) = O \odot ((A_2 + iB) \oplus (A_3 + iB)) = O \odot ((A_2 + A_3) + iB) = O = O \oplus O = (O \odot M_2) \oplus (O \odot M_3) = (M_1 \odot M_2) \oplus (M_1 \odot M_3)$.

γ) If $M_1 = iI$.

Then: $M_1 \odot (M_2 \oplus M_3) = iI \odot (M_2 \oplus M_3) = iI \odot ((A_2 + iB) + (A_3 + iB)) = iI \odot ((A_2 + A_3) + iB) = iI = iI \oplus iI = (iI \odot M_2) \oplus (iI \odot M_3) = (M_1 \odot M_2) \oplus (M_1 \odot M_3)$.

Cases b) − f) can be proved similarly.

b) Consider matrices M_2, M_3, such that $B_2 = B_3$.

c) Exactly one of matrices M_2, M_3 is a zero matrix and the other is from $\mathcal{M}^+$.

d) Exactly one of matrices M_2, M_3 is a zero matrix and the other is iI.

e) Matrices M_2, M_3 are zero matrices and $M_1 \in \mathcal{M}^+$.

f) At least one from matrices M_2, M_3 is equal iI. □

Now, we define a mapping $d : \mathcal{M} \longrightarrow \mathcal{M}$ by

$$d(M) = AB + iB$$

for any $M \in \mathcal{M}^+$ where $M = A + iB$ and $d(O) = O, d(iI) = iI$.

It is easy to see (cf. paper [10]) that the mapping d is an endomorphism of the additive monoid $(\mathcal{M}, \oplus)$, *i. e.*

$$d(M_1 \oplus M_2) = d(M_1) \oplus d(M_2)$$

for any pair of matrices $M_1, M_2 \in \mathcal{M}$. Further, let $\mathcal{N} \subset (M)^+$ be a non-empty subset of complex matrices from set $\mathcal{M}^+$ with pairwise commuting real and imaginary parts of those, *i. e.* for every pair $M_1, M_2 \in \mathcal{N}, M_k = A_k + iB_k, k = 1, 2$ we have $XY = YX$ for any pair $X, Y \in \{A_1, B_1, A_2, B_2\}$. We will denote by $C(\mathcal{N})$ the subalgebra of $(\mathcal{M} - \{iI\}), \oplus, \odot)$ and $C_1(\mathcal{N}) = C(\mathcal{N}) \cup \{iI\}$ with binary operations $\oplus, \odot$ extended as above by $M \oplus iI = iI \oplus M = iI, M \odot iI = iI \odot M = iI$ for any $M \in C(\mathcal{N})$. Consequently $(C_1(\mathcal{N}), \oplus, \odot)$ is a semiring formed by pairwise commuting matrices. There is proven in [10] that the above defined operator d restricted on $C_1(\mathcal{N})$, also satisfies the rule for derivation of a product, *i. e.*

$$d(M_1 \odot M_2) = (d(M_1) \odot M_2) \oplus (M_1 \odot d(M_2))$$

for any pairs $M_1, M_2 \in C_1(\mathcal{N})$. Moreover, the Leibniz formula

$$d^n(M_1 \odot M_2) = \sum_{k=0}^{n}{}^{\oplus} \binom{n}{k} (d^{n-k}(M_1) \odot d^k(M_2)),$$

where the symbol $\sum\limits_{k=0}^{n}{}^{\oplus}$ is a natural extension of sum onto given finite system of matrices with arbitrary $n \in \mathbb{N}$ is valid.

Orbital structure of the derivation d on a certain subsemiring of $C(\mathcal{N})$

Now we describe the structure obtained by the above defined operator d which will be restricted onto a subsemiring of $C(\mathcal{N})$ formed by matrices with nonsingular real and imaginary parts (with the exception of the zero matrix). So, let us denote by $\mathcal{M}_{reg}$ the subset of all matrices $A + iB \in C(\mathcal{N})$ such that $detA \neq 0 \neq detB$, *i. e.* A, B are nonsingular and further let us denote $\mathcal{M}_{reg} = \mathcal{M} \cup O$. (Note that $iI \in \mathcal{M}_{reg}$ as well.) Evidently with respect to binary operations $\oplus, \odot$, the set $\mathcal{M}_{reg}$ is a semiring. Let us consider the usual quasi–ordering $\leqq$, *i. e.* a reflexive and transitive binary relation on the set $\mathcal{M}_{reg}$ defined by the rule:

For a pair $M_1, M_2 \in \mathcal{M}_{reg}$ we set

$$M_1 \leqq_d M_2$$

whenever $M_2 = d^n(M_1)$ for some $n \in \mathbb{N}_0$.

In our case the relation $\leqq_d$ is also antisymmetric. Indeed, let us suppose $M_1 \leqq_d M_2, M_2 \leqq_d M_1$ for a suitable pair of matrices $M_1 = A_1 + iB_1, M_2 = A_2 + iB_2$ from $\mathcal{M}^*_{reg}, O \neq M_k \neq iI$ for $k = 1, 2$. Then there exist integers $m, n \in \mathbb{N}_0$ such that $M_2 = d^n(M_1), M_1 = d^m(M_2)$, thus $M_1 = d^{m+n}(M_1)$, consequently

$$A_1 + iB_1 = A_1 B^{m+n} + iB_1,$$

which implies $B_1^{m+n} = I$. Since B_1 is a matrix with positive entires we have $B_1^{m+n} \neq I$, therefore $d^n(M) \neq M$ for each positive integer n and any matrix $M \in \mathcal{M}^*_{reg} - \{O, iI\}$. Thus $M_1 = M_2$ and we obtain the following assertion.

Lemma 4. The relation $\leqq_d$ is an ordering of the set $\mathcal{M}^*_{reg}$, *i. e.* $(\mathcal{M}^*_{reg}, \leqq_d)$ is an ordered set.

It is clear that the poset $(\mathcal{M}^*_{reg}, \leqq_d)$ has two isolated points (O, iI). We show that the poset satisfies the descending chain condition. Moreover, we describe its structure in the following theorem. Let $(\mathbb{N}, \leqq_d)$ be the chain of all positive integer with the usual natural ordering.

Theorem 5. Let Γ be an antichain of cardinality $2^{\aleph_0}$. The poset $(\mathcal{M}^*_{reg}, \leqq_d)$ is a cardinal sum of a two-element antichain and a set of cardinality $2^{\aleph_0}$ of countable chains isomorphic to $(\mathbb{N}, \leqq)$, more precisely

$$(\mathcal{M}^*_{reg}, \leqq_d) = G + \sum_{\gamma \in \Gamma} (K_\gamma, \leqq_\gamma),$$

where G is a two–elements antichain and $(K_\gamma, \leqq_\gamma) \cong (\mathbb{N}, \leqq)$ for any $\gamma \in \Gamma$.

Proof. Let us show first that the mapping $d : \mathcal{M}^*_{reg} \longrightarrow \mathcal{M}^*_{reg}$ is injective. Suppose on the contrary that $d(M_1) = d(M_2)$, where $M_k = A_k + iB_k \in \mathcal{M}^*_{reg}, k = 1, 2$. Then $A_1B_1 + iB_1 = A_2B_2 + iB_2$ which implies $B_1 = B_2$ and $A_1B_1 = A_2B_1$. Since B_1 is a nonsingular matrix, we have $A_1 = A_1B_1B_1^{-1} = A_2B_1B_1^{-1} = A_2$, consequently

$M_1 = M_2$. This fact implies that for any pair matrices $M_1, M_2 \in \mathcal{M}^*_{reg}$ exactly one of the following cases occurs:

a) $M_1 \leqq M_2$ or $M_2 < M_1$,

b) M_1, M_2 are incomparable and the two-element set $\{M_1, M_2\}$ possesses neither an upper bound nor a lower bound. Therefore any $M \in \mathcal{M}^*_{reg}$ belongs to some subchain of $(\mathcal{M}_{reg}, \leqq_d)$. Furthermore, any chain K_γ, $\gamma \in \Gamma$ has the least element. Indeed, let us suppose $M = A + iB \in \mathcal{M}_{reg}$ is an arbitrary matrix belonging to the chain K_γ. Since $d(X + iY) = XY + iY$, for any matrix $X + iY \in \mathcal{M}_{reg}$, we have

$$A(B^{-1})^n + iB <_d A(B^{-1})^{n-1} + iB <_d \cdots <_d AB^{-1} + iB,$$

thus there exists a positive integer $n \in \mathbb{N}$ such that

$$A(B^{-1})^n + iB \in \mathcal{M}_{reg}$$

and

$$A(B^{-1})^{n+1} + iB \notin \mathcal{M}_{reg},$$

thus the matrix $A(B^{-1})^n + iB$ is the least element of the chain K_γ. The proof is complete. $\square$

From the above proved theorem there follows immediately the following result describing the operator $d : \mathcal{M}^*_{reg} \longrightarrow \mathcal{M}^*_{reg}$ in terms of mono–unary algebras theory see [6, 11, 13]. We denote by $\nu : \mathbb{N} \longrightarrow \mathbb{N}$ the successor function $\nu(n) = n + 1$, *i. e.* $(\mathbb{N}, \nu)$ is the Peano–algebra of all positive integers. Then we have

Theorem 6. Denote by Γ_{min} the set of all minimal elements of the poset $(\mathcal{M}_{reg}, \leqq_d$). Then

$$(\mathcal{M}^*_{reg}, d) = (\{O, iI\}, id) + \sum_{\gamma \in \Gamma_{min}} K_\gamma, d_\gamma,$$

where d_γ is the restriction of d onto K_γ for any $\gamma \in \Gamma_{min}$ and every component $(K_\gamma, d_\gamma), (\gamma \in \Gamma)$ is isomorphic to the Peano–algebra $(\mathbb{N}, \nu)$.

Representations of the mono–unary algebra $(\mathcal{M}^*_{reg}, d)$

Using simple tools we construct in this paragraph mono-unary algebras isomorphic to the algebra $(\mathcal{M}^*_{reg}, d)$. Thus certain representation theorems concerning $(\mathcal{M}^*_{reg}, d)$ will be obtained. Let ∞ be a symbol which does not belong to the set $\mathbb{R}$ of all real numbers with the usual meaning, *i. e.* $r < \infty$ for all $r \in \mathbb{R}$. Let us denote $\bar{J} = \langle 2, \infty) \cup \{\infty\}$, where $\langle 2, \infty) = \{r \in \mathbb{R}; 2 \leqq r\}$. We define a function $\varphi : \bar{J} \longrightarrow \bar{J}$ by the rule $\varphi(x) = 2^x$ for any $x \in (2, \infty)$ and $\varphi(t) = t$ for $t \in \{2, \infty\}$. We are going to construct an isomorphism $F : (\bar{J}, \varphi) \longrightarrow (\mathcal{M}^*_{reg}, d)$ in this way: Let us put $F(2) = O, F(\infty) = iI$. Further, let $\xi : (2, 4) \longleftarrow \Gamma_{min}$ be an arbitrary bijection. For arbitrary $x \in \langle 4, \infty)$ let us denote $n(x) \in \mathbb{N}_0$ and $x_0 \in (2, 4\rangle$ numbers with property $\varphi^{n(x)}(x_0) = x$. Then we set

$$F(x) = d^{n(x)}(\xi(x_0)).$$

In particular $F(x) = \xi(x)$ for every $x \in (2, 4\rangle$.

Lemma 7. The mapping $F : \bar{J} \longrightarrow \mathcal{M}^*_{reg}$ is a bijection.

Proof. Let us denote $\Gamma = \mathcal{M}^*_{reg} - (\{O, iI\} \cup \Gamma_{min})$ and consider the partition of the set $\mathcal{M}^*_{reg}$ in the form

$$\mathcal{M}^*_{reg} = \{O, iI\} \cup \Gamma_{min} \cup \Gamma.$$

If $M = O$ or $M = iI$ then $F^{-1}(M) \in \{2, \infty\}$. If $M \in \Gamma_{min}$, then there exists a unique $x \in (2, 4\rangle$ such that $\xi(M) = x$ Suppose $M \in \Gamma \setminus \Gamma_{min}$. Then also for a suitable $x_0 \in (2, 4\rangle$ we have $M = d^{n(x)}(\xi(x_0))$. Then by definition of the mapping F we obtain

$$F(\varphi^{(n(x))}(x_0)) = M, \varphi^{(n(x))}(x_0) \in \langle 2, \infty),$$

consequently the mapping $F : \bar{J} \longrightarrow \mathcal{M}^*_{reg}$ is surjective. We show that F is also injective. Suppose $x, y \in \bar{J}$ are reals such that $F(x) = F(y)$. Then either
a) $F(x) = F(y) \in \{O, iI\}$ or
b) $F(x) = F(y) \in \mathcal{M}^*_{reg}$.

In case a) we have $x = 2 = y$ or $x = \infty = y$. Suppose the case b) occurs. Let $M_0 \in \Gamma_{min}$ and n be a nonnegative integer such that $F(x) = F(y) = d^n(M_0)$. Then by the definition of the mapping F we have $F^{-1}(M_0) \in (2, 4\rangle$ and $\varphi^n(\xi^{-1}(M_0)) = y$, hence $F : \bar{J} \longrightarrow \mathcal{M}^*_{reg}$ is injective. Therefore F is a bijection. □

Theorem 8. The mono-unary algebra $(\mathcal{M}^*_{reg}, d)$ is isomorphic to the mono-unary algebra $(\bar{J}, \varphi)$, where $\bar{J} = \langle 2, \infty) \cup \{\infty\}$.

Proof. We show that the above defined mapping $F : \bar{J} \longrightarrow \mathcal{M}^*_{reg}$ satisfies the equality

$$F \circ \varphi = d \circ F.$$

Indeed, let $x \in \bar{J}$ be an arbitrary element such that either $x \in \{2, \infty\}$ or $x \in (2, 4\rangle$. In the first case $x = 2$ or $x = \infty$ which implies $F(x) = O$ or $F(x) = iI$, respectively. Then

$$(F \circ \varphi)(x) = F(\varphi(x)) = F(x) = d(F(x)) = (d \circ F)(x).$$

In the second case $\xi(x) \in \Gamma$ and since $F(x) = \xi(x)$ we have

$$(F \circ \varphi)(x) = F(\varphi(x)) = F(2^x) = d(\xi(x)) = (d \circ F)(x).$$

Now let us suppose $x \in (4, \infty), x_0 \in (2, 4\rangle$ and $n(x) \in \mathbb{N}$ are members such that $x = \varphi^{n(x)}(x_0)$. Then

$$(F \circ \varphi)(x) = F(\varphi(x)) = F(\varphi^{n(x)+1}(xo)) == d^{n(x)+1}(\xi(x_0)) = d(F(x)) = (d \circ F)(x).$$

Since the mapping F is bijective, we get that $F : (\bar{J}, \varphi) \longrightarrow (\mathcal{M}^*_{reg}, d)$ is an isomorphism. □

Using the mono-unary algebra $(\bar{J}, \varphi)$ we obtain another representation theorem for the mono-unary algebra $(\mathcal{M}^*_{reg}, d)$. Put

$$S = ((0, 1\rangle \times \mathbb{N}) \cup \{[0, 0], [0, 1]\}$$

and define a function $\psi : S \longrightarrow S$ by

$$\psi([x,y]) = \begin{cases} [x,y+1] & \text{for any pair } [x,y] \in (0,1\rangle \times \mathbb{N} \\ [x,y] & \text{for any } [x,y] \in \{[0,0],[0,1]\} \end{cases}$$

Further we define a mapping $\Phi : S \longrightarrow \bar{J}$ in a similar way as above:
Consider an arbitrary bijection $\eta : (0,1\rangle \longrightarrow (0,4\rangle$ (e.g. $\eta = 2x+2, x \in (0,1\rangle$). For any pair $[x,y] \in (0,1\rangle \times \mathbb{N}$ we denote by $n(x) \in \mathbb{N}$ and $x_0 \in (0,1\rangle$ numbers such that $\eta^{n(x)}(x_0) = x$. Then we define

$$\Phi([x,y]) = \varphi^{n(x)}(\eta(x_0))$$

Moreover

$$\Phi([0,0]) = 2, \Phi([0,1]) = \infty$$

Then we obtain the following assertion, proof of which is rather technical and similar to the proof of Lemma 7. Thus it can be left to the reader.

Lemma 9. The mapping $\Phi : S \longrightarrow \bar{J}$ is bijection.

Now we can easily proved:

Proposition 10. The mapping $\Phi : S \longrightarrow \bar{J}$ is an isomorphism of the mono–unary algebra (S,ψ) onto the mono–unary algebra $(\bar{J},\phi)$.

Proof. We show that the mapping $\Phi : S \longrightarrow \bar{J}$ is a homomorphism.
Let $[x,n] \in S$ be an arbitrary element.If $[x,n] = [0,0]$ or $[x,n] = [0,1]$ then $\Phi([x,n]) = 2$ or $\Phi([x,n]) = \infty$, respectively, and we have $(\Phi \circ \psi)([x,n]) = (\varphi \circ \Phi)([x,n])$. Suppose $[x,n] \in (0,1\rangle \times \mathbb{N}$ is a pair and $0 < x_0 \leqq 1, n(x) \in \mathbb{N}$ are numbers with the property $[x,n] = \psi^{n(x)}(x_0)$ then

$$(\Phi\circ\psi)([x,n]) = \Phi(\psi([x,n])) = \Phi(\psi^{n(x)+1}(x_0)) == \varphi^{n(x)+1}(\eta(x_0)) = \varphi(\Phi([x,n])) = (\varphi\circ\Phi)([x,n]).$$

Consequently with respect to Lemma 9 we have that $F : (S,\psi) \longrightarrow (\bar{J},\varphi)$ is an isomorphism. $\square$

Summarizing the obtained facts (Theorem 8, Proposition 10) we can formulate the main result of this paragraph:

Theorem 11. Let $\bar{J} = \langle 2,\infty) \cup \{\infty\} \subset \tilde{\mathbb{R}}, S = ((0,1\rangle \times \mathbb{N} \cup \{[0,0],[0,1]\} \subset \mathbb{R} \times \mathbb{N}$ and $\varphi : \bar{J} \longrightarrow \bar(J), \psi : S \longrightarrow S$ be functions defined by:

$$\varphi(x) = \begin{cases} 2^x & \text{for any } x \in (2,\infty) \\ x & \text{for any } x \in \{2,\infty\} \end{cases}.$$

$$\psi([x,y]) = \begin{cases} [x,y+1] & \text{for any } [x,y] \in (0,1\rangle \times \mathbb{N} \\ [x,y] & \text{for any } [x,y] \in \{[0,0],[0,1]\} \end{cases}.$$

Then $(\bar{J},\varphi) \cong (\mathcal{M}^*_{reg},d) \cong (S,\psi)$, *i. e.* mono-unary algebras $(\bar{J},\varphi), (\mathcal{M}^*_{reg},d), (S,\psi)$ are mutually isomorphic.

At the end of this paragraph we present — without technical details — another simple representations of $(\mathcal{M}^*_{reg},d)$.

Let us describe some other possible simple representations of the mono-unary algebra $(\mathcal{M}^*_{reg},d)$. Let us denote $X_0 = \{0,1\} \subset \mathbb{C}$, (where $\mathbb{C}$ is Gaussian plane of all complex numbers), $X_1 = \{z = x + yi, x \in (0,1); y \in \mathbb{N}\} \subset \mathbb{C}$ and $X = X_0 \cup X_1$. Let us define the following complex function:

$$f(z) = \begin{cases} z & \text{for } z \in \{0,1\} = X_0 \\ z + i & \text{for } z \in X_1 \end{cases}.$$

Then $(\mathcal{M}^*_{reg}, d) \cong (X, f)$.

Furthermore, let $Y_0 = \{0, i\}, Y_1 = \{z = x + yi, x \in \mathbb{N}, y \in (0,1)\}; Y_1 \subset C$ and $Y = Y_0 \cup Y_1$. We define the function $g(z) : Y \longrightarrow Y$ by the rule:

$$g(z) = \begin{cases} z & \text{for } z \in Y_0 \\ z + i & \text{for } z \in Y_1 \end{cases}.$$

Again $(\mathcal{M}^*_{reg}, d) \cong (Y, g)$.

Let us denote $K = \{1\} \cup (2, \infty)$ and define a real function $u : K \longrightarrow K$ in the following way:

$$u(x) = \begin{cases} x & \text{for } x \in \{1, 2\} \\ 2x & \text{for } x \in (2, \infty) \end{cases}.$$

In this case $(\mathcal{M}^*_{reg}, d) \cong (K, u)$.

The main contribution of the presented paper is divided into two directions: Firstly, a differential semiring of complex matrices with positive real and imaginary parts motivated by the above remark is constructed and secondly: the structure of a mono-unary algebra determined by the constructed differential operator including its representations is clarified.

In matrix theory including its various applications the concept of a pseudo-inverse matrix plays an important role [8]. One of so called More-Penrose conditions can be expressed also in terms of the semigroup theory. More precisely, a semigroup $(S, \cdot)$ is said to be regular, if for any $a \in S$ there exists $b \in S$ such that $aba = a$ [6, 7, 15]. It is easy to see, that the endomorphism monoid $\mathrm{End}(\mathcal{M}_{reg^*}, d)$ is not regular. However, if we restrict our solves onto $(\mathcal{M}_{reg}, d)$, then it is possible to construct a certain unique extension $(\bar{\mathcal{M}}_{reg}, \bar{d})$ of this unar such that the monoid $\mathrm{End}(\bar{\mathcal{M}}_{reg}, \bar{d})$ is regular. But this seems to be a topic for an additional paper.

References

[1] Bapat, R. B., *Linear Algebra and Linear Models*, 2nd ed., Springer Verlag, New York, etc., 2000.

[2] Bellman, R., *Introduction to Matrix Analysis*, McGraw-Hill Book Company, Inc., New York–Toronto–London, 1960.

[3] Dupač V. and Dupačová, J., *Markoff Processes*, SPN, Praha, 1980. (Czech)

[4] Gantmacher, F. R., *Matrizentheorie*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1986.

[5] Chvalina, J., *Fractional iterations of derivatives on a certain semiring of complex numbers*, Department Mathematics Report Series **8** (2000), University of South Bohemia, České Budějovice, 77–86..

[6] ———, *Functional Graphs, Quasi-ordered Sets and Commutative Hypergroups*, Masaryk University, Brno, 1995.

[7] Moučka, J., *Connected functional graphs with regular endomorpihsm monoids and their hypergroups*, Matematika a didaktika matematiky, Sborník prací PdF MU Brno 152, Masaryk University, Brno, 2000, pp. 53–59.

[8] Nagy, J., *Systems of Ordinary Differintial Equations*, SNTL, Prague, 1980. (Czech)

[9] Nashed, Z., *Generalized Inverses and Applications*, Academic Press, 1976.

[10] Novotná, J., *Applications of Matrices in Economy*, II. konf. o matematice a fyzice na vysokých školách technických, Vojenská akademie v Brně, 2002, pp. 69–74.
[11] Novotný, M., *Über Abbildungen von Mengen*, Pacific J. Math. **13** (1963), no. 4, 1359–1369.
[12] ———, *Commutativity of endomorphismus of linear spaces*, Čas. pěst. mat. **107** (1982), 24 – 138.
[13] ———, *Mono-unary algebras in the work of Czechoslovak mathematicians*, Arch. Math **26** (1990), Brno, 155–164.
[14] ———, *Construction of all homomorphisms of mono-unary algebras*, Seminář OAS PřF MU, Masarykova Universita, Brno, 1995.
[15] Skornjakov, L. A., *Unary algebras with regular endomorphism monoids*, Acta Sci. Math. **40** (1978), 375–381.
[16] Šik, F., *Linear Algebra directed to Numerical Analysis*, Masaryk University, Brno, 1998.
[17] Šmarda, B., *Linear Programming in Exercises*, SPN, Praha, 1983.
[18] Targonski, G., *Topics in Iteration Theory*, Vandenhoeck et Ruprecht, Gottingen - Zurich, 1981.

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