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UPPER BOUND FOR QUEUE NUMBER OF SHUFFLE-EXCHANGE GRAPH

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Abstract. Shuffle-Exchange network has some good properties in parallel data processing. Its graph abstraction called Shuffle-Exchange graph is well known in the area of VLSI design. The concept of queue number is the abstraction of some problems from computer science, such as the design of fault-tolerant processor arrays, a problem of sorting with parallel queues, and a problem of scheduling parallel processors. In this paper we prove that Shuffle-Exchange graph has a 3-queue layout, while it is known that at least 2 queues are necessary. This value provides upper bound for queue number of Shuffle-Exchange graph.

Introduction

Shuffle-Exchange Graph

Definition 1. The *d-dimensional shuffle-exchange graph* (SE_d) has 2^d nodes. Each vertex is numbered by unique binary string of length d. The edges are defined as follows. Vertex represented by binary string αa, where $\alpha \in \{0,1\}^{(d-1)}$ and $a \in \{0,1\}$, is connected with vertex $\alpha\neg a$ and $a\alpha$ (where $\neg a$ is negation of a). Edges directions, multiple edges and loops are ignored.

The edges between vertices αa and $\alpha\neg a$ are called *exchange* edges and the edges between vertices αa and $a\alpha$ are called *shuffle* edges.

The Shuffle-Exchange network provides suitable interconnection patterns for implementation of parallel algorithms like : *polynomial evaluation, fast Fourier transform, sorting* and *matrix transposition.*

Linear Layout of a Graph

The linear layout of a graph is such a layout in which the vertices are drawn on a horizontal line in some order (desginated by σ in this paper). Although the graph is undirected, we consider the edges orientation given by the ordering of vertex set.

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K-Queue Layout and Queue Number

A *k-queue layout* of an undirected graph $G = (V, E)$ has two aspects. The first aspect is linear order of V(which we think of as being on a horizontal line). The second aspect is an assignment of each edge in E into one of k queues in such a way that the set of edges assigned to each queue obeys a first-in/first-out discipline. Each queue q_j operates as follows. The vertices of V are scanned in left-to-right ascending order. When vertex i is encountered, any edges assigned to q_j that have vertex i as their right endpoint must be at the front of that queue; they are removed(dequeued). Any edges assigned to q_j that have vertex i as their left endpoint are placed on the back of that queue (enqueued). Queue number (qn) is smallest k such that G has *k-queue layout.* [heath]

This layout problem abstracts design problem of fault-tolerant processor arrays, a problem of sorting with parallel queues, and a problem of scheduling parallel processors.

The question of queue number of Shuffle-Exchange graph is still open, although it is known for deBruijn graph (close relative of Shuffle-Exchange). Heath and Rosenberg made the characterization of 1-queue graphs as arched leveled-planar graphs[heath]. Queue numbers of some typical graphs are also in [heath].

Leighton showed that crossing number of Shuffle-Exchange graph is $\Theta(N^2/log^2 N)$ [leighton]. SE_d graph is therefore not planar in general (with exception for $d \leq 3$) and for its quenumber holds $qn(SE_d) \geq 2$.

K-Rainbow set of edges

Definition 2. Suppose we have a linear graph layout (all vertices are on the horizontal line) with some vertices ordering σ. Then a *k-rainbow* is a set of k edges $e_i = (a_i, b_i), 1 \leq i \leq k$ such that

$$a_1 <_\sigma a_2 <_\sigma \cdots <_\sigma a_k <_\sigma b_k <_\sigma b_{k-1} <_\sigma \cdots <_\sigma b_2 <_\sigma b_1.$$

In other words, a rainbow is a *nested* matching. A rainbow is an obstacle for a queue layout because no two nested edges can be assigned to the same queue.[heath]

Alternative definition of SE_d

Definition 3. Let $G(V, E)$ be the graph with 2^d vertices. Label the vertices with numbers 0,1,2,...,$2^d - 1$. The edges are defined as follows. Vertex with number n will be connected

- with vertex $n + 1$ in case of even n or $n = 0$,
- with vertex $\frac{n}{2}$ in case of even n,
- with vertex $\frac{n-1}{2} + 2^{d-1}$ in case of odd n.

This definition is a modification of SE_d definition from [akl].

Theorem 1. Definitions 1 and 3 are equivalent.

Proof. From both definitions the vertex sets contain the same members only with different labelling. It is sufficient to prove that edge generating formulas will generate the edges of Shuffle-Exchange graph. In following two points we simply make the conversion from binary defintion of edges to decadic one.

(1) Exchange edges $\alpha a \rightarrow \alpha \neg a$
  (1a) $\alpha 0 \rightarrow \alpha 1 \iff n \rightarrow (n+1)$
  Note that number $\alpha 0$ is even. So this will work for even n or $n = 0$.
  (1b) $\alpha 1 \rightarrow \alpha 0 \iff n \rightarrow (n-1)$
  $\alpha 1$ is odd number. n is also odd. The edges of the form (a) and (b) are identical with different directions. Since we ignore edge directions in SE_d we use only first form (a) of edges in our definition.
(2) Shuffle edges $\alpha a \rightarrow a\alpha$
  (2a) $\alpha 0 \rightarrow 0\alpha \iff n \rightarrow \frac{n}{2}$ $\alpha 0$ is even.
  Result of shuffle operation in this case is α. This operation is division by the base of binary system. Equivalent operation in decadic system is $\frac{n}{2}$.
  (2b) $\alpha 1 \rightarrow 1\alpha \iff \frac{n-1}{2} + 2^{d-1}$.
  The operation $\alpha 1 \rightarrow 1\alpha$ is equivalent to $(\alpha 1 \rightarrow \alpha 0 \rightarrow 0\alpha \rightarrow 1\alpha)$. According to this operation equivalent formula in decadic system is $\frac{n-1}{2} + 2^{d-1}$. □

Upper bound

Lemma 2. The queue number $qn(G)$ of a graph G is a minimum, taken over all vertex orderings σ of G, of a maximum size of a rainbow in σ.[heath]

Lemma 3. Let $p(n), n \in N$ be the non-descending sequence and let G be the graph $G = (V, E)$, where $V = 0, 1, 2, ..., n$ and all edges from E are of type $(v, p(v)), v \in V$. Then G has one-queue layout with natural $(0, 1, ..., n)$ ordering of vertices.

Proof. Let $m_1, m_2, ..., m_n; m_i \in N$ be the vertex indexes. ¿From non-descending sequence $p(m_i)$ we have

$$m_1 < m_2 < ... < m_n \Rightarrow p(m_1) \leq p(m_2) \leq ... \leq p(m_n).$$

Comparing this property with definition of *k-rainbow set* we see that in this type of graph the *k-rainbow set* can not exists with $k > 1$. According to *Lemma 2* we have ordering with maximum rainbow size of 1, and therefore with queue number 1. □

Theorem 4. The SE_d graph has 3-queue layout with natural vertices ordering $0, 1, ..., 2^d - 1$. The edges will be assigned to queues as follows.

(1) queue : edges of type $(m, m+1)$ for even m or $m = 0$.
(2) queue : edges of type $(m, \frac{m}{2})$ for even m.
(3) queue : edges of type $(m, \frac{m-1}{2} + 2^{d-1})$ for odd m.

Proof. By assigning the edges into three queues we have three subgraphs of SE_d. It is sufficent to prove that these subgraphs have 1-queue layout with vertices ordering $0, 1, 2, ..., 2^d - 1$.

(1) queue : edges of type $(m, m+1)$ for even m or $m = 0$.
Edges of this subgraph are generated by the formula $p(m) = m + 1$. It is ascending sequence and according to *Lemma 3* graph with such edges has 1-queue layout.

(2) queue : edges of type $(m, \frac{m}{2})$ for even m.
Edges generating formula is $p(m) = \frac{m}{2}$. Again, it is ascending sequence and according to *Lemma 3* graph with such edges has 1-queue layout.

(3) queue : edges of type $(m, \frac{m-1}{2} + 2^{d-1})$ for odd m.
Edges genarating formula is $p(m) = \frac{m-1}{2} + 2^{d-1}$. Analogue to previous points graph with such edges has 1-queue layout. □

Corollary 1. The SE_d graph layed out on horizontal line with natural vertices ordering has k-rainbow set of edges with $k = 3$.

Proof. From our alternative definition of SE_d we have three types of edges. From proof of *Theorem 4* only edges of different types can nest. It means that maximal nested matching can have degree 3 (with respect to natural ordering of vertices). It is trivial to find such rainbow. For example one exists in SE_4 and consists of edges $\{(8, 9), (6, 12), (7, 11)\}$. In SE_d where $d > 4$ can be found for example this rainbow: $\{(8, 9), (6, 12), (1, 2^{d-1})\}$. □

Theorem 4 gives upper bound for queue number of Shuffle-Exchange graph. The final value can be 3 (from *Theorem 4*) or 2 as a lower bound, since Shuffle Exchange can not have queue number 1 due to its non-planarity.

3-queue layout from *Theorem 4* is according to *Corollary 1* minimal. In other words, there exists no 2-queue layout of Shuffle-Exchange graph with natural order of vertices because of existence of 3-rainbow set of edges.

Open problem. To prove that $qn(SE_d) = 2$ or $qn(SE_d) > 2$.

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