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ON THE STRUCTURE OF POLYHEDRAL GRAPHS WITH PRESCRIBED EDGE AND DUAL EDGE WEIGHT

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Abstract. We consider families of polyhedral graphs with prescribed minimum vertex degree δ, minimum face degree ρ, minimum edge weight w and dual edge weight w^*. We determine all quadruples (δ, ρ, w, w^*) for which the asociated family is nonempty.

Introduction

Throughout this paper we consider connected plane graphs without loops or multiple edges. For a plane graph G, $V = V(G), E = E(G)$ and $F = F(G)$ denotes the set of its vertices, edges and faces, respectively. A *k-vertex* (*k-face*) will stand for a vertex (a face) of degree k, a $\geq$ *k-vertex*/$\leq$ *k-vertex* ($\geq$ *k-face*/$\leq$ *k-face*) for those of degree at least k/at most k. For an edge e being incident with an a-vertex and a b-vertex, and with a c-face and a d-face, the type of e is (a, b, c, d) where $a \leq b, c \leq d$. The weight $w(e)$ of an edge $e = uv$ is the sum $\deg_G(u) + \deg_G(v)$. The edge weight $w(G)$ of a plane graph G is equal to $\min_{uv \in E(G)} \{\deg_G(u) + \deg_G(v)\}$; the dual edge weight $w^*(G)$ of G is the edge weight of the dual of the graph G. Let $\mathcal{G}_c(\delta, \rho, w, w^*)$ be the family of all c-connected plane graphs with minimum vertex degree at least δ, minimum face degree at least ρ, edge weight at least w and dual edge weight at least w^*; for $c = 3$, we will use the notation $\mathcal{G}(\delta, \rho, w, w^*)$.

It is well known that every plane graph contains a vertex of degree at most 5. Among numerous generalizations of this result (see [6]), the fundamental role plays the Kotzig's theorem [7] stating that for each polyhedral graph G, $w(G) \leq 13$, and if G is of minimum degree at least 4, then $w(G) \leq 11$; both these bounds are sharp. Thus, the family of all polyhedral graphs with minimum edge weight at least 14 (and of the ones with minimum degree at least 4 and minimum edge weight at least 12) is empty. Considering the dual graphs, we obtain the analogical results for dual weight constraints. Note that the Kotzig's theorem provides no information about degrees of two faces incident with an edge that attains the minimum weight in a graph, but there are generalisations of theorem taking this aspect into account. For example, in [1] Borodin extended the Kotzig's result showing that each normal plane map (that is, a plane pseudograph having no vertices or faces of degree less than

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three) contains either a 3-face incident with an edge of weight at most 13, or a 4-face incident with an edge of weight at most 8, or else a 5-face incident with an edge of weight 6; all these bounds being sharp. Another partial results are contained in the classical paper [8]. It is also possible to consider some combinations of constraints based on minimum vertex degree, face degree, edge weight and dual edge weight; to our knowledges, the simultaneous combinations of all four mentioned constraints were not studied in deeper details.

The aim of this paper is give the complete characterization of quadruples (δ, ρ, w, w^*) for which the family $\mathcal{G}(\delta, \rho, w, w^*)$ is empty. The Euler theorem implies that $(4, 4, 8, 8)$ yields the empty family. From the Kotzig theorem, it follows that for quadruples $(3, 3, 14, 6), (4, 3, 12, 6)$, the corresponding families are empty, and from [L] follows the emptiness of families determined by quadruples $(4, 3, 8, 9)$ and $(3, 5, 7, 10)$. Using the duality, we get that the corresponding families are empty also for $(3, 3, 6, 14), (3, 4, 9, 8), (3, 4, 6, 12), (5, 3, 10, 7)$.

We prove

Theorem 1. The families $\mathcal{G}(3, 3, 7, 10), \mathcal{G}(3, 3, 8, 9), \mathcal{G}(3, 4, 7, 9)$ are empty.

In each of three cases of theorem, we proceed by contradiction, thus, assuming the non-emptiness of specified family, we consider its representant G with specified minimum vertex degree, face size, edge and dual edge weight. At this graph G, the discharging method is used. We define the charge $c : V \cup E \cup F \to \mathbb{Z}$ by the following assignments:

$$(\forall v \in V)\ c(v) = \deg_G(v) - 6$$
$$(\forall \alpha \in F)\ c(\alpha) = 2 \cdot \deg_G(\alpha) - 6$$
$$(\forall e \in E)\ c(e) = 0.$$

From the Euler Theorem, it follows that $\sum\limits_{x \in V \cup E \cup F} c(x) = -12.$

Next, we define the local redistribution of charges between the elements of G such that the total sum of charges remains the same. This is performed by certain rules which specify the charge transfers from elements to another elements in specific situations. After such redistribution, we obtain a new charge $\widetilde{c} : V \cup E \cup F \to \mathbb{Q}$. Then, we prove that for any element $x \in V \cup E \cup F$, $\widetilde{c}(x) \geq 0$ (hence, $\sum\limits_{x \in V \cup E \cup F} \widetilde{c}(x) \geq 0$). This contradiction shows that G cannot exist.

The family $\mathcal{G}(3, 3, 7, 10)$

The discharging rules are the following:

Rule 1: Each k-vertex x sends $\frac{c(x)}{k}$ to each incident edge.

Rule 2: Each k-face α, $k \notin \{7, 8, 9, 10, 11\}$ sends $\frac{c(\alpha)}{k}$ to each incident edge.

Rule 3: Each l-face β, $l \in \{7, 8, 9, 10, 11\}$ sends to each incident edge e of type (a, b, c, d) the following charge:

(a) $\frac{3}{2}$ if $a = 3, 3 \leq c \leq 4$,
(b) $\frac{4}{5}$ if $a = 3, c = 5$,
(c) $\frac{3}{4}$ if $a = 3, c \geq 6$,
(d) 1 otherwise.

For proving the nonnegativity of final charges, firstly observe that all vertices and all k-faces, $k \notin \{7,8,9,10,11\}$ are discharged to zero. Now we analyze the final charge of the remaining faces and edges.

1. Let β be an l-face, $7 \le l \le 11$. Then β is incident with at most five 3-vertices (since $w(G) \ge 7$).
   (a) If β is not incident with a 3-vertex then $\widetilde{c}(\beta) \ge 2l - 6 - l \cdot 1 = l - 6 > 0$ by Rule 3(d).
   (b) Let β be incident with exactly t 3-vertices, $1 \le t \le 5$. Then $2t$ transfers from β are by Rule 3(a), (b), or (c), and the remaining ones are by Rule 3(d). Moreover, for transfers through a pair of edges of β with common 3-vertex, Rule 3(a) may be used only with Rule 3(c) (since $w^*(G) \ge 10$). From this fact we have that the maximum charge transferred from β is in the case when each of Rules 3(a) and 3(c) is used t times; then $\widetilde{c}(\beta) \ge 2l - 6 - t\frac{3}{2} - t\frac{3}{4} - (l-2t) \cdot 1 = l - 6 - \frac{t}{4}$. Hence, $\widetilde{c}(\beta) \ge 0$ for $t \ge 4$; for $t = 5$, we have $l \in \{10, 11\}$ and so $\widetilde{c}(\beta) > 0$.

2. Let e be an edge of G of the type (a,b,c,d); note that $a + b \ge 7$ since $w(G) \ge 7$.
   (a) If $a = 3, 3 \le c \le 4, 7 \le d \le 11$, then $\widetilde{c}(e) \ge -1 - \frac{1}{2} + \frac{3}{2} = 0$ by Rules 1 and 3(a).
   (b) If $a = 3, 3 \le c \le 4, d \ge 12$, then $\widetilde{c}(e) \ge -1 - \frac{1}{2} + \frac{2 \cdot d - 6}{d} \ge 0$ by Rules 1 and 2.
   (c) If $a = 3, c = 4, d = 6$, then $\widetilde{c}(e) \ge -1 - \frac{1}{2} + \frac{1}{2} + \frac{2 \cdot 6 - 6}{6} = 0$ by Rules 1 and 2.
   (d) If $a = 3, c = 5, 5 \le d \le 6$, then $\widetilde{c}(e) \ge -1 - \frac{1}{2} + \frac{4}{5} + \frac{2 \cdot d - 6}{d} > 0$ by Rules 1 and 2.
   (e) If $a = 3, c = 5, 7 \le d \le 11$, then $\widetilde{c}(e) \ge -1 - \frac{1}{2} + \frac{4}{5} + \frac{4}{5} > 0$ by Rules 1, 2 and 3(b).
   (f) If $a = 3, c = 5, d \ge 12$, then $\widetilde{c}(e) \ge -1 - \frac{1}{2} + \frac{4}{5} + \frac{2d - 6}{d} > 0$ by Rules 1 and 2.
   (g) If $a \ge 3, c \ge 6, d \ge 6$, then $\widetilde{c}(e) \ge -1 - \frac{1}{2} + 1 + \frac{3}{4} > 0$ by Rules 1 and 2 (or 3(c) or 3(d)).

The family $\mathcal{G}(3,3,8,9)$

The discharging rules are the following:

Rule 1: Each k-vertex x sends $\frac{c(x)}{k}$ to each incident edge.

Rule 2: Each k-face α, $k \notin \{6,7\}$ sends $\frac{c(\alpha)}{k}$ to each incident edge.

Rule 3: Each k-face β, $k \in \{6,7\}$ sends to each incident edge e of type (a,b,c,d) the following charge:
   (a) $\frac{6}{5}$ if $a = 3, b \ge 5, 3 \le c \le 4$,
   (b) 1 if $a \ge 4, b \ge 4, c \ge 3$,
   (c) $\frac{3}{5}$ if $a = 3, b \ge 5, c \ge 5$.

Like in previous proof, all vertices and all k-faces, $k \notin \{6,7\}$ are discharged to zero. Now we analyze the final charge of the remaining faces and edges.]

1. Let β be an l-face, $6 \le l \le 7$. Then β is incident with at most three 3-vertices (since $w(G) \ge 8$).
   (a) If β is not incident with a 3-vertex then $\widetilde{c}(\beta) \ge 2l - 6 - l \cdot 1 = l - 6 \ge 0$ by Rule 3(b).

(b) Let β be incident with exactly t 3-vertices, $1 \le t \le 3$. Like in previous proof, $2t$ transfers from β are by Rule 3(a) or (c), and the remaining ones are by Rule 3(b). Again, for transfers through a pair of edges of β with common 3-vertex, Rule 3(a) may be used only with Rule 3(c) (since $w^*(G) \ge 9$); this yields that the maximum charge transferred from β is in the case when each of Rules 3(a) and 3(c) is used t times. Hence, $\widetilde{c}(\beta) \ge 2l - 6 - \frac{6}{5}t - \frac{3}{5}t - (l - 2t) \cdot 1 = l - 6 + \frac{t}{5} > 0$.

2. Let e be an edge of G of the type (a, b, c, d); as $w(G) \ge 8$, we have $a + b \ge 8$.
   (a) If $a = 3, b \ge 5, 4 \le c \le 5, d = 5$ then $\widetilde{c}(e) \ge -1 - \frac{1}{5} + \frac{4}{5} + \frac{1}{2} = \frac{1}{10} > 0$ by Rules 1 and 2.
   (b) If $a \ge 4, b \ge 4, c \ge 4, d \ge 5$ then $\widetilde{c}(e) \ge 2 \cdot \left(-\frac{1}{2}\right) + \frac{4}{5} + \frac{1}{2} = \frac{3}{10} > 0$ by Rules 1 and 2.
   (c) If $a = 3, b \ge 5, 3 \le c \le 4, 6 \le d \le 7$ then $\widetilde{c}(e) \ge -1 - \frac{1}{5} + \frac{6}{5} = 0$ by Rules 1 and 3(a).
   (d) If $a \ge 4, b \ge 4, c \ge 3, 6 \le d \le 7$ then $\widetilde{c}(e) \ge 2 \cdot \left(-\frac{1}{2}\right) + 1 = 0$ by Rules 1 and 3(b).
   (e) If $a = 3, b \ge 5, c = 5, 6 \le d \le 7$ then $\widetilde{c}(e) \ge -1 - \frac{1}{5} + \frac{4}{5} + \frac{3}{5} = \frac{1}{5} > 0$ by Rules 1, 2 and 3(c).
   (f) If $a = 3, b \ge 5, 6 \le c \le 7, 6 \le d \le 7$ then $\widetilde{c}(e) \ge -1 - \frac{1}{5} + 2 \cdot \frac{3}{5} = 0$ by Rules 1 and 3(c).
   (g) If $a = 3, b \ge 5, c \ge 3, d \ge 8$ then $\widetilde{c}(e) \ge -1 - \frac{1}{5} + \frac{2 \cdot 8 - 6}{8} = \frac{1}{20} > 0$ by Rules 1 and 2.
   (h) If $a \ge 4, b \ge 4, c \ge 3, d \ge 8$ then $\widetilde{c}(e) \ge 2 \cdot \frac{-1}{2} + \frac{2 \cdot 8 - 6}{8} = \frac{1}{4} > 0$ by Rules 1 and 2.

The family $\mathcal{G}(3, 4, 7, 9)$

The discharging rules are the following:

Rule 1: Each k-vertex x sends $\frac{c(x)}{k}$ to each incident edge.

Rule 2: Each k-face α, $k \neq 5$ sends $\frac{c(\alpha)}{k}$ to each incident edge.

Rule 3: Each 5-face β sends to each incident edge e of type (a, b, c, d) the following charge:
   (a) 1 if $a = 3, b \ge 4, c = 4$,
   (b) $\frac{3}{4}$ if $a = 3, b \ge 4, c \ge 5$,
   (c) $\frac{1}{2}$ if $a \ge 4, b \ge 4, c \ge 4$.

All vertices and all faces except a 5-face are discharged to zero. Consider the final charge of 5-faces and edges:

1. Let β be a 5-face. If β is not incident with a 3-vertex then $\widetilde{c}(\beta) \ge 2 \cdot 5 - 6 - 5 \cdot \frac{1}{2} = \frac{3}{2} > 0$. Otherwise, β is incident with t 3-vertices, $1 \le t \le 2$. Again, due to the fact that $w^*(G) \ge 9$, for transfers through a pair of edges of β sharing common 3-vertex, Rule 3(a) may be used only in the combination with Rule 3(c). Thus, $\widetilde{c}(\beta) \ge 2 \cdot 5 - 6 - 1 \cdot t - \frac{3}{4} \cdot t - (5 - 2t)\frac{1}{2} = \frac{3}{2} - \frac{3t}{4} \ge 0$.
2. Let e be an edge of G of the type (a, b, c, d); as $w(G) \ge 7$, $a + b \ge 7$.
   (a) If $a = 3, b \ge 4, c = 4, d \ge 5$ then $\widetilde{c}(e) \ge -1 - \frac{1}{2} + 1 + \frac{1}{2} = 0$ by Rules 1, 2 and 3(a).
   (b) If $a = 3, b \ge 4, c = 5, d \ge 5$ then $\widetilde{c}(e) \ge -1 - \frac{1}{2} + 2 \cdot \frac{3}{4} = 0$ by Rules 1 and 3(b).

(c) If $a \geq 4, b \geq 4, c \geq 4, d \geq 5$ then $\widetilde{c}(e) \geq 2 \cdot \left(-\frac{1}{2}\right) + 2 \cdot \frac{1}{2} = 0$ by Rules 1 and 3(c).

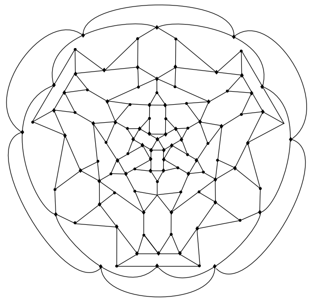

Fig. 1

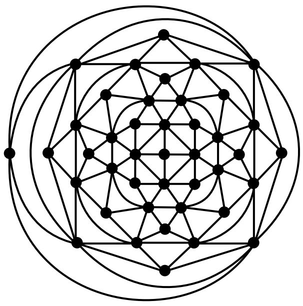

Fig. 2

Concluding remarks

1. Concerning the quadruples $(3,3,6,13)$, $(3,3,7,9)$, $(3,3,8,8)$, $(3,4,6,11)$, $(3,4,7,8)$ and $(3,5,6,11)$ (and the quadruples derived from them by swapping the first entry with the second one, and the third with the fourth one), it is easy to show that the corresponding families are nonempty (and, in fact, infinite); the examples are: the truncated dodecahedron, the graph of Fig. 1, the icosidodecahedron, the dual of the graph of Fig. 2, the rhombic dodecahedron and the truncated icosahedron (for the names of these polyhedra, see [9]). In this sense, our results are best possible. The diagram on Fig. 3 presents the hierarchy of all nonempty families generated by quadruples (δ, ρ, w, w^*) under the set inclusion partial ordering.

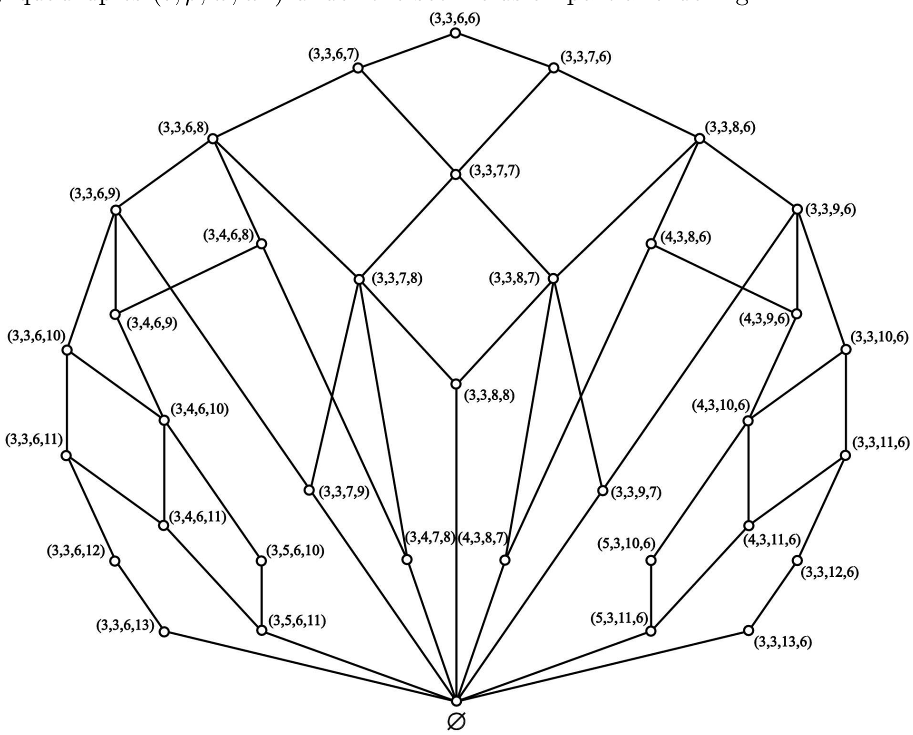

Fig. 3

2. The Kotzig theorem was further generalized in many ways, one of which considered, for the specified family of polyhedral graphs, the existence of longer paths with the degrees of their vertices bounded above by a finite constant that depends only of the specified family and of path length; such paths are called light. For the related results regarding various families, see [2], [3], [4] or [5]. In the connection with the results presented in this paper, one may consider, for given integer $k \geq 1$, the families of polyhedral graphs with prescribed weight of i-paths and dual i-paths for all $i \in \{1, \ldots, k\}$. To our knowledge, currently there are no results involving both normal and dual weights for $k \geq 3$.

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