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SCALAR CARDINALITIES OF IF SETS

PAVOL KRÁL'

Abstract. There are several approaches to the cardinality of fuzzy sets. One group of them are constructive approaches. Following these approaches we get single numbers (scalar cardinalities) or convex fuzzy sets (fuzzy cardinalities) as cardinalities of the fuzzy sets. In our contribution we will present possible extensions of the scalar cardinality to the case of IF sets.

Introduction

Cardinality is a very important characteristic of a crisp set. We would like to get a similar characteristic also for fuzzy sets. There is rather theoretical motivation for dealing with the notion of the cardinality of fuzzy sets. Nevertheless, measuring the cardinality of fuzzy sets has also many applications, especially in the case of finite fuzzy sets. For instance in communication with databases, we mean the problem of a satisfactory and adequate answer to queries of the form: "How many elements are p? " or "Are there more elements which are p than elements which are q?", where p, q are arbitrary properties. Those queries are about cardinalities or comparisons of cardinalities of fuzzy sets. An example of a query in this form is:

"Are there more students who are blond than students who are tall?"

There are several approaches to the cardinality of fuzzy sets. One group of them are constructive approaches. Following these approaches we get a single number, or alternatively, a fuzzy set as a cardinality of fuzzy sets. In many applications one prefers a simple scalar approximation of the cardinality of fuzzy sets. The complete axiomatic theory of scalar cardinality of fuzzy sets can be found in Wygralak's book [8]. The axiomatic theory of fuzzy cardinality was introduced by Casasnovas and Torrens in [3]. We introduce here the scalar cardinality for IF sets as a generalization of cardinality of fuzzy sets and a special case of cardinality of LFS. We use the following basic definitions:

Definition 1. (Zadeh 1965) A fuzzy set (the collection of all fuzzy sets we denote by FS) A on a universe X is a function $A : X \to [0, 1]$.

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Definition 2. (Atanassov 1983) An IF set (the collection of all IF sets we denote by IFS) on a universe X is an object of the form

$$A = \{(x, \mu_A(x), \nu_A(x))| \ \ x \in X\},$$

where μ_A and ν_A satisfy the following condition

$$(\mu_A(x) + \nu_A(x) \leq 1) \text{ for all } x \in X.$$

$\mu_A(x) \in [0,1]$ and $\nu_A(x) \in [0,1]$ are called the membership degree and the non-membership degree, respectively, of $x \in A$.

Definition 3. (Goguen 1967) An L-fuzzy set (the collection of all L-fuzzy sets we denote by LFS) A on a universe X is a function $A : X \to L$, where L is a complete distributive lattice equipped with the standard operations $\vee, \wedge$, bottom element $\mathbf{0}$, top element $\mathbf{1}$ and a unary, involutive, order-reversing operator $\mathbf{N}$.

Let us define basic relations for fuzzy sets, IF sets and L-fuzzy sets:

Definition 4. The union, intersection and complementare defined as follows:

- If A, $B \in$ FS. Then
  $A \cup B(x) = \max(A(x), B(x))$, for all $x \in X$,
  $A \cap B(x) = \min(A(x), B(x))$, for all $x \in X$,
  co$A(x) = 1 - A(x)$, for all $x \in X$.
- If A, $B \in$ IFS. Then
  $A \cup B = \{x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)) \mid x \in X\}$,
  $A \cap B = \{x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)) \mid x \in X\}$,
  co$A = \{(x, \nu_A(x), \mu_A(x)) | x \in X\}$.
- If A, $B \in$ LFS. Then
  $A \cup B(x) = \sup(A(x), B(x))$, for all $x \in X$,
  $A \cap B(x) = \inf(A(x), B(x))$, for all $x \in X$,
  co$A(x) = \mathbf{N}(A(x))$, for all $x \in X$.

We use the following basic definitions:
α-cut set of $A \in$ LFS is the set

$$A_\alpha = \{x \in X : A(x) \geq \alpha\} \text{ with } \alpha \in L \backslash \{\mathbf{0}\},$$

sharp α-cut set of $A \in LFS$ is the set

$$A^\alpha = \{x \in X : A(x) > \alpha\} \text{ with } \alpha \in L \backslash \{\mathbf{1}\},$$

the core of $A \in LFS$ is the set $core(A) = A_1$,
and the support of $A \in LFS$ is the set $supp(A) = A^0$.

If $supp(A)$ is finite, we say that A is a finite L-fuzzy set (the collection of all finite L-fuzzy sets we denote by LFFS). We use the following notations: The cardinality of a crisp set B will be denoted by $|B|$. The cardinality of a fuzzy set A will be denoted by $card(A)$. The cardinality of an IF set A will be denoted by $card_I(A)$. The cardinality of a L-fuzzy set A will be denoted by $card_L(A)$.

The concept of LFS (introduced by Goguen in [4]) is a generalization of the concept of fuzzy sets and includes the latter as a special case when $L = [0, 1]$. There are several different kinds of understanding the concept of an L-fuzzy set, distinguished by how the lattice L is specified.

Following [8], the cardinality of finite LFS was defined in [5]:

Definition 5. An element a of the lattice L is irreducible if $a = b \vee c$ implies $b = a$ or $c = a$. An irreducible element a of the lattice L is maximal irreducible if, for each irreducible b, $a \leq b$ implies $a = b$.

Definition 6. Let n be the number of maximal irreducible elements of a lattice L. A function $card_L : LFFS \to [0, \infty)^n$ will be called a scalar cardinality if the following postulates are satisfied for each a, $b \in L$, x, $y \in X$, and A, $B \in$ LFFS:

(1) **coincidence**

$$card_L(\mathbf{1}/x) = (\underbrace{1, ..., 1}_{n})$$

(2) **monotonicity**

$$a \leq b \Rightarrow card_L(a/x) \leq card_L(b/y)$$

(3) **additivity**

$$\text{if } supp(A) \cap supp(B) = \emptyset, \text{ then}$$

$$card_L(A \cup B) = card_L(A) + card_L(B).$$

Proposition 1. Let A, $B \in$ LFFS and $A_i \in$ LFFS for each index $i \in J$(J is finite). The following properties hold for every scalar cardinality:

(1) **additivity**

$$card_L(\bigcup_{i \in J} A_i) = \sum_{i \in J} card_L(A_i)$$

whenever $supp(A_i) \cap supp(A_j) = \emptyset$ for each $i \neq j$,

(2) **coincidence** $card_L(A) = (\underbrace{|core(A)|, ..., |core(A)|}_{n})$ if A is a crisp set,

(3) **monotonicity** $card_L(A) \leq card_L(B)$ if $A \subset B$,

(4) **boundedness**

$$(\underbrace{|core(A)|, ..., |core(A)|}_{n}) \leq card_L(A) \leq (\underbrace{|supp(A)|, ..., |supp(A)|}_{n}),$$

(5) **shiftability** $card_L(A) = card_L(B)$ iff there exists a bijection $b : supp(A) \to$ $\to supp(B)$ such that $A(x) = B(b(x))$ for each $x \in supp(A)$.

It is easy to see that these properties are immediate consequences of the axioms from definition 8.

The following theorem brings a useful characterization of a scalar cardinality of LFS:

Theorem 1. Let n be a number of maximal irreducible elements of a lattice L. A mapping $card_L : LFFS \to [0,\infty)^n$ is a scalar cardinality iff there exists a function $f_L : L \to [0,1]^n$ fulfilling the conditions:

(1) $f_L(\mathbf{0}) = (\underbrace{0,...,0}_{n})$, $f_L(\mathbf{1}) = (\underbrace{1,...,1}_{n})$, and

(2) $f_L(a) \le f_L(b)$ whenever $a \le_L b$ and such that

$$card_L(A) = \sum_{x \in supp(A)} f_L(A(x))$$

for each $A \in LFFS$.

The proof can be found in [5].

Each function f_L satisfying the conditions from the previous theorem will be called an L-cardinality pattern as it expresses our understanding of the scalar cardinality of a singleton.

The concept of IF sets (Atanassov in [1]) is a generalization of the concept of fuzzy sets, and a special case of LFS (we assume IF sets with finite support).

An IF set can be understood as an L-fuzzy set for a complete lattice $(L^*, \le_L)$ with $\mathbf{N}$ defined by $L = \{(\mu,\nu) \in [0;1]^2 | \ \mu \le 1-\nu\}$, $(\mu_1,\nu_1) \le (\mu_2,\nu_2) \iff \mu_1 \le \mu_2 \wedge \wedge \ \nu_1 \ge \nu_2$, $\mathbf{N}(\mu,\nu) = (\nu,\mu)$, the top element $\mathbf{1} = (1,0)$ and the bottom element $\mathbf{0} = (0,1)$.

Deschrijver, Cornelis and Kerre (see [2]) have extended a triangular norm, a triangular conorm, and a negator to the lattice L^*.

Definition 7. [2] A negator on L^* is any decreasing mapping $\mathbf{N} : L^* \to L^*$ satisfying $\mathbf{N}(\mathbf{0}) = \mathbf{1}$ and $\mathbf{N}(\mathbf{1}) = \mathbf{0}$. If $\mathbf{N}(\mathbf{N}(x)) = x$, for all $x \in L^*$, then $\mathbf{N}$ is called an involutive negator.

Remark 1. [2] Any involutive negator on L^* can be represented using an involutive negator on $[0,1]$, where a negator on $[0,1]$ is a decreasing mapping $N : [0,1] \to [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$. It holds

$$\mathbf{N}(a) = (N(1-a_2), 1-N(a_1)),$$

for all $a \in L^*$.

Definition 8. [2] A triangular norm (t-norm) on L^* is a mapping $\mathbf{T} : (L^*)^2 \to L^*$ satisfying the following conditions for all $x, y, x', y', z \in L^*$:

- $\mathbf{T}(x,\mathbf{1}) = x$,
- $\mathbf{T}(x,y) = \mathbf{T}(y,x)$,
- $\mathbf{T}(x,\mathbf{T}(y,z)) = \mathbf{T}(\mathbf{T}(x,y),z)$,
- $x \le_{L^*} x'$ and $y \le_{L^*} y' \Rightarrow \mathbf{T}(x,y) \le_{L^*} \mathbf{T}(x',y')$.

Definition 9. [2] A triangular conorm (t-conorm) on L^* is a mapping $\mathbf{S} : (L^*)^2 \to \to L^*$ satisfying the following conditions for all $x, y, x', y', z \in L^*$:

- $\mathbf{S}(x,\mathbf{0}) = x$,
- $\mathbf{S}(x,y) = \mathbf{S}(y,x)$,
- $\mathbf{S}(x,\mathbf{S}(y,z)) = \mathbf{S}(\mathbf{S}(x,y),z)$,
- $x \le_{L^*} x'$ and $y \le_{L^*} y' \Rightarrow \mathbf{S}(x,y) \le_{L^*} \mathbf{S}(x',y')$).

In IF set theory, negators are used to model the complement of an IF set, t-norms are used to model the intersection, and t-conorms are used to model the union of two IF sets. Let A, $B \in$ IFS:
$A \cup_{\mathbf{T}} B(x) = \mathbf{T}(A(x), B(x)),$ for all $x \in X$,
$A \cap_{\mathbf{S}} B(x) = \mathbf{S}(A(x), B(x)),$ for all $x \in X$,
$\mathrm{co}A(x) = \mathbf{N}(A(x)),$ for all $x \in X$.
Some triangular norms and conorms on L^* can be characterized using t-norms and t-conorms on [0,1]:

Definition 10. [2] A t-norm $\mathbf{T}$ on L^* is called t-representable iff there exist a t-norm T and a t-conorm S on [0,1] such that, for all $x, y \in L^*$,

$$\mathbf{T}(x, y) = (T(x_1, y_1), S(x_2, y_2)).$$

A t-conorm $\mathbf{S}$ on L^* is called t-representable iff there exist a t-norm T and a t-conorm S on [0,1] such that, for all $x, y \in L^*$,

$$\mathbf{S}(x, y) = (S(x_1, y_1), T(x_2, y_2)).$$

The following examples contain some non t-representable t-norms and t-conorms on L^* [2]:

(1) $\mathbf{T}_W(x, y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + 1 - y_1, y_2 + 1 - x_1))$
$\mathbf{S}_W(x, y) = (\min(1, x_2 + 1 - y_1, y_2 + 1 - x_1), \max(0, x_2 + y_2 - 1))$
(2) $\mathbf{T}_1(x, y) = (\max(0, x_1 + y_1 - x_2 y_2 - 1), \min(1, x_2 + y_2))$
$\mathbf{S}_1(x, y) = (\min(1, x_1 + y_1), \max(0, x_2 + y_2 - x_1 y_1 - 1))$
(3) $\mathbf{T}_2(x, y) = (\max(0, \min(x_1 - y_2, y_1 - x_2), \min(1, x_2 + y_2))$
$\mathbf{S}_2(x, y) = (\min(1, x_1 + y_1), \max(0, \min(x_2 - y_1, y_2 - x_1))$
(4) $\mathbf{T}_3(x, y) = (\max(0, x_1 + y_1 - 1), \min(1, y_2 + 2(1 - x_1), x_2 + 2(1 - y_1), 1 - -x_1 + 1 - y_1))$
$\mathbf{S}_3(x, y) = (\min(1, y_1 + 2(1 - x_2), x_1 + 2(1 - y_2), 1 - x_2 + 1 - y_2), \max(0, x_2 + +y_2 - 1))$
(5) $\mathbf{T}_4(x, y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + y_2 + \frac{1}{2}, 1 - x_1 + +y_2, 1 - y_1 + x_2))$
$\mathbf{S}_4(x, y) = (\min(1, x_1 + y_1 + \frac{1}{2}, 1 - x_2 + y_1, 1 - y_2 + x_1), \max(0, x_2 + +y_2 - 1))$

The following cardinalities of IF set are widely used:

Definition 11. [6] Let A be an IF set in X. The least cardinality of A is equal to the sigma-count, and it is called the *minCount*

$$minCount(A) = \sum_{x \in supp(A)} \mu_A(x).$$

The biggest cardinality of A is called the maxCount

$$maxCount(A) = \sum_{x \in supp(A)} (1 - \nu_A(x)).$$

We will give a generalization of the previous concept of cardinality of IF sets. Let A be an IF set. For each element (μ, ν) of a lattice L^* we have $(\mu, \nu) = (\mu, 1-\mu) \vee (0, \nu)$. The cardinality of IF sets can be defined naturally in the following way:

Definition 12. Let A be an IF set. A scalar cardinality of IF set A is a function $card_I : X \to [0,\infty)^2$ defined as follows

$$card_I(A) = (card(\mu), card(1-\nu)),$$

where $card(\mu), card(1-\nu)$ are cardinalities of fuzzy sets μ and $1-\nu$, respectively.

The cardinality pattern of IF sets can be characterized as follows:

Theorem 2. A mapping $f_I : L^* \to [0,1]^2$ is a scalar cardinality pattern of IF set A iff there exists a cardinality pattern of fuzzy sets $f : [0,1] \to [0,1]$ fulfilling the condition

$$f_I(a) = (f(a_1), f(1-a_2)),$$

where $a = (a_1, a_2) \in L^*$.

Proof.

" $\Rightarrow$ " Let f_I be a cardinality pattern of IF sets. Let $f : [0,1] \to [0,1]$ be a function such that $f(a_1) = f_I(a)_1$, $f(1-a_2) = f_I(a)_2$ for $a = (a_1, a_2) \in L^*$. It is easy to see that f fulfils the required conditions:

(1) $f(0) = f_I((0,1))_2 = 0$, $f(1) = f_I((1,0))_1 = 1$,
(2) $f(a) \le f(b)$ whenever $a \le b$,
(3) $\sum\limits_{x \in supp(A)} f(A(x)) = card(A)$ for each $A \in FFS$.

" $\Leftarrow$ " The converse part of the statement is obvious. $\square$

We present some instances of cardinality patterns of IF sets in the next example.

Example 1.

a) Let $a \in L^*$,

$$f_{I_p}(a) = \begin{cases} (1,1) & if\ a_1 \ge p_1, a_2 \le p_2, \\ (0,1) & if\ a_1 < p_1, a_2 \le p_2, \\ (0,0) & otherwise, \end{cases}$$

for $p = (p_1, p_2) \in L^*$, where $p_2 = 1 - p_1$. It is easy to see that

$$f_{I_{p_1}}(a) = (f_{p_1}(a_1), f_{p_1}(1-a_2)),$$

where

$$f_{p_1}(a) = \begin{cases} 1 & if\ a \ge p_1, \\ 0 & otherwise. \end{cases}$$

We obtain the smallest cardinality pattern for $p = \mathbf{1}$

$$f_{I*}(a) = \begin{cases} (1,1) & if\ a = \mathbf{1}, \\ (0,1) & if\ a_1 < 1, a_2 = 0, \\ (0,0) & otherwise, \end{cases}$$

where

$$f_{I*}(a) = (f_*(a_1), f_*(1-a_2))$$

and

$$f_*(a) = \begin{cases} 1 & if\ a = 1, \\ 0 & otherwise. \end{cases}$$

b) Let $a \in L^*$,

$$f_I{}^p(a) = \begin{cases} (1,1) & if\ a_1 > p_1, a_2 < p_2, \\ (0,1) & if\ a_1 \le p_1, a_2 < p_2, \\ (0,0) & otherwise, \end{cases}$$

for $p = (p_1, p_2) \in L^*$, where $p_2 = 1 - p_1$. Evidently

$$f_I{}^{p_1}(a) = (f^{p_1}(a_1), f^{p_1}(1 - a_2)),$$

where

$$f^{p_1}(a) = \begin{cases} 1 & if\ a > p_1, \\ 0 & otherwise. \end{cases}$$

We obtain the largest cardinality pattern for $p = \mathbf{0}$

$$f_I{}^*(a) = \begin{cases} (1,1) & if\ a > \mathbf{0}, \\ (0,1) & if\ a_1 = 0, a_2 < 1, \\ (0,0) & otherwise. \end{cases}$$

where

$$f_I{}^*(a) = (f^*(a_1), f^*(1 - a_2)),$$

$$f^*(a) = \begin{cases} 1 & if\ a = 1, \\ 0 & otherwise. \end{cases}$$

c) Let $a \in L^* \setminus \{\mathbf{0}, \mathbf{1}\}$,

$$f_{I\frac{1}{2}}(a) = \begin{cases} (0, 0.5) & if\ a_1 = 0, 0 < a_2 < 1, \\ (0,1) & if\ a_1 = 0, a_2 = 0, \\ (0.5, 0.5) & if\ 0 < a < 1, 0 < a_2 < 1, \\ (0.5, 1) & if\ 0 < a_1 < 1, a_2 = 0, \end{cases}$$

It is easy to see that

$$f_{I\frac{1}{2}}(a) = (f_{\frac{1}{2}}(a_1), f_{\frac{1}{2}}(1 - a_2)),$$

where

$$f_{\frac{1}{2}}(a) = \begin{cases} 1 & if\ a = 1 \\ 0.5 & if\ a \ne 0, \ne 1, \\ 0 & otherwise. \end{cases}$$

d) Let $a \in L^*$,

$$f_I{}^\circ(a) = \begin{cases} (0,0) & if\ a_1 < 0.5, a_2 > 0.5, \\ (0, 0.5) & if\ a_1 < 0.5, a_2 = 0.5, \\ (0,1) & if\ a_1 < 0.5, a_2 < 0.5, \\ (0.5, 0.5) & if\ a = (0.5, 0.5), \\ (0.5, 1) & if\ a_1 = 0.5, a_2 < 0.5, \\ (1,1) & if\ a_1 > 0.5, a_2 < 0.5. \end{cases}$$

Evidently
$$f_I{}^{\circ}(a) = (f^{\circ}(a_1), f^{\circ}(1 - a_2)),$$
where
$$f^{\circ}(a) = \begin{cases} 0 & if\ a < 0.5 \\ 0.5 & if\ a = 0.5, \\ 1 & otherwise. \end{cases}$$

Important features of sets and their cardinalities are the well-known valuation property, subadditivity and complementarity rules. We can formulate the subadditivity property, valuation property, complementarity and cartesian product rules for IF sets using t-norms, t-conorms and negators on lattice L^*.

Valuation property

For each A,$B \in IFFS$
$$card_I(A \cap_{\mathbf{T}} B) + card_I(A \cup_{\mathbf{S}} B) = card_I(A) + card_I(B).$$

Subadditivity property

For each $(A_i)_{i \in J} \in IFFS$
$$card_I(\bigcup_{i \in J}^{\mathbf{S}} A_i) \leq \sum_{i \in J} card_I(A_i).$$

Complementarity rule

For each $A \in IFFS$
$$card_I(A) + card_I(\text{co}A) = |X|$$
with a negator $\mathbf{N}$.

Cartesian product rule

For each A,$B \in IFFS$
$$card_I(A \times_{\mathbf{T}} B) = card_I(A).card_I(B).$$

Proposition 2. The scalar cardinality induced by a cardinality pattern f_I satisfies the valuation property iff a t-norm $\mathbf{T}$ and t-conorm $\mathbf{S}$ are such that for each a, $b \in L^*$:
$$f_I(\mathbf{T}(a, b)) + f_I(\mathbf{S}(a, b) = f_I(a) + f_I(b).$$

Proof. " $\Leftarrow$ " Following Theorem 1 we obtain
$$card_I(A \cap_{\mathbf{T}} B) + card_I(A \cup_{\mathbf{S}} B) = \sum_{x \in X}(f_I(\mathbf{T}(A(x), B(x))) + f_I(\mathbf{S}(A(x), B(x))) =$$
$$= \textstyle\sum_{x \in X}(f_I(A(x) + f_I(B(x)) = \sum_{x \in supp(A)} f_I(A(x)) + \sum_{x \in supp(B)} f_I(B(x)) =$$
$$= card_I(A) + card_I(B).$$
" $\Rightarrow$ " The converse part of the proof is obvious. □

Example 2. We present examples of triples $(f_I, \mathbf{T}, \mathbf{S})$ satisfying the valuation property:

a) $(f_I, \wedge, \vee)$ with any cardinality pattern.

b) $(id, \mathbf{T} = (T_{F,\lambda}, S_{F,\lambda}), \mathbf{S} = (S_{F,\lambda}, T_{F,\lambda}))$, where $\lambda \in [0, \infty]$ $(T_{F,\lambda}, S_{F,\lambda}$ are Frank t-norms and conorms). It is easy to see that for identity there is no other
t-representable t-norm and t-conorm satisfying valuation property up to Franks t-norms (and their ordinal sums) and related t-conorms.

c) $(f_{I*}, \mathbf{T} = (T, S), \mathbf{S} = (S, T))$, where S has no zero divisors and $T(a, b) \leq$ $\leq 1 - S(1 - a, 1 - b)$ for $a, b \in [0, 1]$.

d) $(f_I{}^*, \mathbf{T} = (T, S), \mathbf{S} = (S, T))$, where T has no zero divisors and $T(a, b) \leq$ $\leq 1 - S(1 - a, 1 - b)$ for $a, b \in [0, 1]$.

Proposition 3. The subadditivity property is satisfied by a cardinality pattern f_I and a t-conorm $\mathbf{S}$ iff, for each $a, b \in L^*$,

$$f_I(\mathbf{S}(a, b)) \leq f_I(a) + f_I(b).$$

Proof. " $\Leftarrow$ " Following Theorem 1 and the equality $supp(A \cup B) = supp(A) \cup supp(B)$, we have

$$\begin{aligned} card_I(A \cup_{\mathbf{S}} B) &= \sum_{x \in supp(A \cup B)} f_I(\mathbf{S}(A(x), B(x))) \leq \\ \leq \sum_{x \in supp(A \cup B)} f_I(A(x)) &+ \sum_{x \in supp(A \cup B)} f_I(B(x)) = \\ = \sum_{x \in supp(A)} f_I(A(x)) &+ \sum_{x \in supp(B)} f_I(B(x)) = \\ &= card_I(A) + card_I(B) \end{aligned}$$

for each $A, B \in IFFS$, which implies subadditivity.

" $\Rightarrow$ " The converse part of the proof is obvious. □

The following example contains instances of couples $(f_I, \mathbf{S})$ satisfying the subadditivity property.

Example 3.

a) Couples from the previous example.

b) $(id, \mathbf{S} = (S, T))$, where S is any t-conorm which is weaker than $S_{\mathbf{L}}$.

c) $(id, \mathbf{S} = (S_{H,2}, T_{H,2}))$ $(S_{H,2}, T_{H,2}$ are the Hamacher t-norm and t-conorm) is an example of a t-representable conorm of IF sets for which the triple $(id, \mathbf{S} =$
$= (S_{H,2}, T_{H,2}), \mathbf{T} = (T_{H,2}, S_{H,2}))$ does not fulfil the valuation property.

Proposition 4. The complementarity rule holds for a cardinality pattern f_I and a negator $\mathbf{N}$ iff

$$f_I(a) + f_I(\mathbf{N}(a)) = (1, 1) \text{ for each } a \in L^*.$$

Proof. " $\Leftarrow$ " If condition $f_I(a) + f_I(\mathbf{N}(a)) = (1, 1)$ is satisfied, then

$$\sum_{x \in X} f_I(A(x)) + \sum_{x \in X} f_I(\mathbf{N}(A(x))) = card_I(A) + card_I(\text{co}A) = |X|.$$

" $\Rightarrow$ " The converse part of the proof is obvious. □

Proposition 5. There is no cardinality pattern satisfying the complementarity rule with an involutive negation $\mathbf{N}$.

Proof. If $a = (0,0)$, then $f_I((0,0)) = (0,1)$. From the Remark 1 it follows $\mathbf{N}((0,0)) = (0,0)$. It is easy to see that using the condition from the Proposition 4 we obtain $f_I((0,0)) = (\frac{1}{2}, \frac{1}{2})$, which is a contradiction. □

Proposition 6. The cartesian product rule holds iff a cardinality pattern f_I and a t-norm $\mathbf{T}$ are such that for each a, $b \in L^*$

$$f_I(\mathbf{T}(a,b)) = f_I(a).f_I(b).$$

Proof. " $\Leftarrow$ " If the condition $f_I(\mathbf{T}(a,b)) = f_I(a).f_I(b)$ is satisfied, then

$$card_I(A \times_{\mathbf{T}} B) = \sum_{(x,y)\in X\times X} f_I(\mathbf{T}(A(x), B(y))) = \sum_{(x,y)\in X\times X} f_I(A(x)).f_I(B(x)) =$$

$$= \sum_{(x,y)\in supp(A\times_{\mathbf{T}} B)} f_I(\mathbf{T}(A(x), B(y))) = \sum_{(x,y)\in supp(A)\times supp(B)} f_I(A(x)).f_I(B(x)) =$$

$$= card_I(A).card_I(B)$$

" $\Rightarrow$ " The converse part of the proof is obvious. □

The next example shows couples $(f_I, \mathbf{T})$ which satisfy the cartesian product rule.

Example 5.

a) $(f_{I*}, \wedge)$, where f_{I*} is the smallest cardinality pattern from Example 1.
b) $(f_I{}^*, \wedge)$, where $f_I{}^*$ is the largest cardinality pattern from Example 1.

Conclusion

We have extended the notion of cardinality to IF sets. We have proven some important results which are natural generalizations of the axiomatic cardinality theory of fuzzy sets. The complete description of t-norms (t-representable or not) satisfying the valuation property, subadditivity property, complementarity rule and cartesian product rule for IF sets will be the subject of a future research.

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Department of Applied Informatics; Faculty of Economics; Matej Bel University; Tajovskeho 10; SK–975 90 Banska Bystrica; Slovakia

E-mail: pavol.kral@fpv.umb.sk