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NATURAL DUALITIES FOR STRUCTURES

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Dedicated to George Grätzer and E. Tamás Schmidt on their 70th birthdays.

Abstract. Following on from results of Hofmann [27], we investigate the extension of the theory of natural dualities to quasivarieties generated by finite structures that can have operations, partial operations and relations in their type. It turns out that the usual proofs of the Second Duality Theorem, the Duality Compactness Theorem and the NU Duality Theorem extend to this setting with only minor changes. We present simple proofs of Two-for-One Full Duality and Strong Duality Theorems, and show how our techniques can be applied to yield new dualities from known strong dualities by simply swapping the topology from one side to the other.

While developing the theory of natural dualities, the author and his various coauthors made a conscious decision to aim the theory at those who use universal algebraic ideas, for whom a duality may be a useful tool; see, for example, Davey and Werner [20, 21, 22], Davey and Priestley [18], Clark and Davey [5] and Pitkethly and Davey [30]. This required a careful choice of an appropriate level of generality for both the starting algebraic category $\boldsymbol{\mathcal{A}}$ and the topological dual category $\boldsymbol{\mathfrak{X}}$. To maximise the algebraic content we decided to concentrate on the case where $\boldsymbol{\mathcal{A}}$ was an $\mathbb{ISP}$-closed class of algebras generated by a single finite (total) algebra. Later, we extended the theory to include multi-sorted dualities, where $\boldsymbol{\mathcal{A}}$ is an $\mathbb{ISP}$-closed class generated by a finite set of finite algebras; see [18] and [5, Chapter 7]. We were aware that the restriction to *finite* generating algebras was not necessary; see, for example, Davey [12], the appendix of Davey and Werner [20], Davey and Werner [21, 22], Davey and Priestley [18], and Clark and Davey [5, Exercise 2.9]. Nevertheless, we felt that the theory for finitely generated classes was sufficiently rich to warrant special attention. Based on our experience with many examples, we chose the dual category $\boldsymbol{\mathfrak{X}}$ to be the $\mathbb{IS}_c\mathbb{P}^+$-closed class generated by a finite topological structure whose type allowed finitary total operations, partial operations and relations.

The lack of symmetry between the allowable types of structures on the original algebraic side and on the dual topological side was intentional but to a large extent unnecessary. Much of the theory goes through if we allow total operations, partial operations and relations on both sides. The first development in this direction was by Hofmann [27], who presented a generalisation of the Duality Compactness Theorem and a Two-for-One Full Duality Theorem. Hofmann's results make an important contribution to the theory of natural dualities. He generalises the

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basic setting of natural dualities by allowing both the "algebraic" and "topological" categories to be determined in an appropriate way by finitary limit sketches. (See [27] for the details and Adámek and Rosický [1] for the required background on sketches.) In this way, Hofmann eliminates the asymmetry inherent in the original theory of natural dualities. In particular, his approach allows structures with operations, partial operations and relations on both sides of the duality.

Hofmann's results suggest many avenues for further investigation within the theory of natural dualities. For example,

(1) extend results in the original theory to the sketch-based setting,
(2) extend results in the original theory to the case where the algebraic category $\boldsymbol{\mathcal{A}}$ and the topological category $\boldsymbol{\mathfrak{X}}$ consist of structures, with operations, partial operations and relations,
(3) give examples of dualities for categories that are truly sketch-based and not covered by the extension of the theory to classes of structures,
(4) give new examples of dualities for classes of structures.

This paper addresses (2) and (4). These are important as they make the theory, expanded to structures, available to users of natural dualities without requiring them first to absorb the theory of sketches. Moreover, work on (2) and (4) will set the boundaries of the theory of natural dualities for structures and thereby indicate what might be possible in (1) and (3).

As a contribution to (2), we present the basics of the theory of natural dualities for structures. We show that a generalisation of the Second Duality Theorem (Davey and Werner [20, 1.16]), the version of Hofmann's Duality Compactness Theorem that applies to structures, and a generalisation of the NU Duality Theorem (Davey and Werner [20, 1.19]) can all be obtained by easy modifications of the proofs given in Clark and Davey [5]. We also present proofs of Hofmann's Two-for-One Full Duality Theorem for structures and of a generalisation of Clark and Davey's Two-for-One Strong Duality Theorem [4]. As a contribution to (4), we close the paper with a number of applications to quasivarieties of structures of the two-for-one duality theorems and an application of the generalised NU Duality Theorem.

1. Structures of type $\langle G, H, R\rangle$

This section is a brief refresher on quasivarieties and universal Horn classes generated by finite structures. We present just enough to meet our needs. For detailed treatments in the setting of partial algebras and more general categories we refer to Burmeister [3] and Adámek and Rosický [1], respectively.

We begin with sets G of finitary total operation symbols, H of finitary partial operation symbols, and R of finitary relation symbols. The total operation symbols may be nullary, but, to remove unnecessary complications, we shall assume that the arity of each partial operation and relation symbol is positive. A *structure*,

$$\mathbf{A} = \langle A; G^{\mathbf{A}}, H^{\mathbf{A}}, R^{\mathbf{A}}\rangle,$$

of type $\langle G, H, R\rangle$ is defined in the usual way; see Clark and Davey [5]. If H is empty, then we refer to $\mathbf{A}$ as a *total structure*; if both H and R are empty, then we simply refer to $\mathbf{A}$ as an *algebra*. We allow the underlying set A of $\mathbf{A}$ to be empty only if there are no nullary symbols in G. An *atomic formula* of type $\langle G, H, R\rangle$ is an expression of the form

$$t_1 \approx t_2 \quad \text{or} \quad r(t_1, \dots, t_n),$$

where $t_1, t_2, \ldots, t_n$ are terms of type $G \cup H$ and $r \in R$ is n-ary. For a detailed discussion of the validity of first-order formulæ in a structure, see Burmeister's survey article [3]. Note that in [3], no distinction is made between total and partial operation symbols in the type. A brief introduction suitable to our needs can be found in Clark and Davey [5, page 25]. Two important features to note are that, given a term t of type $G \cup H$, the domain of the corresponding term function $t^{\mathbf{A}}$ on A is its maximum domain, and that, given n-ary terms t_1 and t_2 and $a_1, \ldots, a_n \in A$, the structure $\mathbf{A}$ satisfies $t_1^{\mathbf{A}}(a_1, ..., a_n) = t_2^{\mathbf{A}}(a_1, ..., a_n)$ if and only if both sides are defined and equal. In particular, $\mathbf{A}$ satisfies $t^{\mathbf{A}}(a_1, ..., a_n) = t^{\mathbf{A}}(a_1, ..., a_n)$ if and only if $(a_1, \ldots, a_n) \in \operatorname{dom}(t^{\mathbf{A}})$. A *universal Horn sentence* of type $\langle G, H, R\rangle$ is a universally quantified formula of the form

$$\varphi, \qquad \Big(\bigwedge_{i=1}^{n} \varphi_i\Big) \to \varphi, \quad \text{or} \quad \bigvee_{i=1}^{n} \neg\varphi_i,$$

where φ and all the φ_i's are atomic formulæ of type $\langle G, H, R\rangle$, and $n \in \mathbb{N}$. We shall refer to universal Horn sentences of the first and second kinds as *atomic sentences* and *quasi-atomic sentences*, respectively.

Let Σ be a set of universal Horn sentences of type $\langle G, H, R\rangle$. Then $\operatorname{Mod}(\Sigma)$ denotes the class consisting of all *non-empty* models of Σ, while $\operatorname{Mod}^0(\Sigma)$ includes the empty structure $\boldsymbol{\varnothing}$ of type $\langle G, H, R\rangle$ in the case that G contains no nullary symbols. A class $\boldsymbol{\mathcal{A}}$ of structures is called a *universal Horn class* if $\boldsymbol{\mathcal{A}} = \operatorname{Mod}(\Sigma)$ or $\boldsymbol{\mathcal{A}} = \operatorname{Mod}^0(\Sigma)$, for some set Σ of universal Horn sentences. A class defined by atomic and quasi-atomic sentences is called a *quasivariety*, and a class defined by atomic sentences is called either an *atomic class* or a *variety*.

Let $\mathbb{I}$, $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$ be the usual class operators. We adopt the normal algebraic convention that $\mathbb{S}(\boldsymbol{\mathcal{K}})$ denotes the class consisting of all *non-empty* substructures of structures from $\boldsymbol{\mathcal{K}}$. Note that $\mathbb{P}(\boldsymbol{\mathcal{K}})$ includes products over an empty index set, whence $\mathbb{P}(\boldsymbol{\mathcal{K}})$ includes the *complete one-element structure* $\mathbf{1}$ of type $\langle G, H, R\rangle$ with underlying set $\{\varnothing\}$ and every relation and the domain of every partial operation non-empty. We also require two further operators, namely $\mathbb{S}^0$, which includes empty substructures in case G contains no nullary operation symbols, and $\mathbb{P}^+$, which excludes products over an empty index set.

Theorem 1.1. *Let $\boldsymbol{\mathcal{K}}$ be a finite non-empty set of finite structures of type $\langle G, H, R\rangle$.*

- (i) *The smallest universal Horn class containing $\boldsymbol{\mathcal{K}}$ is the class $\mathbb{ISP}^+(\boldsymbol{\mathcal{K}})$, if the empty structure is not allowed, and is $\mathbb{IS}^0\mathbb{P}^+(\boldsymbol{\mathcal{K}})$, if the empty structure is allowed.*
- (ii) *The smallest quasivariety containing $\boldsymbol{\mathcal{K}}$ is the class $\mathbb{ISP}(\boldsymbol{\mathcal{K}})$, if the empty structure is not allowed, and is $\mathbb{IS}^0\mathbb{P}(\boldsymbol{\mathcal{K}})$, if the empty structure is allowed.*
- (iii) *The smallest atomic class containing $\boldsymbol{\mathcal{K}}$ is the class $\mathbb{HSP}(\boldsymbol{\mathcal{K}})$, if the empty structure is not allowed, and is $\mathbb{HS}^0\mathbb{P}(\boldsymbol{\mathcal{K}})$, if the empty structure is allowed.*

Proof. This is the finitely generated version of Theorems 3.4(i)(iii) and 4.5 of Burmeister [3]. It is a good exercise to write out a direct proof based on the version for algebras in Clark and Davey [5, 1.3.4 and Appendix A]. □

If and when to include the empty structure in a class of structures is a matter of some debate. It still leads to heated discussions between category theorists, who want their categories to be complete and cocomplete, and algebraists, who are usually happy to live without the free structure generated by the empty set when it happens to be empty, and even without the complete one-element structure when considering universal Horn classes. Our usual convention will be to exclude the empty structure when considering algebras ($H = R = \varnothing$), and to include the empty structure when considering purely relational structures ($G = H = \varnothing$), and to make a decision on a case-by-case basis otherwise. Moreover, we almost always allow the empty structure when working with topological structures; see the discussion at the start of Section 6 and particularly Lemma 6.2.

The following result is a completely standard but often-required characterisation of the structures in the class $\mathbb{ISP}^+(\boldsymbol{\mathcal{K}})$. It is the non-topological version of the Separation Theorem [5, 1.4.4], see Lemma 3.1 below.

Lemma 1.2. *Let $\boldsymbol{\mathcal{K}}$ be a non-empty set of structures of type $\langle G, H, R\rangle$ and let $\mathbf{A}$ be a non-empty structure of the same type.*

(i) *The complete one-element structure $\mathbf{1}$ belongs to $\mathbb{ISP}^+(\boldsymbol{\mathcal{K}})$ if and only if some $\mathbf{M} \in \boldsymbol{\mathcal{K}}$ has a substructure isomorphic to $\mathbf{1}$.*

(ii) *Assume that $\mathbf{A}$ is not isomorphic to the complete one-element structure $\mathbf{1}$. Then $\mathbf{A} \in \mathbb{ISP}^+(\boldsymbol{\mathcal{K}})$ if and only if, for all $r \in \{=\} \cup \{\, \mathrm{dom}(h) \mid h \in H\,\} \cup R$ of arity n, and all $a_1, \dots, a_n \in A$ with $(a_1, \dots, a_n) \notin r^{\mathbf{A}}$, there exists $\mathbf{M} \in \boldsymbol{\mathcal{K}}$ and a homomorphism $\varphi : \mathbf{A} \to \mathbf{M}$ such that $(\varphi(a_1), \dots, \varphi(a_n)) \notin r^{\mathbf{M}}$.*

Later we shall need the fact that every universal Horn class of structures is closed under direct limits. Indeed, the usual construction for algebras still applies. Let $\boldsymbol{\mathcal{A}}$ be a class of structures of type $\langle G, H, R\rangle$, let $\mathbf{S} = \langle S; \leqslant\rangle$ be a non-empty directed ordered set and let $\{\, \mathbf{A}_s \mid s \in S\,\}$ be a direct system in $\boldsymbol{\mathcal{A}}$ with connecting homomorphisms $\varphi_{st} : \mathbf{A}_s \to \mathbf{A}_t$, for all $s \leqslant t$ in $\mathbf{S}$. Define an equivalence relation $\equiv$ on the disjoint union $\dot{\bigcup}\{\, A_s \mid s \in S\,\}$ as follows: for $a \in A_s$ and $b \in A_t$, define $a \equiv b$ if there exists an upper bound u of $\{s, t\}$ in $\mathbf{S}$ such that $\varphi_{su}(a) = \varphi_{tu}(b)$, and denote the equivalence class of a by $[a]$. We convert $B := \dot{\bigcup}\{\, A_s \mid s \in S\,\}/\equiv$ into a structure $\mathbf{B}$ of type $\langle G, H, R\rangle$ as follows. For each $h \in H$ of arity n, and all $s_1, \dots, s_n \in S$ and $a_1, \dots, a_n$, with $a_i \in A_{s_i}$, define

$$([a_1], \dots, [a_n]) \in \mathrm{dom}(h^{\mathbf{B}}) \iff \begin{cases} (\varphi_{s_1 t}(a_1), \dots, \varphi_{s_n t}(a_n)) \in \mathrm{dom}(h^{\mathbf{A}_t}), \\ \text{for some upper bound } t \text{ of } \{s_1, \dots, s_n\} \text{ in } \mathbf{S}, \end{cases}$$

in which case

$$h^{\mathbf{B}}([a_1], \dots, [a_n]) := \big[h^{\mathbf{A}_t}(\varphi_{s_1 t}(a_1), \dots, \varphi_{s_n t}(a_n))\big].$$

For each $g \in G$ and each $r \in R$, the total operation $g^{\mathbf{B}}$ and the relation $r^{\mathbf{B}}$ are defined analogously.

Theorem 1.3. *Let $\boldsymbol{\mathcal{A}}$ be a universal Horn class of structures of type $\langle G, H, R\rangle$. Let $\mathbf{S} = \langle S; \leqslant\rangle$ be a non-empty directed ordered set and let $\{\, \mathbf{A}_s \mid s \in S\,\}$ be a direct system in $\boldsymbol{\mathcal{A}}$ with connecting homomorphisms $\varphi_{st} : \mathbf{A}_s \to \mathbf{A}_t$, for all $s \leqslant t$ in $\mathbf{S}$. Then the structure $\mathbf{B}$ defined above belongs to $\boldsymbol{\mathcal{A}}$ and is a direct limit in $\boldsymbol{\mathcal{A}}$ of the direct system $\{\, \mathbf{A}_s \mid s \in S\,\}$.*

Proof. It is an easy exercise to show that a universal Horn sentence satisfied by all of the structures $\mathbf{A}_s$, for $s \in S$, is also satisfied by $\mathbf{B}$, whence $\mathbf{B}$ belongs to $\boldsymbol{\mathcal{A}}$.

A simple argument now shows that $\mathbf{B}$ satisfies the universal mapping definition of the direct limit in $\boldsymbol{\mathcal{A}}$ of the direct system. $\square$

Henceforth, we denote the structure $\mathbf{B}$ constructed above by $\varinjlim_{s\in S} \mathbf{A}_s$. The following useful fact is well known for algebras (see Grätzer [24]). Its proof is no more difficult in this more general setting, and we leave it for the reader.

Lemma 1.4. *Let* $\mathbf{A}$ *be a structure of type* $\langle G, H, R\rangle$. *Then* $\mathbf{A}$ *is isomorphic to* $\varinjlim_{s\in S} \mathbf{A}_s$, *where* $\{\, \mathbf{A}_s \mid s \in S \,\}$ *is the direct system consisting of the finitely generated substructures of* $\mathbf{A}$ *ordered by inclusion.*

A structure $\mathbf{A}$ is said to be *locally finite* if every finitely generated substructure of $\mathbf{A}$ is finite, and a class $\boldsymbol{\mathcal{A}}$ is locally finite if every structure in $\boldsymbol{\mathcal{A}}$ is locally finite. As in the case of algebras, finitely generated quasivarieties are locally finite.

Lemma 1.5. *Let* $\boldsymbol{\mathcal{K}}$ *be a finite non-empty set of finite structures of type* $\langle G, H, R\rangle$. *Then the quasivariety* $\mathbb{ISP}(\boldsymbol{\mathcal{K}})$ *generated by* $\boldsymbol{\mathcal{K}}$ *is locally finite.*

Proof. Let $\boldsymbol{\mathcal{K}} = \{\mathbf{M}_1, \dots, \mathbf{M}_k\}$. Every n-ary term t of type $G \cup H$ induces a k-tuple $(t^{\mathbf{M}_1}, \dots, t^{\mathbf{M}_k})$, where $t^{\mathbf{M}_i}$ is an n-ary partial operation on M_i. The number of such k-tuples of n-ary term functions is bounded above by $\ell := m^{2^{m^n}k}$, where m is the maximum size of a structure in $\boldsymbol{\mathcal{K}}$. Hence, there exist n-ary terms $t_1, t_2, \dots, t_s$ of type $G \cup H$, with $s \leqslant \ell$, such that, for every n-ary term t of type $G \cup H$, there exists $i \in \{1, \dots, s\}$ such that, for every $\mathbf{M} \in \boldsymbol{\mathcal{K}}$, we have $t^{\mathbf{M}} = t_i^{\mathbf{M}}$. So the class $\boldsymbol{\mathcal{K}}$ satisfies the quasi-equations

$$
\begin{aligned}
t_i(x_1, \dots, x_n) \approx t_i(x_1, \dots, x_n) &\implies t(x_1, \dots, x_n) \approx t(x_1, \dots, x_n), \\
t(x_1, \dots, x_n) \approx t(x_1, \dots, x_n) &\implies t(x_1, \dots, x_n) \approx t_i(x_1, \dots, x_n),
\end{aligned}
$$

which express the fact that t and t_i induce identical term functions on every structure in $\boldsymbol{\mathcal{K}}$. Since every structure in $\mathbb{ISP}(\boldsymbol{\mathcal{K}})$ satisfies every quasi-equation true in $\boldsymbol{\mathcal{K}}$, we conclude that t and t_i induce identical term functions on every structure in $\mathbb{ISP}(\boldsymbol{\mathcal{K}})$. It now follows easily that, if $\mathbf{A} \in \mathbb{ISP}(\boldsymbol{\mathcal{K}})$ is n-generated, then $|A| \leqslant s$. $\square$

The following standard result is a useful consequence of the previous three results. As usual, we denote the finite members of a class $\boldsymbol{\mathcal{C}}$ by $\boldsymbol{\mathcal{C}}_{\text{fin}}$.

Lemma 1.6. *Let* $\mathbf{M} = \langle M; G, H, R\rangle$ *be a finite structure and let* Σ *be a set of universal Horn sentences of type* $\langle G, H, R\rangle$. *If* $[\mathbb{ISP}^+(\mathbf{M})]_{\text{fin}} = [\mathrm{Mod}(\Sigma)]_{\text{fin}}$ *and every finitely generated model of* Σ *is finite, then* $\mathbb{ISP}^+(\mathbf{M}) = \mathrm{Mod}(\Sigma)$.

Proof. By Theorem 1.3 and Lemma 1.4, every universal Horn class is uniquely determined by its finitely generated structures. By Lemma 1.5, the universal Horn class $\mathbb{ISP}^+(\mathbf{M})$ is locally finite and, by assumption, the universal Horn class $\mathrm{Mod}(\Sigma)$ is locally finite. It follows at once that $[\mathbb{ISP}^+(\mathbf{M})]_{\text{fin}} = [\mathrm{Mod}(\Sigma)]_{\text{fin}}$ implies $\mathbb{ISP}^+(\mathbf{M}) = \mathrm{Mod}(\Sigma)$. $\square$

In order to extend the First and Second Duality Theorems (see Clark and Davey [5, 2.2.2 and 2.2.7]) to this more general setting, we need to know that the usual description of free algebras in the class $\mathbb{ISP}(\mathbf{M})$ generated by an algebra $\mathbf{M}$ extends to this setting. The proof of the following result is an easy modification of the proof of the corresponding result for algebras (see, for example, Appendix A of [5]). Let $\mathbf{M} = \langle M; G, H, R\rangle$ be a structure. The term function induced by an n-ary term of type $G \cup H$ is an n-ary partial operation $t^{\mathbf{M}} : D \to M$, where $D \subseteq M^n$. If $D = M^n$, then we refer to $t^{\mathbf{M}} : D \to M$ as a *total n-ary term function of* $\mathbf{M}$. In this case, even though $t^{\mathbf{M}}$ is a total operation, the term t may include partial operation symbols in H for which the corresponding partial operation $h^{\mathbf{M}}$ is not total. Let S be a non-empty set. A function $f : M^S \to M$ is a *total S-ary term function of* $\mathbf{M}$ if, for some $n \geqslant 0$, there exist $s_1, \dots, s_n \in S$ and a total n-ary term function $t^{\mathbf{M}} : M^n \to M$ of $\mathbf{M}$ such that $f(a) = t^{\mathbf{M}}(a(s_1), \dots, a(s_n))$, for all $a \in M^S$. Let $\mathbf{F_M}(S)$ denote the substructure of $\mathbf{M}^{M^S}$ consisting of the total S-ary term functions of $\mathbf{M}$. Clearly, $\mathbf{F_M}(S)$ is the substructure of $\mathbf{M}^{M^S}$ generated by the projections.

Lemma 1.7. *Let* $\mathbf{M}$ *be a non-empty structure and let* $\mathcal{V} := \mathbb{HSP}(\mathbf{M})$ *be the atomic class generated by* $\mathbf{M}$. *For every non-empty set* S, *the structure* $\mathbf{F_M}(S)$ *is freely generated in* $\mathcal{V}$ *by the set* $\{\, \pi_s : M^S \to M \mid s \in S \,\}$ *of projections.*

2. Alter egos

In this section we indicate how the definition of an alter ego of a finite algebra can be extended to finite structures. Let

$$\mathbf{M}_1 = \langle M; G_1, H_1, R_1\rangle \quad \text{and} \quad \mathbf{M}_2 = \langle M; G_2, H_2, R_2\rangle$$

be two structures defined on the same set M. To avoid technicalities, we assume that M is non-empty and that the relations in R_1 and R_2 and the domains of the partial operations in H_1 and H_2 are non-empty. Then $\mathbf{M}_2$ is said to be *compatible with* $\mathbf{M}_1$ if

(a) for all $n \geqslant 0$, each n-ary operation $g \in G_2$ is a homomorphism from $\mathbf{M}_1^n$ to $\mathbf{M}_1$,
(b) for all $n \geqslant 1$ and each n-ary partial operation $h \in H_2$, the domain of h forms a substructure $\mathbf{dom}(h)$ of $\mathbf{M}_1^n$ and h is a homomorphism from $\mathbf{dom}(h)$ to $\mathbf{M}_1$, and
(c) for all $n \geqslant 1$, each n-ary relation $r \in R_2$ forms a substructure of $\mathbf{M}_1^n$.

Note that it follows from (a) that, if $\mathbf{M}_2$ is compatible with $\mathbf{M}_1$ and c is (the value of) a nullary operation of $\mathbf{M}_2$, then $\{c\}$ forms a substructure of $\mathbf{M}_1$ isomorphic to the complete one-element structure $\mathbf{1}_1$ of type $\langle G_1, H_1, R_1\rangle$.

Lemma 2.1. *Let* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *be structures defined on the same underlying set. Then* $\mathbf{M}_2$ *is compatible with* $\mathbf{M}_1$ *if and only if* $\mathbf{M}_1$ *is compatible with* $\mathbf{M}_2$.

Proof. This is a symbol-pushing exercise. The crux of the proof is the fact that, given partial operations h_1 and h_2 of arities m and n on a set M, the domain of h_1 is closed under h_2 and h_1 preserves h_2 if and only if the same thing holds with h_1 and h_2 interchanged. □

Since compatibility is a symmetric relation, we shall simply say that $\mathbf{M}_1$ and $\mathbf{M}_2$ are *compatible*. If $\mathbf{M} = \langle M; G, H, R\rangle$ is a finite structure, then $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ will denote the topological structure obtained by adding the discrete topology $\mathcal{T}$ to $\mathbf{M}$. If M is a finite non-empty set and $\mathbf{M}_1$ and $\mathbf{M}_2$ are compatible structures with underlying set M, then we shall refer to the discrete topological structure $\underset{\sim}{\mathbf{M}}_2$ as an *alter ego* of the structure $\mathbf{M}_1$. (In the case that $\mathbf{M}_1$ is an algebra, instead of saying that $\underset{\sim}{\mathbf{M}}_2$ is an alter ego of $\mathbf{M}_1$, many authors say that $\underset{\sim}{\mathbf{M}}_2$ is *algebraic over* $\mathbf{M}_1$.)

It is now completely straightforward to check that the basics of the theory of natural dualities extend to structures. We will sketch the details.

Let $\mathbf{M}_1 = \langle M; G_1, H_1, R_1 \rangle$ be a finite structure and let $\underset{\sim}{\mathbf{M}}_2 = \langle M; G_2, H_2, R_2, \mathcal{T} \rangle$ be an alter ego of $\mathbf{M}_1$. Define $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ to be the quasivariety generated by $\mathbf{M}_1$, and let $\boldsymbol{\mathfrak{X}} := \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$ be the class consisting of all topological structures of the same type as $\underset{\sim}{\mathbf{M}}_2$ that are isomorphic to a possibly empty closed substructure of a non-zero power of $\underset{\sim}{\mathbf{M}}_2$. With a slight abuse of terminology, the class $\boldsymbol{\mathfrak{X}}$ is usually referred to as the *topological quasivariety* generated by $\underset{\sim}{\mathbf{M}}_2$. We also denote by $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathfrak{X}}$ the corresponding categories obtained by adding as morphisms all homomorphisms and all continuous homomorphisms, respectively. (As an aid to the reader, we shall refer to morphisms in $\boldsymbol{\mathcal{A}}$ as homomorphisms and reserve the name morphism for the category $\boldsymbol{\mathfrak{X}}$.)

The fact that the structures $\mathbf{M}_1$ and $\mathbf{M}_2$ are compatible guarantees that we can set up a dual adjunction $\langle \mathrm{D}, \mathrm{E}, e, \varepsilon \rangle$ between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathfrak{X}}$. The verification of the many claims below, both implicit and explicit, is straightforward. (See Section 1.5 of [5] for the details in the case that $\mathbf{M}_1$ is an algebra.) Define contravariant hom-functors $\mathrm{D} : \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathfrak{X}}$ and $\mathrm{E} : \boldsymbol{\mathfrak{X}} \to \boldsymbol{\mathcal{A}}$ as follows:

- for each structure $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, the *dual of* $\mathbf{A}$ is the topologically closed substructure $\mathrm{D}(\mathbf{A})$ of $\underset{\sim}{\mathbf{M}}_2^A$ formed by the set $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}_1)$ of all homomorphisms from $\mathbf{A}$ to $\mathbf{M}_1$,
- for each structure $\mathbf{X} \in \boldsymbol{\mathfrak{X}}$, the *dual of* $\mathbf{X}$ is the substructure $\mathrm{E}(\mathbf{X})$ of $\mathbf{M}_1^X$ formed by the set $\boldsymbol{\mathfrak{X}}(\mathbf{X}, \underset{\sim}{\mathbf{M}}_2)$ of all morphisms from $\mathbf{X}$ to $\underset{\sim}{\mathbf{M}}_2$,
- given a homomorphism $u : \mathbf{A} \to \mathbf{B}$ in $\boldsymbol{\mathcal{A}}$, the morphism $\mathrm{D}(u) : \mathrm{D}(\mathbf{B}) \to \mathrm{D}(\mathbf{A})$ is defined by $\mathrm{D}(u)(x) := x \circ u$, for all $x \in \boldsymbol{\mathcal{A}}(\mathbf{B}, \mathbf{M}_1)$,
- given a morphism $\psi : \mathbf{X} \to \mathbf{Y}$ in $\boldsymbol{\mathfrak{X}}$, the homomorphism $\mathrm{E}(\psi) : \mathrm{E}(\mathbf{Y}) \to \mathrm{E}(\mathbf{X})$ is defined by $\mathrm{E}(\psi)(\alpha) := \alpha \circ \psi$, for all $\alpha \in \boldsymbol{\mathfrak{X}}(\mathbf{Y}, \underset{\sim}{\mathbf{M}}_2)$.

For each $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ and each $\mathbf{X} \in \boldsymbol{\mathfrak{X}}$, define the *evaluation maps*

$$e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A}) \quad \text{and} \quad \varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathrm{DE}(\mathbf{X})$$

by $e_{\mathbf{A}}(a)(x) := x(a)$, for all $a \in A$ and all $x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}_1)$, and $\varepsilon_{\mathbf{X}}(x)(\alpha) := \alpha(x)$, for all $x \in X$ and all $\alpha \in \boldsymbol{\mathfrak{X}}(\mathbf{X}, \underset{\sim}{\mathbf{M}}_2)$. This defines a pair of natural transformations $e : \mathrm{id}_{\boldsymbol{\mathcal{A}}} \to \mathrm{ED}$ and $\varepsilon : \mathrm{id}_{\boldsymbol{\mathfrak{X}}} \to \mathrm{DE}$. Moreover, the construction of $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathfrak{X}}$ via $\mathbb{ISP}$ and $\mathbb{IS}_c^0\mathbb{P}^+$, respectively, ensures that the maps $e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ and $\varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ are embeddings, for all $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ and all $\mathbf{X} \in \boldsymbol{\mathfrak{X}}$. (For us, *embedding* in $\boldsymbol{\mathcal{A}}$ means 'isomorphism onto a substructure' and in $\boldsymbol{\mathfrak{X}}$ means 'isomorphism onto a topologically closed substructure'.)

The following theorem summarises the basic properties of this construction that we shall need later.

Theorem 2.2. *Let* $\mathbf{M}_1$ *be a finite structure, let* $\underset{\sim}{\mathbf{M}}_2$ *be an alter ego of* $\mathbf{M}_1$ *and define* $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M}_1)$ *and* $\boldsymbol{\mathfrak{X}} = \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$. *Then*

(i) *products in both* $\boldsymbol{\mathcal{A}}$ *and* $\boldsymbol{\mathfrak{X}}$ *are constructed pointwise,*
(ii) $\langle \mathrm{D}, \mathrm{E}, e, \varepsilon \rangle$*, as defined above, is a dual adjunction between* $\boldsymbol{\mathcal{A}}$ *and* $\boldsymbol{\mathfrak{X}}$*,*
(iii) *for all* $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ *and* $\mathbf{X} \in \boldsymbol{\mathfrak{X}}$*, the maps* $e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ *and* $\varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ *are embeddings,*

(iv) *for every non-empty set S, the natural bijection between* $\mathcal{A}(\mathbf{F}_{\mathbf{M}_1}(S), \mathbf{M}_1)$ *and* M^S *is an isomorphism between* $\mathrm{D}(\mathbf{F}_{\mathbf{M}_1}(S))$ *and* $\underset{\sim}{\mathbf{M}}_2^S$*,*

(v) *if* $u : \mathbf{A} \to \mathbf{B}$ *is a surjection in* $\mathcal{A}$*, then* $\mathrm{D}(u) : \mathrm{D}(\mathbf{B}) \to \mathrm{D}(\mathbf{A})$ *is an embedding in* $\mathcal{X}$*,*

(vi) *if* $\psi : \mathbf{X} \to \mathbf{Y}$ *is a surjection in* $\mathcal{X}$*, then* $\mathrm{E}(\psi) : \mathrm{E}(\mathbf{Y}) \to \mathrm{E}(\mathbf{X})$ *is an embedding in* $\mathcal{A}$*.*

Proof. Part (iv) follows from the fact that a dual adjunction maps coproducts to products, along with the additional fact that $\mathbf{F}_{\mathbf{M}_1}(S)$ is an S-fold coproduct of copies of $\mathbf{F}_{\mathbf{M}_1}(1)$ and $\mathrm{D}(\mathbf{F}_{\mathbf{M}_1}(1))$ is isomorphic to $\underset{\sim}{\mathbf{M}}_2$. The remaining parts of the theorem are straightforward calculations. □

If the map $e_{\mathbf{A}}$ is surjective and therefore an isomorphism, for all $\mathbf{A} \in \mathcal{A}$, then we say that the alter ego $\underset{\sim}{\mathbf{M}}_2$ *yields a duality on* $\mathcal{A}$ or simply that $\underset{\sim}{\mathbf{M}}_2$ *dualises* $\mathbf{M}_1$. If $\underset{\sim}{\mathbf{M}}_2$ yields a duality on $\mathcal{A}$, then $\mathcal{A}$ is dually equivalent to a full subcategory of the category $\mathcal{X}$. In this case, we have a representation for $\mathcal{A}$: each structure $\mathbf{A}$ in $\mathcal{A}$ is isomorphic to the structure $\mathrm{ED}(\mathbf{A})$ consisting of all continuous homomorphisms from $\mathrm{D}(\mathbf{A})$ to $\underset{\sim}{\mathbf{M}}_2$. If $\underset{\sim}{\mathbf{M}}_2$ yields a duality on $\mathcal{A}$ and, in addition, $\varepsilon_{\mathbf{X}}$ is surjective and therefore an isomorphism, for all $\mathbf{X}$ in $\mathcal{X}$, then we say that $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality on* $\mathcal{A}$ or, more fully, that $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $\mathcal{A}$ *and* $\mathcal{X}$ or, more simply, that $\underset{\sim}{\mathbf{M}}_2$ *fully dualises* $\mathbf{M}_1$. In this case, the functors D and E give a dual equivalence between the categories $\mathcal{A}$ and $\mathcal{X}$. If $\underset{\sim}{\mathbf{M}}_2$ yields a full duality between $\mathcal{A}$ and $\mathcal{X}$ and, moreover, $\underset{\sim}{\mathbf{M}}_2$ is injective in the category $\mathcal{X}$, then we say that $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality on* $\mathcal{A}$ or that $\underset{\sim}{\mathbf{M}}_2$ *strongly dualises* $\mathbf{M}_1$. (Recall that $\underset{\sim}{\mathbf{M}}_2$ is *injective* in a subclass $\mathfrak{C}$ of $\mathcal{X}$ if $\underset{\sim}{\mathbf{M}}_2 \in \mathfrak{C}$ and, for every embedding $\varphi : \mathbf{X} \to \mathbf{Y}$, with $\mathbf{X}, \mathbf{Y} \in \mathfrak{C}$, every morphism $\alpha : \mathbf{X} \to \underset{\sim}{\mathbf{M}}_2$ extends to a morphism $\beta : \mathbf{Y} \to \underset{\sim}{\mathbf{M}}_2$, that is, $\beta \circ \varphi = \alpha$.)

Remark 2.3. In the case that G_1 contains no nullary symbols, we may wish to include the empty structure $\varnothing_1$ of type $\langle G_1, H_1, R_1 \rangle$ in $\mathcal{A}$. In that case, we must also add (all isomorphic copies of) the complete structure $\mathbf{1}_2$ of type $\langle G_2, H_2, R_2 \rangle$ to $\mathcal{X}$. Thus, we would redefine $\mathcal{A}$ and $\mathcal{X}$ to be $\mathcal{A} := \mathbb{IS}^0\mathbb{P}(\mathbf{M}_1)$ and $\mathcal{X} := \mathbb{IS}_c^0\mathbb{P}(\underset{\sim}{\mathbf{M}}_2)$. Some care must be taken, as this simple change can destroy a duality. Indeed, assume that $\mathbf{M}_1$ has no nullary operations but does have constant total unary term functions. Then the empty structure $\varnothing_1$ and the substructure $\mathbf{C}_1^1$ of $\mathbf{M}_1$, consisting of the values of the constant total unary term functions of $\mathbf{M}_1$, satisfy $\mathrm{D}(\varnothing_1) \cong \mathrm{D}(\mathbf{C}_1^1) \cong \mathbf{1}_2$, with $\varnothing_1 \ncong \mathbf{C}_1^1$. See Lemma 6.2 below for further details.

3. Axiomatizing topological quasivarieties

Let $\mathbf{X} = \langle X; G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}, \mathcal{T}^{\mathbf{X}} \rangle$ be a structure of type $\langle G, H, R \rangle$ with a topology added. We say that $\mathbf{X}$ is a *Boolean topological structure* of type $\langle G, H, R \rangle$ if

- $\mathcal{T}^{\mathbf{X}}$ is a Boolean topology on X (that is, $\mathcal{T}^{\mathbf{X}}$ is compact, Hausdorff and has a basis of clopen sets),
- every relation in $R^{\mathbf{X}}$ and the domain of each partial operation in $H^{\mathbf{X}}$ is a closed subspace of the appropriate power of $\langle X; \mathcal{T}^{\mathbf{X}} \rangle$, and
- every map in $G^{\mathbf{X}} \cup H^{\mathbf{X}}$ is continuous with respect to $\mathcal{T}^{\mathbf{X}}$.

The following result, known as the Separation Theorem [5, 1.4.4], is the topological version of Lemma 1.2 above. While completely elementary, it is a basic tool when trying to describe the class $\mathbb{IS}_c^0\mathbb{P}^+(\mathcal{Y})$ generated by a class $\mathcal{Y}$ of Boolean topological structures.

Lemma 3.1. *Let $\mathcal{Y}$ be a non-empty set of Boolean topological structures of type $\langle G, H, R\rangle$ and let $\mathbf{X} = \langle X; G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}, \mathcal{T}^{\mathbf{X}}\rangle$ be a non-empty structure of the same type with a compact Hausdorff topology added.*

(i) *The complete one-element structure $\mathbf{1}$ belongs to $\mathbb{IS}_c^0\mathbb{P}^+(\mathcal{Y})$ if and only if some $\mathbf{Y} \in \mathcal{Y}$ has a substructure isomorphic to $\mathbf{1}$.*

(ii) *Assume that $\mathbf{X}$ is not isomorphic to the complete one-element structure $\mathbf{1}$. Then $\mathbf{X} \in \mathbb{IS}_c^0\mathbb{P}^+(\mathcal{Y})$ if and only if, for all $r \in \{=\} \cup \{\, \mathrm{dom}(h) \mid h \in H \,\} \cup R$ of arity n, and all $x_1, \dots, x_n \in X$ with $(x_1, \dots, x_n) \notin r^{\mathbf{X}}$, there exist $\mathbf{Y} \in \mathcal{Y}$ and a morphism $\psi : \mathbf{X} \to \mathbf{Y}$ such that $(\psi(x_1), \dots, \psi(x_n)) \notin r^{\mathbf{Y}}$.*

Recently, a number of authors have addressed the question of how to describe the topological quasivariety $\mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ generated by a finite discrete topological structure $\underset{\sim}{\mathbf{M}}$; see Clark, Davey, Haviar, Pitkethly and Talukder [7], Clark, Davey, Freese and Jackson [6], Davey and Talukder [19] and Clark, Davey, Jackson and Pitkethly [8]. A number of powerful techniques have been developed, but we shall restrict our attention to just one that will be particularly useful in the examples considered in Section 7.

Let Σ be a set of universal Horn sentences of type $\langle G, H, R\rangle$. A topological structure $\mathbf{X} = \langle X; G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}, \mathcal{T}^{\mathbf{X}}\rangle$ is a *Boolean topological model of* Σ if

- $\mathbf{X}$ is a Boolean topological structure of type $\langle G, H, R\rangle$ and
- the structure $\langle X; G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}\rangle$ is a model of Σ.

The class consisting of all non-empty Boolean topological models of Σ is denoted by $\mathrm{Mod}_{\mathrm{Bt}}(\Sigma)$, while $\mathrm{Mod}^0_{\mathrm{Bt}}(\Sigma)$ includes the empty structure if G contains no nullaries.

Let $\mathbf{M}$ be a finite structure and consider the topological quasivariety $\mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ generated by the corresponding discrete topological structure $\underset{\sim}{\mathbf{M}}$. By Theorem 1.1, there is a set Σ of universal Horn sentences with $\mathbb{ISP}^+(\mathbf{M}) = \mathrm{Mod}(\Sigma)$. Perhaps the simplest description of $\mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ arises when $\mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}^0_{\mathrm{Bt}}(\Sigma)$, that is, when $\mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ is precisely the class of (possibly empty) Boolean topological models of Σ. The papers referred to in the first paragraph of this section give many examples where this is true and many where it fails.

The following important positive result applies to all of our examples in Section 7. In order to state it, we need to make precise what we mean by a quotient of a total structure. Let $\mathbf{A} = \langle A; G, R\rangle$ be a total structure. A congruence θ on the algebraic reduct $\langle A; G\rangle$ of $\mathbf{A}$ determines a quotient structure $\mathbf{A}/\theta$: for each $r \in R$, we have $(a_1/\theta, \dots, a_m/\theta) \in r^{\mathbf{A}/\theta}$ provided there are $b_1, \dots, b_m \in A$ such that $(a_i, b_i) \in \theta$, for $i = 1, 2, \dots, m$, and $(b_1, \dots, b_m) \in r^{\mathbf{A}}$. We say that a class $\mathcal{C}$ of total structures is *closed under finite quotients* if, whenever $\mathbf{A} \in \mathcal{C}$ and θ is a finite-index congruence on the algebraic reduct of $\mathbf{A}$, the quotient structure $\mathbf{A}/\theta$ belongs to $\mathcal{C}$.

Theorem 3.2. [8, 2.13], [6, 4.3 and 6.9] *Let $\mathbf{M} = \langle M; G, R\rangle$ be a finite total structure. Assume that the quasivariety $\mathbb{ISP}(\mathbf{M})$ generated by $\mathbf{M}$ is closed under finite quotients and that the variety generated by the algebraic reduct of $\mathbf{M}$ is congruence distributive.*

(i) *If Σ is a set of universal Horn sentences such that $\mathbb{ISP}^+(\mathbf{M}) = \mathrm{Mod}(\Sigma)$, then $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}_{\mathrm{Bt}}(\Sigma)$ and $\mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}^0_{\mathrm{Bt}}(\Sigma)$.*

(ii) *If Σ is a set of quasi-atomic sentences such that $\mathbb{ISP}(\mathbf{M}) = \mathrm{Mod}(\Sigma)$, then $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}_{\mathrm{Bt}}(\Sigma)$ and $\mathbb{IS}_c^0\mathbb{P}(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}^0_{\mathrm{Bt}}(\Sigma)$.*

Remark 3.3. This is a particularly powerful result. It applies, in particular, to every finite algebra $\mathbf{M}$ with a lattice reduct such that $\mathbb{ISP}(\mathbf{M})$ is closed under homomorphic images. For example, when applied to the two-element bounded lattice $\mathbf{2}$, Theorem 3.2(ii) tells us that $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{2}})$ is the class consisting of all Boolean topological bounded distributive lattices, a result first proved by Numakura [29].

We would also like to be able to derive a first-order axiomatization of the quasivariety $\mathbb{ISP}(\mathbf{M})$ from a non-first-order description of the topological quasivariety $\mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$. The following simple observation will suffice.

Lemma 3.4. *Let $\mathbf{M} = \langle M; G, H, R\rangle$ be a finite structure and let Σ be a set of universal Horn sentences of type $\langle G, H, R\rangle$. Assume that every finitely generated model of Σ is finite and that $[\mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}})]_{\mathrm{fin}} = [\mathrm{Mod}^0_{\mathrm{Bt}}(\Sigma)]_{\mathrm{fin}}$. Then $\mathbb{ISP}^+(\mathbf{M}) = \mathrm{Mod}(\Sigma)$.*

Proof. It follows from the assumptions that $[\mathbb{ISP}^+(\mathbf{M})]_{\mathrm{fin}} = [\mathrm{Mod}(\Sigma)]_{\mathrm{fin}}$, and hence $\mathbb{ISP}^+(\mathbf{M}) = \mathrm{Mod}(\Sigma)$ by Lemma 1.6. ☐

Remark 3.5. Often we have a description of the topological quasivariety $\mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ of the form $\mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}^0_{\mathrm{Bt}}(\Sigma_0 \cup \Phi)$, where Φ is some non-first-order topological condition. If we can find some set Σ_1 of universal Horn sentences such that the finite models of $\Sigma_0 \cup \Sigma_1$ are precisely the finite models of $\Sigma_0 \cup \Phi$ and every finitely generated model of $\Sigma_0 \cup \Sigma_1$ is finite, then we have $\mathbb{ISP}^+(\mathbf{M}) = \mathrm{Mod}(\Sigma)$, where $\Sigma := \Sigma_0 \cup \Sigma_1$. For example, $\mathbf{X} \models \Phi$ might be the statement that $\mathbf{X} = \langle X; \leqslant, \mathcal{T}\rangle$ is a Priestley space, in which case the natural choice for Σ_1 would be the axioms for an ordered set.

4. Three basic duality theorems

In this section, we present generalisations of the three theorems that have been used to establish most natural dualities: the Second Duality Theorem, the Duality Compactness Theorem and the NU Duality Theorem (see Clark and Davey [5, 2.2.7, 2.2.11 and 2.3.4]). All three theorems are concerned with lifting up a duality from the finite level. Let $\mathbf{M}_1$ be a finite structure, let $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ and let $\underset{\sim}{\mathbf{M}}_2$ be an alter ego of $\mathbf{M}_1$. If $e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ is an isomorphism, for all $\mathbf{A} \in \boldsymbol{\mathcal{A}}_{\mathrm{fin}}$, then we say that $\underset{\sim}{\mathbf{M}}_2$ *yields a duality on* $\boldsymbol{\mathcal{A}}_{\mathrm{fin}}$, or that $\underset{\sim}{\mathbf{M}}_2$ *yields a duality at the finite level*; we also say that $\underset{\sim}{\mathbf{M}}_2$ *dualises* $\mathbf{M}_1$ *at the finite level*.

Now assume that $\mathbf{M}_1$ is an algebra and define $\boldsymbol{\mathcal{X}} := \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$. The Second Duality Theorem is due to Davey and Werner [20]. It says that, if $\underset{\sim}{\mathbf{M}}_2$ has no partial operations and only a finite number of relations in its type and $\underset{\sim}{\mathbf{M}}_2$ yields a duality at the finite level and is injective in $\boldsymbol{\mathcal{X}}_{\mathrm{fin}}$, then $\underset{\sim}{\mathbf{M}}_2$ yields a duality on $\boldsymbol{\mathcal{A}}$ and is injective in $\boldsymbol{\mathcal{X}}$. The Duality Compactness Theorem is due independently to Willard [33] and Zádori [34]. It says that, if $\underset{\sim}{\mathbf{M}}_2$ is of finite type and yields a duality at the finite level, then $\underset{\sim}{\mathbf{M}}_2$ yields a duality on $\boldsymbol{\mathcal{A}}$. The NU Duality Theorem was proved by Davey and Werner [20] and tells us that if $\mathbf{M}_1$ has a $(k{+}1)$-ary near-unanimity term, then the purely relational alter ego $\underset{\sim}{\mathbf{M}}_2 := \langle M; R, \mathcal{T}\rangle$, where R is the set of all non-empty subuniverses of $\mathbf{M}_1^k$, yields a duality on $\boldsymbol{\mathcal{A}}$.

The proofs of these theorems given in Clark and Davey [5] extend with only the obvious changes (replace *algebra* by *structure*, etc) to the case where $\mathbf{M}_1$ is an arbitrary finite structure, though in the case of the NU Duality Theorem we need to assume that $\mathbf{M}_1$ is a total structure. We state the structure-theoretic versions of the required results from Chapter 2 of [5], and refer to [5] for the proofs. In each case, we indicate the corresponding result in [5] in square brackets at the start of the statement.

Let $\mathbf{M}_1 = \langle M; G_1, H_1, R_1\rangle$ be a finite structure, let $\underset{\sim}{\mathbf{M}}_2 = \langle M; G_2, H_2, R_2, \mathfrak{T}\rangle$ be an alter ego of $\mathbf{M}_1$, and let $\langle \mathrm{D}, \mathrm{E}, e, \varepsilon\rangle$ be the induced dual adjunction between $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ and $\boldsymbol{\mathfrak{X}} := \mathbb{IS}_{\mathrm{c}}^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$. We say that (CLO) *holds*, or more precisely, that $\mathbf{M}_2$ *satisfies* (CLO) *with respect to* $\mathbf{M}_1$ if

(CLO) for each $n \in \mathbb{N}$, every homomorphism $t : \mathbf{M}_2^n \to \mathbf{M}_2$ is a (total) n-ary term function of $\mathbf{M}_1$.

The fact that $\mathbf{M}_1$ and $\mathbf{M}_2$ are compatible guarantees that, for every non-empty set S, each total S-ary term function of $\mathbf{M}_1$ is a morphism from $\underset{\sim}{\mathbf{M}}_2^S$ to $\underset{\sim}{\mathbf{M}}_2$. Thus (CLO) says exactly that, for all $n \in \mathbb{N}$, the total n-ary term functions of $\mathbf{M}_1$ and the homomorphisms from $\mathbf{M}_2^n$ to $\mathbf{M}_2$ agree, that is, the structure $\mathbf{M}_2$ determines the clone of total finitary term functions of the structure $\mathbf{M}_1$. The addition of the discrete topology to $\mathbf{M}_2$ extends this to arbitrary non-zero arities.

Theorem 4.1. [5, 2.2.3] *Let $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ be the quasivariety generated by the finite structure $\mathbf{M}_1$ and let $\underset{\sim}{\mathbf{M}}_2$ be an alter ego of $\mathbf{M}_1$.*

(i) *Fix a non-empty set S and let $\mathbf{F} := \mathbf{F}_{\mathbf{M}_1}(S)$. The map $e_{\mathbf{F}} : \mathbf{F} \to \mathrm{ED}(\mathbf{F})$ is an isomorphism if and only if*

(CLO)$_S$ *every morphism $t : \underset{\sim}{\mathbf{M}}_2^S \to \underset{\sim}{\mathbf{M}}_2$ is a (total) S-ary term function of $\mathbf{M}_1$.*

(ii) *The following are equivalent:*
  1. (CLO) *holds;*
  2. (CLO)$_S$ *holds, for every non-empty set S;*
  3. $e_{\mathbf{F}} : \mathbf{F} \to \mathrm{ED}(\mathbf{F})$ *is an isomorphism, for every finitely generated $\boldsymbol{\mathcal{A}}$-free structure $\mathbf{F}$;*
  4. $e_{\mathbf{F}} : \mathbf{F} \to \mathrm{ED}(\mathbf{F})$ *is an isomorphism, for every $\boldsymbol{\mathcal{A}}$-free structure $\mathbf{F}$.*

We note that the First Duality Theorem [5, 2.2.2] holds in the present setting provided, as above, *term function of* $\mathbf{M}_1$ is replaced by *total term function of* $\mathbf{M}_1$. While this theorem has the advantage that it gives necessary and sufficient conditions for $\underset{\sim}{\mathbf{M}}_2$ to yield a duality on $\boldsymbol{\mathcal{A}}$, we will not state it here as it is rarely used in practice. Instead, we state a corollary of the First Duality Theorem that provides sufficient conditions for $\underset{\sim}{\mathbf{M}}_2$ to dualise $\mathbf{M}_1$.

Theorem 4.2. [5, 2.2.2] *Let $\mathbf{M}_1$ be a finite structure, let $\underset{\sim}{\mathbf{M}}_2$ be an alter ego of $\mathbf{M}_1$ and define $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ and $\boldsymbol{\mathfrak{X}} := \mathbb{IS}_{\mathrm{c}}^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$. Then $\underset{\sim}{\mathbf{M}}_2$ yields a duality on $\boldsymbol{\mathcal{A}}$ provided* (CLO) *holds and $\underset{\sim}{\mathbf{M}}_2$ is injective in $\boldsymbol{\mathfrak{X}}$.*

By combining (CLO) with the injectivity of $\underset{\sim}{\mathbf{M}}_2$ in $\boldsymbol{\mathfrak{X}}_{\mathrm{fin}}$, we obtain a natural interpolation condition. We say that $\mathbf{M}_2$ *satisfies the interpolation condition* (IC) *with respect to* $\mathbf{M}_1$ if

(IC) for each $n \in \mathbb{N}$ and each substructure $\mathbf{X}$ of $\mathbf{M}_2^n$, every homomorphism $\alpha : \mathbf{X} \to \mathbf{M}_2$ extends to a total n-ary term function of $\mathbf{M}_1$.

This condition is sufficient to guarantee that $\underset{\sim}{\mathbf{M}}_2$ dualises $\mathbf{M}_1$ at the finite level.

IC Lemma 4.3. [5, 2.2.5] *Let $\mathbf{M}_1$ be a finite structure, let $\underset{\sim}{\mathbf{M}}_2$ be an alter ego of $\mathbf{M}_1$ and define $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ and $\boldsymbol{\mathfrak{X}} := \mathbb{IS}_{\mathrm{c}}^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$. The following are equivalent:*

1. $\underset{\sim}{\mathbf{M}}_2$ *yields a duality on $\boldsymbol{\mathcal{A}}_{\mathrm{fin}}$ and is injective in $\boldsymbol{\mathfrak{X}}_{\mathrm{fin}}$;*

(2) $\mathbf{M}_2$ *satisfies* (CLO) *with respect to* $\mathbf{M}_1$ *and* $\underset{\sim}{\mathbf{M}}_2$ *is injective in* $\boldsymbol{\mathcal{X}}_{\text{fin}}$*;*
(3) $\mathbf{M}_2$ *satisfies* (IC) *with respect to* $\mathbf{M}_1$*.*

Assume that $\mathbf{M}_2$ satisfies (IC) with respect to $\mathbf{M}_1$. By Theorem 4.2, to show that the duality on $\boldsymbol{\mathcal{A}}_{\text{fin}}$ lifts to a duality on $\boldsymbol{\mathcal{A}}$, we need to know that the injectivity of $\underset{\sim}{\mathbf{M}}_2$ in $\boldsymbol{\mathcal{X}}$ follows from its injectivity in $\boldsymbol{\mathcal{X}}_{\text{fin}}$.

Injectivity Lifting Lemma 4.4. [5, 2.2.7] *Let* $\mathbf{M} = \langle M; G, R\rangle$ *be a finite total structure with* R *finite and define* $\boldsymbol{\mathcal{X}} := \mathbb{IS}_{\mathrm{c}}^{0}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}})$*. If* $\underset{\sim}{\mathbf{M}}$ *is injective in* $\boldsymbol{\mathcal{X}}_{\text{fin}}$*, then* $\underset{\sim}{\mathbf{M}}$ *is injective in* $\boldsymbol{\mathcal{X}}$*.*

Combining this result with the previous two yields our first major lift-from-the-finite-level duality theorem.

Second Duality Theorem 4.5. [5, 2.2.7] *Let* $\mathbf{M}_1$ *be a finite structure and let* $\underset{\sim}{\mathbf{M}}_2 = \langle M; G_2, R_2, \mathcal{T}\rangle$ *be an alter ego of* $\mathbf{M}_1$ *that is a total structure with* R_2 *finite. Define* $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ *and* $\boldsymbol{\mathcal{X}} := \mathbb{IS}_{\mathrm{c}}^{0}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}}_2)$*. If* $\mathbf{M}_2$ *satisfies* (IC) *with respect to* $\mathbf{M}_1$*, then* $\underset{\sim}{\mathbf{M}}_2$ *yields a duality on* $\boldsymbol{\mathcal{A}}$ *and is injective in* $\boldsymbol{\mathcal{X}}$*.*

We turn now to the Duality Compactness Theorem. The following lemma is proved by an easy application of the fact that the inverse limit of an inverse system of non-empty finite sets is non-empty.

Lemma 4.6. [5, 2.2.9] *Let* $\mathbf{B}$ *be a non-empty substructure of a locally finite structure* $\mathbf{A}$*, let* $\mathbf{D}$ *be a finite structure and let* $h : \mathbf{B} \to \mathbf{D}$ *be a homomorphism. If, for every finite substructure* $\mathbf{F}$ *of* $\mathbf{A}$ *that intersects* $\mathbf{B}$*, there is a homomorphism* $k : \mathbf{F} \to \mathbf{D}$ *that agrees with* h *on* $B \cap F$*, then there is a homomorphism* $g : \mathbf{A} \to \mathbf{D}$ *that extends* h*.*

We state the following immediate corollary more generally than it is stated in [5]. While this corollary is not needed in the proof of the generalised Duality Compactness Theorem, we include it because of the important role that injectivity plays in the theory of natural dualities.

Corollary 4.7. [5, 2.2.10] *Let* $\boldsymbol{\mathcal{A}}$ *be a locally finite class of structures and assume that* $\boldsymbol{\mathcal{A}}$ *is closed under forming substructures. If* $\mathbf{D}$ *is injective in* $\boldsymbol{\mathcal{A}}_{\text{fin}}$*, then* $\mathbf{D}$ *is injective in* $\boldsymbol{\mathcal{A}}$*.*

Let $\mathbf{M}$ be a finite structure and let $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M})$. By Lemma 1.5, the quasi-variety $\boldsymbol{\mathcal{A}}$ is locally finite. It follows from Corollary 4.7 that if $\mathbf{D}$ is injective in $\boldsymbol{\mathcal{A}}_{\text{fin}}$, then $\mathbf{D}$ is injective in $\boldsymbol{\mathcal{A}}$. Similarly, if $\mathbf{M}$ is a finite total structure and $\mathbf{D}$ is injective in $\mathbb{HSP}(\mathbf{M})_{\text{fin}}$, then $\mathbf{D}$ is injective in $\mathbb{HSP}(\mathbf{M})$.

The conversion of the proof of the Duality Compactness Theorem given in Clark and Davey [5] to the present setting is a minor search-and-replace exercise and is left to the reader. Recall that a structure $\mathbf{M} = \langle M; G, H, R\rangle$ is of *finite type* if $G \cup H \cup R$ is finite.

Duality Compactness Theorem 4.8. [5, 2.2.11] *Let* $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ *be the quasi-variety generated by the finite structure* $\mathbf{M}_1$*. If* $\underset{\sim}{\mathbf{M}}_2$ *is an alter ego of* $\mathbf{M}_1$ *of finite type that yields a duality at the finite level, then* $\underset{\sim}{\mathbf{M}}_2$ *yields a duality on* $\boldsymbol{\mathcal{A}}$*.*

This theorem is a special case of Hofmann's Theorem 2.3, [27]. Combining the IC Lemma 4.3 with the Duality Compactness Theorem yields the following immediate corollary.

IC Duality Theorem 4.9. *Let $\mathcal{A} := \mathbb{ISP}(\mathbf{M}_1)$ be the quasivariety generated by the finite structure $\mathbf{M}_1$ and let $\underset{\sim}{\mathbf{M}}_2$ be an alter ego of $\mathbf{M}_1$ of finite type. If $\mathbf{M}_2$ satisfies* (IC) *with respect to $\mathbf{M}_1$, then $\underset{\sim}{\mathbf{M}}_2$ yields a duality on $\mathcal{A}$.*

Some care is required when extending other basic results in the theory of natural dualities to this more general setting. For example, in the case that $\mathbf{M}_1$ is an algebra, it is a common practice when seeking a duality to work interchangeably with partial operations and their graphs in the type of the alter ego $\underset{\sim}{\mathbf{M}}_2$. If $\mathbf{M}_1$ includes partial operations or relations in its type, this is no longer possible as the graph of a partial operation can be compatible with $\mathbf{M}_1$ while the partial operation itself is not.

Another common trick that is used in the case that $\mathbf{M}_1$ is an algebra is to take a finite number of homomorphisms $x_1, \dots, x_n : \mathbf{A} \to \mathbf{M}_1$, for some $\mathbf{A} \in \mathbb{ISP}(\mathbf{M}_1)$, and then use image of the product map $x_1 \sqcap \cdots \sqcap x_n : \mathbf{A} \to \mathbf{M}_1^n$, that is, the n-ary relation $\{\, (x_1(a), \dots, x_n(a)) \mid a \in A \,\}$, in an alter ego of $\mathbf{M}_1$. While this trick is still available when $\mathbf{M}_1$ is a total structure, it cannot be used when the type of $\mathbf{M}_1$ includes partial operations, as the image of $x_1 \sqcap \cdots \sqcap x_n$ may not be a substructure of $\mathbf{M}_1^n$. Two important results whose proofs utilise this trick are the Brute Force Duality Theorem [5, 2.3.1] and the NU Duality Theorem [5, 2.3.4]. These theorems continue to hold when $\mathbf{M}_1$ is a total structure.

We close this section with the statement of the NU Duality Theorem as it applies to total structures. For $n \geqslant 3$, a function $t : M^n \to M$ is a *near unanimity function* on the set M if it satisfies $t(a, \dots, a, b) = t(a, \dots, a, b, a) = \cdots = t(b, a, \dots, a) = a$, for all $a, b \in M$.

NU Duality Theorem 4.10. [5, 2.3.4] *Let $k \geqslant 2$ and assume that $\mathbf{M}_1$ is a finite total structure that has a $(k+1)$-ary near unanimity term function. Let $\underset{\sim}{\mathbf{M}}_2 = \langle M; R \rangle$, where R is the set of all non-empty subuniverses of $\mathbf{M}_1^k$, and define $\mathcal{A} := \mathbb{ISP}(\mathbf{M}_1)$ and $\mathfrak{X} := \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$. Then $\underset{\sim}{\mathbf{M}}_2$ satisfies* (IC) *with respect to $\mathbf{M}_1$, yields a duality on $\mathcal{A}$ and is injective in $\mathfrak{X}$.*

5. Lifting full duality up from the finite level

The theorems in the previous section give conditions under which a duality for the class $\mathcal{A}_{\text{fin}}$ can be lifted up to a duality for the class $\mathcal{A}$. In this section we turn our attention to finding conditions under which a full duality for $\mathcal{A}_{\text{fin}}$ can be lifted up to a full duality for $\mathcal{A}$. The results are a refinement and simplification, in our restricted setting, of the presentation given by Hofmann [27].

Let $\mathbf{M}_1$ be a finite structure, let $\underset{\sim}{\mathbf{M}}_2$ be an alter ego of $\mathbf{M}_1$ and consider the classes $\mathcal{A} := \mathbb{ISP}(\mathbf{M}_1)$ and $\mathfrak{X} := \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$. If $e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ is an isomorphism, for all $\mathbf{A} \in \mathcal{A}_{\text{fin}}$, and $\varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ is an isomorphism, for all $\mathbf{X} \in \mathfrak{X}_{\text{fin}}$, then we say that $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between $\mathcal{A}_{\text{fin}}$ and $\mathfrak{X}_{\text{fin}}$*, or simply that $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality at the finite level*, or that $\underset{\sim}{\mathbf{M}}_2$ *fully dualises $\mathbf{M}_1$ at the finite level*. In this case, the functors D and E yield a dual category equivalence between the categories $\mathcal{A}_{\text{fin}}$ and $\mathfrak{X}_{\text{fin}}$. If $\underset{\sim}{\mathbf{M}}_2$ yields a full duality between $\mathcal{A}_{\text{fin}}$ and $\mathfrak{X}_{\text{fin}}$ and, moreover, $\underset{\sim}{\mathbf{M}}_2$ is injective in $\mathfrak{X}_{\text{fin}}$, then we say that $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality between $\mathcal{A}_{\text{fin}}$ and $\mathfrak{X}_{\text{fin}}$*, or simply that $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality at the finite level*, or that $\underset{\sim}{\mathbf{M}}_2$ *strongly dualises $\mathbf{M}_1$ at the finite level*.

The class $\mathfrak{X}$ is closed under forming inverse limits. Indeed, let $\mathbf{S} = \langle S; \leqslant \rangle$ be a non-empty directed ordered set and let $\{\, \mathbf{X}_s \mid s \in S \,\}$ be an inverse system in $\mathfrak{X}$

with connecting morphisms $\eta_{st} : \mathbf{X}_s \to \mathbf{X}_t$, for all $s \geqslant t$ in $\mathbf{S}$. Then the inverse limit in $\boldsymbol{\mathcal{X}}$ of the system is the closed substructure of $\prod_{s\in S} \mathbf{X}_s$ on the set

$$\{\, x \in \textstyle\prod_{s\in S} X_s \mid (\forall s, t \in S)\ s \geqslant t \implies \eta_{st}(x(s)) = x(t) \,\}$$

and is denoted by $\varprojlim_{s\in S} \mathbf{X}_s$. For a subclass $\boldsymbol{\mathcal{Y}}$ of $\boldsymbol{\mathcal{X}}$, we shall use the notation $\varprojlim \boldsymbol{\mathcal{Y}}$ to denote the full subcategory of $\boldsymbol{\mathcal{X}}$ whose objects are (isomorphic copies of) inverse limits of structures in $\boldsymbol{\mathcal{Y}}$. The following lemma is a simple piece of category theory and can be formulated much more generally (see, for example, Banaschewski [2] and Hofmann [27]).

Lemma 5.1. *Let* $\mathbf{M}_1$ *be a finite structure and let* $\underset{\sim}{\mathbf{M}}_2$ *be an alter ego of* $\mathbf{M}_1$*. Define* $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ *and* $\boldsymbol{\mathcal{X}} := \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$ *and let* $\boldsymbol{\mathcal{Y}}$ *be a subclass of* $\boldsymbol{\mathcal{X}}$*. Assume that* $\underset{\sim}{\mathbf{M}}_2$ *yields a duality on* $\boldsymbol{\mathcal{A}}$ *and that* $\varepsilon_{\mathbf{Y}} : \mathbf{Y} \to \mathrm{DE}(\mathbf{Y})$ *is an isomorphism, for all* $\mathbf{Y} \in \boldsymbol{\mathcal{Y}}$*. Then* $\varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ *is an isomorphism, for all* $\mathbf{X} \in \varprojlim \boldsymbol{\mathcal{Y}}$*.*

Proof. Let $\mathbf{X} \in \varprojlim \boldsymbol{\mathcal{Y}}$. To prove that $\varepsilon_{\mathbf{X}}$ is an isomorphism, it suffices to show that $\mathbf{X} \cong \mathrm{D}(\mathbf{A})$, for some structure $\mathbf{A} \in \boldsymbol{\mathcal{A}}$. Indeed, if $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ and $\varphi : \mathbf{X} \to \mathrm{D}(\mathbf{A})$ is an isomorphism, then since $\langle \mathrm{D}, \mathrm{E}, e, \varepsilon \rangle$ is a dual adjunction between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$, we have $\varphi = \mathrm{D}(\mathrm{E}(\varphi) \circ e_{\mathbf{A}}) \circ \varepsilon_{\mathbf{X}}$ (see the triangular commutative diagrams on page 5 of Clark and Davey [5], for example). Since φ and $e_{\mathbf{A}}$ are isomorphisms it follows immediately that $\varepsilon_{\mathbf{X}}$ is also an isomorphism.

We have $\mathbf{X} = \varprojlim_{s\in S} \mathbf{Y}_s$, for some inverse system $\{\, \mathbf{Y}_s \mid s \in S \,\}$ in $\boldsymbol{\mathcal{Y}} \subseteq \boldsymbol{\mathcal{X}}$ with connecting morphisms $\eta_{st} : \mathbf{Y}_s \to \mathbf{Y}_t$. The structures $\mathrm{E}(\mathbf{Y}_s)$ and the connecting maps $\mathrm{E}(\eta_{st})$, with $t \leqslant s$, form a direct system of structures in $\boldsymbol{\mathcal{A}}$. Let $\mathbf{A} := \varinjlim_{s\in S} \mathrm{E}(\mathbf{Y}_s)$ be the direct limit calculated in the quasivariety $\boldsymbol{\mathcal{A}}$. Since $\langle \mathrm{D}, \mathrm{E}, e, \varepsilon \rangle$ is a dual adjunction between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$, the functor D maps direct limits in $\boldsymbol{\mathcal{A}}$ to inverse limits in $\boldsymbol{\mathcal{X}}$ (see Mac Lane [28, V.5]). Thus

$$\mathrm{D}(\mathbf{A}) = \mathrm{D}(\varinjlim_{s\in S} \mathrm{E}(\mathbf{Y}_s)) \cong \varprojlim_{s\in S} \mathrm{DE}(\mathbf{Y}_s) \cong \varprojlim_{s\in S} \mathbf{Y}_s = \mathbf{X},$$

as $\mathbf{Y}_s \in \boldsymbol{\mathcal{Y}}$ and therefore $\mathrm{DE}(\mathbf{Y}_s) \cong \mathbf{Y}_s$. □

We would like to be able to use this observation to lift a full duality at the finite level up to a full duality between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$. The first step in this process is another general category-theoretic observation that can be stated more generally.

Lifting Full Duality Lemma 5.2. *Let* $\mathbf{M}_1$ *be a finite structure and let* $\underset{\sim}{\mathbf{M}}_2$ *be an alter ego of* $\mathbf{M}_1$*. Define* $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ *and* $\boldsymbol{\mathcal{X}} := \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$ *and assume that* $\underset{\sim}{\mathbf{M}}_2$ *yields a duality on* $\boldsymbol{\mathcal{A}}$*. Then* $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $\boldsymbol{\mathcal{A}}$ *and* $\boldsymbol{\mathcal{X}}$ *if and only if* $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality at the finite level and* $\boldsymbol{\mathcal{X}} = \varprojlim \boldsymbol{\mathcal{X}}_{\mathrm{fin}}$*.*

Proof. First assume that the alter ego $\underset{\sim}{\mathbf{M}}_2$ yields a full duality between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$. Then it certainly yields a full duality between $\boldsymbol{\mathcal{A}}_{\mathrm{fin}}$ and $\boldsymbol{\mathcal{X}}_{\mathrm{fin}}$. By Lemma 1.4, every structure in $\boldsymbol{\mathcal{A}}$ is the direct limit of its finitely generated substructures. As $\boldsymbol{\mathcal{A}}$ is locally finite (by Lemma 1.5), every structure in $\boldsymbol{\mathcal{A}}$ is therefore a direct limit of structures from $\boldsymbol{\mathcal{A}}_{\mathrm{fin}}$. Since the functors D and E give a dual equivalence between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$, it follows that each structure $\mathbf{X}$ in $\boldsymbol{\mathcal{X}}$ is an inverse limit of structures from $\boldsymbol{\mathcal{X}}_{\mathrm{fin}}$. Thus, the forward direction holds. The backward direction is an immediate consequence of the previous lemma. □

In order to lift a full duality at the finite level up to a full duality between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$, we now need answers to the following two questions.

(I) If the alter ego $\underset{\sim}{\mathbf{M}}_2$ yields a duality between $\boldsymbol{\mathcal{A}}_{\text{fin}}$ and $\boldsymbol{\mathcal{X}}_{\text{fin}}$, does it follow that it yields a duality between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$?

(II) If the alter ego $\underset{\sim}{\mathbf{M}}_2$ yields a duality between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$ and a full duality between $\boldsymbol{\mathcal{A}}_{\text{fin}}$ and $\boldsymbol{\mathcal{X}}_{\text{fin}}$, does it follow that $\boldsymbol{\mathcal{X}} = \varprojlim \boldsymbol{\mathcal{X}}_{\text{fin}}$?

The Duality Compactness Theorem 4.8 tells us that the answer to (I) is 'yes', provided the type of $\underset{\sim}{\mathbf{M}}_2$ is finite. A restriction on the alter ego is necessary here. For example, if $\mathbf{I}$ is the two-element implication algebra, then no alter ego yields a duality on the class $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{I})$, yet the alter ego consisting of all finitary relations that are algebraic over $\mathbf{I}$ yields a duality at the finite level (see [5] for details). Clark, Davey, Jackson and Pitkethly [8, Corollary 2.4] prove that the answer to (II) is 'yes' in the case that $\underset{\sim}{\mathbf{M}}_2 = \langle M; G, R, \mathcal{T}\rangle$ is a total structure. Thus we obtain the following result that can be viewed as a limited *Full Duality Compactness Theorem.*

Total Structure Full Duality Theorem 5.3. *Let $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ be the quasi-variety generated by the finite structure $\mathbf{M}_1$ and let $\underset{\sim}{\mathbf{M}}_2 = \langle M; G, R, \mathcal{T}\rangle$ be an alter ego of $\mathbf{M}_1$ that is a total structure.*

(i) *If $\underset{\sim}{\mathbf{M}}_2$ yields a duality on $\boldsymbol{\mathcal{A}}$ and yields a full duality at the finite level, then $\underset{\sim}{\mathbf{M}}_2$ yields a full duality on $\boldsymbol{\mathcal{A}}$.*

(ii) *If $\underset{\sim}{\mathbf{M}}_2$ is of finite type and yields a full duality at the finite level, then $\underset{\sim}{\mathbf{M}}_2$ yields a full duality on $\boldsymbol{\mathcal{A}}$.*

Proof. Part (i) follows immediately from the previous lemma and the fact, proved in [8], that $\boldsymbol{\mathcal{X}} = \varprojlim \boldsymbol{\mathcal{X}}_{\text{fin}}$ provided the type of $\underset{\sim}{\mathbf{M}}_2$ includes no partial operations. The Duality Compactness Theorem 4.8 guarantees that, if the type of $\underset{\sim}{\mathbf{M}}_2$ is finite, then a duality at the finite level lifts to a duality on $\boldsymbol{\mathcal{A}}$. Thus, (ii) follows from (i). □

This is a special case of Theorem 2.5 in Hofmann's paper [27]. Where we have assumed that $\underset{\sim}{\mathbf{M}}_2$ is a total structure, Hofmann assumes that $\boldsymbol{\mathcal{X}}$ has Sur-Inj factorizations. This amounts to assuming that the image of every morphism in $\boldsymbol{\mathcal{X}}$ is a substructure, a condition obviously guaranteed by our assumption that $\underset{\sim}{\mathbf{M}}_2$ is a total structure. In fact, in the case that $\mathbf{M}_1$ is an algebra, Exercise 6.5 of Clark and Davey [5] shows that, in the presence of a full duality, the image of every morphism in $\boldsymbol{\mathcal{X}}$ is a substructure if and only if the alter ego $\underset{\sim}{\mathbf{M}}_2$ is structurally equivalent to a total structure.

As usual, the true role of partial operations in the proof of Theorem 5.3 is somewhat mysterious. The following example shows that, in the presence of partial operations, the answer to (II) can be 'no': it is possible for $\underset{\sim}{\mathbf{M}}_2$ to yield a duality on $\mathbb{ISP}(\mathbf{M}_1)$ and yield a full duality at the finite level and yet satisfy $\varprojlim \boldsymbol{\mathcal{X}}_{\text{fin}} \subsetneq \boldsymbol{\mathcal{X}}$.

Let $\mathbf{3}_L = \langle \{0, a, 1\}; \vee, \wedge, 0, 1\rangle$ be the three-element bounded lattice. Then $\boldsymbol{\mathcal{D}}^{01} := \mathbb{ISP}(\mathbf{3}_L)$ is the class of all bounded distributive lattices. The non-identity endomorphisms of $\mathbf{3}_L$ are f and g, given by

$$f(0) = f(a) = 0,\ f(1) = 1 \quad \text{and} \quad g(0) = 0,\ g(a) = g(1) = 1.$$

Davey, Haviar and Priestley [15] proved that $\underset{\sim}{\mathbf{3}}_{fg} := \langle \{0, a, 1\}; f, g, \mathcal{T}\rangle$ yields a duality on the class $\boldsymbol{\mathcal{D}}^{01}$ of bounded distributive lattices. Subsequently, Davey, Haviar and Willard [17] proved that the alter ego $\underset{\sim}{\mathbf{3}}_{fgh} := \langle \{0, a, 1\}; f, g, h, \mathcal{T}\rangle$,

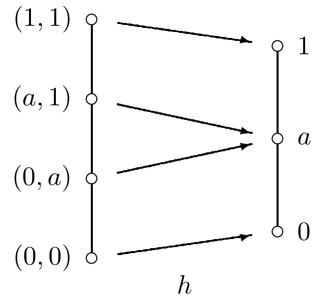

FIGURE 1. The partial operation h

where h is the binary partial operation shown in Figure 1, yields a duality on $\mathbf{\mathcal{D}}^{01}$ that is full at the finite level (but not strong at the finite level).

Example 5.4. *Let* $\mathbf{3}_L = \langle \{0, a, 1\}; \vee, \wedge, 0, 1\rangle$ *and* $\underset{\sim}{\mathbf{3}}_{fgh} = \langle \{0, a, 1\}; f, g, h, \mathcal{T}\rangle$ *be as above and let* $\mathbf{\mathcal{X}} := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{3}}_{fgh})$*. Then*

(i) $\underset{\sim}{\mathbf{3}}_{fgh}$ *yields a duality on the class* $\mathbf{\mathcal{D}}^{01}$ *of bounded distributive lattices,*
(ii) $\underset{\sim}{\mathbf{3}}_{fgh}$ *yields a full duality on the class* $\mathbf{\mathcal{D}}^{01}_{\mathrm{fin}}$ *of finite bounded distributive lattices, but*
(iii) *not every topological structure in* $\mathbf{\mathcal{X}}$ *is an inverse limit of finite structures in* $\mathbf{\mathcal{X}}$*.*

Proof. Since (i) and (ii) are proved in [15] and [17], we turn to (iii). We shall utilise a construction used in [17]. Let $\mathbf{X} \in \mathbf{\mathcal{X}}$ and define $P_{\mathbf{X}} := \mathrm{fix}(f) = \mathrm{fix}(g) \subseteq X$. Endow $P_{\mathbf{X}}$ with the subspace topology and define a binary relation $\preccurlyeq$ on $P_{\mathbf{X}}$ by

$$u \preccurlyeq v \iff (\exists x \in X)\, f(x) = u \;\;\&\;\; g(x) = v.$$

Davey, Haviar and Willard [17] proved that the structure $\mathbf{P_X} := \langle P_{\mathbf{X}}; \preccurlyeq, \mathcal{T}\rangle$ is an ordered Boolean space, that is, $\langle P_{\mathbf{X}}; \preccurlyeq\rangle$ is an ordered set, $\mathcal{T}$ is a Boolean topology on $P_{\mathbf{X}}$ and $\preccurlyeq$ is a closed subset of $P_{\mathbf{X}} \times P_{\mathbf{X}}$. In fact, it is easily seen that $F : \mathbf{X} \mapsto \mathbf{P_X}$ is (the object half of) a functor from $\mathbf{\mathcal{X}}$ to the category $\mathbf{\mathcal{Z}}_{\leqslant}$ of ordered Boolean spaces. Since f (and g) are calculated pointwise in a product of structures from $\mathbf{\mathcal{X}}$ and since inverse limits are calculated pointwise in both $\mathbf{\mathcal{X}}$ and $\mathbf{\mathcal{Z}}_{\leqslant}$, it follows by a simple calculation that F preserves inverse limits. Let $\mathbf{X} = \varprojlim_{s\in S} \mathbf{X}_s$ be an inverse limit in $\mathbf{\mathcal{X}}$ with $\mathbf{X}_s \in \mathbf{\mathcal{X}}_{\mathrm{fin}}$, for all $s \in S$. Then

$$\mathbf{P_X} = F(\mathbf{X}) = F(\varprojlim_{s\in S} \mathbf{X}_s) \cong \varprojlim_{s\in S} F(\mathbf{X}_s) = \varprojlim_{s\in S} \mathbf{P}_{\mathbf{X}_s}.$$

Thus, since an inverse limit of finite ordered sets is a Priestley space, it follows that $\mathbf{P_X}$ is a Priestley space, for all $\mathbf{X} \in \varprojlim \mathbf{\mathcal{X}}_{\mathrm{fin}}$. In [17] an example is given of a structure $\mathbf{Y}$ in $\mathbf{\mathcal{X}}$ for which the ordered Boolean space $\mathbf{P_Y}$ is not a Priestley space. The argument just given shows that $\mathbf{Y}$ does not belong to $\varprojlim \mathbf{\mathcal{X}}_{\mathrm{fin}}$. $\square$

The following observation adds to the mystery. While $H = \varnothing$ is a sufficient condition for $\mathbf{\mathcal{X}} = \varprojlim \mathbf{\mathcal{X}}_{\mathrm{fin}}$, it is certainly not necessary. Let $\mathbf{M}_1$ be a finite, strongly dualisable algebra that is not injective in the quasivariety it generates. For example, let $\mathbf{M}_1$ be the four-element Heyting chain (see Example 7.5 and the discussion preceding it). Let $\underset{\sim}{\mathbf{M}}_2$ be an alter ego that strongly dualises $\mathbf{M}_1$. The general theory tells us that $\underset{\sim}{\mathbf{M}}_2$ *must* have partial operations in its type (see the Total Structure Theorem [5, 6.1.2]). Since $\underset{\sim}{\mathbf{M}}_2$ yields a full duality on $\mathbb{ISP}(\mathbf{M}_1)$, Lemma 5.2 guarantees that $\boldsymbol{\mathcal{X}} = \varprojlim \boldsymbol{\mathcal{X}}_{\text{fin}}$.

We close this section with a result that gives conditions under which a strong duality can be lifted up from the finite level. Note that by using the Second Duality Theorem 4.5 instead of the Duality Compactness Theorem 4.8, we have weakened the assumption that $\underset{\sim}{\mathbf{M}}_2$ is of finite type.

Total Structure Strong Duality Theorem 5.5. *Let* $\mathbf{M}_1$ *be a finite structure and let* $\underset{\sim}{\mathbf{M}}_2 = \langle M; G_2, R_2, \mathcal{T}\rangle$ *be an alter ego of* $\mathbf{M}_1$ *that is a total structure with* R_2 *finite. Define* $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ *and* $\boldsymbol{\mathcal{X}} := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$*. If* $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality between* $\boldsymbol{\mathcal{A}}_{\text{fin}}$ *and* $\boldsymbol{\mathcal{X}}_{\text{fin}}$*, then* $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality between* $\boldsymbol{\mathcal{A}}$ *and* $\boldsymbol{\mathcal{X}}$*.*

Proof. Assume that $\underset{\sim}{\mathbf{M}}_2$ yields a strong duality at the finite level. By the IC Lemma 4.3, $\mathbf{M}_2$ satisfies (IC) with respect to $\mathbf{M}_1$ and hence $\underset{\sim}{\mathbf{M}}_2$ yields a duality on $\boldsymbol{\mathcal{A}}$ and is injective in $\boldsymbol{\mathcal{X}}$, by the Second Duality Theorem 4.5. It remains to show that this duality is full. But this follows immediately from the Total Structure Full Duality Theorem 5.3(i). □

6. Two-for-one duality theorems

In this section, we investigate when it is possible to remove the topology from $\underset{\sim}{\mathbf{M}}_2$, add it to $\mathbf{M}_1$, and thereby convert a full duality for $\mathbb{ISP}(\mathbf{M}_1)$, induced by the alter ego $\underset{\sim}{\mathbf{M}}_2$, into a full duality for $\mathbb{ISP}(\mathbf{M}_2)$, induced by the alter ego $\underset{\sim}{\mathbf{M}}_1$. When this is possible, we get two dualities for the price of one. The following lemma shows that, in order for this to work, the types of both $\mathbf{M}_1$ and $\mathbf{M}_2$ must include enough nullary operations. First, we need some notation and a definition.

Let $\mathbf{M}$ be a structure. Define C^0 to be the subuniverse of $\mathbf{M}$ consisting of the values of the nullary term functions of $\mathbf{M}$, and define C^1 to be the subuniverse of $\mathbf{M}$ consisting of the values of the constant total unary term functions of $\mathbf{M}$. Obviously we have $C^0 \subseteq C^1$. If every element of M that is the value of a constant total unary term function of $\mathbf{M}$ is the value of a nullary term function of $\mathbf{M}$, that is, if $C^1 = C^0$, then we say that $\mathbf{M}$ *has named constants*. Note that $C^1 = C^0$ if and only if either $C^0 \neq \varnothing$ or $C^1 = \varnothing$.

Now assume that $\mathbf{M}_1$ and $\mathbf{M}_2$ are compatible structures. For $i \in \{1, 2\}$, define C^0_i and C^1_i as above, and let K_i be the set consisting of all elements of M that form complete one-element substructures of $\mathbf{M}_i$. Since $\mathbf{M}_1$ and $\mathbf{M}_2$ are compatible, K_1 forms a substructure of $\mathbf{M}_2$ and K_2 forms a substructure of $\mathbf{M}_1$. Moreover, if t is a constant total unary term function of $\mathbf{M}_2$, then t is a constant endomorphism of $\mathbf{M}_1$. Since we have a base assumption that, on $\mathbf{M}_1$, the relations in R_1 and the domains of the partial operations in H_1 are non-empty, it follows that the image of t forms a complete one-element substructure of $\mathbf{M}_1$. So $C^1_2 \subseteq K_1$, and, similarly, $C^1_1 \subseteq K_2$. Thus we have

$$C^0_1 \subseteq C^1_1 \subseteq K_2 \quad \text{and} \quad C^0_2 \subseteq C^1_2 \subseteq K_1.$$

In the presence of a full duality, all but one of these inclusions become equalities.

Lemma 6.1. *Assume that* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are compatible structures.*

(i) *If* $\mathbf{M}_2$ *satisfies* $(\mathrm{CLO})_1$ *with respect to* $\mathbf{M}_1$ *(in particular, if* $\underset{\sim}{\mathbf{M}}_2$ *dualises* $\mathbf{M}_1$ *at the finite level), then every element of* M *that forms a complete one-element substructure of* $\mathbf{M}_2$ *is the value of a constant total unary term function of* $\mathbf{M}_1$*, that is,* $C_1^1 = K_2$.

(ii) *If* $\underset{\sim}{\mathbf{M}}_2$ *fully dualises* $\mathbf{M}_1$ *at the finite level, then* $\mathbf{M}_2$ *has named constants.*

(iii) *If* $\underset{\sim}{\mathbf{M}}_2$ *fully dualises* $\mathbf{M}_1$ *at the finite level, then* $C_1^1 = K_2$ *and* $C_2^0 = C_2^1 = K_1$.

Proof. Assume that $\mathbf{M}_2$ satisfies $(\mathrm{CLO})_1$ with respect to $\mathbf{M}_1$ and that $a \in M$ forms a complete one-element substructure of $\mathbf{M}_2$. Then the constant map φ from M to M with value a is a morphism from $\mathbf{M}_2$ to $\mathbf{M}_2$. Since $\mathbf{M}_2$ satisfies $(\mathrm{CLO})_1$ with respect to $\mathbf{M}_1$, the map φ is a constant total unary term function of $\mathbf{M}_1$. This proves (i).

Now define $\boldsymbol{\mathcal{A}}_i := \mathbb{ISP}(\mathbf{M}_i)$ and $\boldsymbol{\mathfrak{X}}_i := \mathbb{IS}_\mathrm{c}^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_i)$, for $i \in \{1,2\}$, and assume that $\underset{\sim}{\mathbf{M}}_2$ yields a full duality between $(\boldsymbol{\mathcal{A}}_1)_\mathrm{fin}$ and $(\boldsymbol{\mathfrak{X}}_2)_\mathrm{fin}$. Let $\mathrm{D}_2 : \boldsymbol{\mathcal{A}}_1 \to \boldsymbol{\mathfrak{X}}_2$ and $\mathrm{E}_2 : \boldsymbol{\mathfrak{X}}_2 \to \boldsymbol{\mathcal{A}}_1$ be the functors induced by the alter ego $\underset{\sim}{\mathbf{M}}_2$ of $\mathbf{M}_1$. Suppose that $C_2^0 \neq C_2^1$. Then we must have $C_2^0 = \varnothing$ and $C_2^1 \neq \varnothing$. Let $\mathbf{C}_2^0$ and $\mathbf{C}_2^1$ be the corresponding substructures of $\mathbf{M}_2$. Then, by definition, $\mathrm{E}_2(\mathbf{C}_2^0) = \mathrm{E}_2(\boldsymbol{\varnothing}_2) = \mathbf{1}_1$, where $\boldsymbol{\varnothing}_2$ denotes the empty structure of type $\langle G_2, H_2, R_2\rangle$ and $\mathbf{1}_1$ denotes the complete one-element structure of type $\langle G_1, H_1, R_1\rangle$. The only element of $\mathrm{E}_2(\mathbf{C}_2^1)$ is the inclusion map of C_2^1 into M. An easy calculation, using the fact that, on $\mathbf{M}_1$, each relation in R_1 and the domain of every partial operation in H_1 is non-empty, shows that $\mathrm{E}_2(\mathbf{C}_2^1) \cong \mathbf{1}_1$. So

$$\mathbf{C}_2^0 \cong \mathrm{D}_2\mathrm{E}_2(\mathbf{C}_2^0) \cong \mathrm{D}_2(\mathbf{1}_1) \cong \mathrm{D}_2\mathrm{E}_2(\mathbf{C}_2^1) \cong \mathbf{C}_2^1,$$

a contradiction. Thus, (ii) holds.

We now prove (iii). By (i) and (ii), it remains to prove that $C_2^1 = K_1$. First, we shall prove that $\underset{\sim}{\mathbf{M}}_1$ yields a duality between $(\boldsymbol{\mathcal{A}}_2)_\mathrm{fin}$ and $(\boldsymbol{\mathfrak{X}}_1)_\mathrm{fin}$. To simplify the notation, our remaining calculations are modulo the obvious addition or removal of the discrete topology. Thus, we regard $(\boldsymbol{\mathcal{A}}_2)_\mathrm{fin}$ as a subclass of $(\boldsymbol{\mathfrak{X}}_2)_\mathrm{fin} \cup \{\mathbf{1}_2\}$. Let $\mathrm{D}_1 : \boldsymbol{\mathcal{A}}_2 \to \boldsymbol{\mathfrak{X}}_1$ and $\mathrm{E}_1 : \boldsymbol{\mathfrak{X}}_1 \to \boldsymbol{\mathcal{A}}_2$ be the functors induced by the alter ego $\underset{\sim}{\mathbf{M}}_1$ of $\mathbf{M}_2$. Let $\mathbf{A} \in (\boldsymbol{\mathcal{A}}_2)_\mathrm{fin}$. If $\mathbf{A} \ncong \mathbf{1}_2$ or if $\mathbf{A} \cong \mathbf{1}_2$ and $\mathbf{1}_2 \in \boldsymbol{\mathfrak{X}}_2$, then $\mathbf{A} \in \boldsymbol{\mathfrak{X}}_2$ and hence $\mathrm{E}_1\mathrm{D}_1(\mathbf{A}) = \mathrm{D}_2\mathrm{E}_2(\mathbf{A}) \cong \mathbf{A}$, as $\underset{\sim}{\mathbf{M}}_2$ fully dualises $\mathbf{M}_1$ at the finite level. Otherwise, $\mathbf{A} \cong \mathbf{1}_2$ and $\mathbf{1}_2 \notin \boldsymbol{\mathfrak{X}}_2$, in which case $\mathrm{D}_1(\mathbf{A}) \cong \mathrm{D}_1(\mathbf{1}_2) = \boldsymbol{\varnothing}_1$ and hence $\mathrm{E}_1\mathrm{D}_1(\mathbf{A}) = \mathrm{E}_1(\boldsymbol{\varnothing}_1) \cong \mathbf{1}_2 \cong \mathbf{A}$. It follows that $\underset{\sim}{\mathbf{M}}_1$ yields a duality between $(\boldsymbol{\mathcal{A}}_2)_\mathrm{fin}$ and $(\boldsymbol{\mathfrak{X}}_1)_\mathrm{fin}$. An application of (i) gives $C_2^1 = K_1$, as required. □

When can we extend a full duality between $\mathbb{ISP}(\mathbf{M}_1)$ and $\mathbb{IS}_\mathrm{c}^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$ to a full duality between $\mathbb{IS}^0\mathbb{P}(\mathbf{M}_1)$ and $\mathbb{IS}_\mathrm{c}^0\mathbb{P}(\underset{\sim}{\mathbf{M}}_2)$? The next lemma answers this question. As adding the complete one-element structure $\mathbf{1}_2$ to $\mathbb{IS}_\mathrm{c}^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$ cannot affect the injectivity of $\underset{\sim}{\mathbf{M}}_2$, the lemma also holds with 'full' replaced by 'strong'.

Lemma 6.2. *Let* $\underset{\sim}{\mathbf{M}}_2$ *be an alter ego of a finite structure* $\mathbf{M}_1$*. Assume that* $\mathbf{M}_1$ *has no constant total unary term functions and that* $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $\mathbb{ISP}(\mathbf{M}_1)$ *and* $\mathbb{IS}_\mathrm{c}^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$*. Then* $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $\mathbb{IS}^0\mathbb{P}(\mathbf{M}_1)$ *and* $\mathbb{IS}_\mathrm{c}^0\mathbb{P}(\underset{\sim}{\mathbf{M}}_2)$.

Proof. Since $\underset{\sim}{\mathbf{M}}_2$ yields a full duality between $\mathbb{ISP}(\mathbf{M}_1)$ and $\mathbb{IS}_\mathrm{c}^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$, we have $C_1^1 = K_2$, by Lemma 6.1. As C_1^1 is empty, by assumption, we have $K_2 = \varnothing$. Hence

$\mathbf{M}_2$ has no complete one-element substructures and consequently $\mathbf{1}_2 \notin \mathbb{IS}^0_\mathrm{c}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$, where $\mathbf{1}_2$ denotes the complete one-element structure of type $\langle G_2, H_2, R_2\rangle$. As the empty structure $\boldsymbol{\varnothing}_1$ in $\mathbb{IS}^0\mathbb{P}(\mathbf{M}_1)$ and the complete one-element structure $\mathbf{1}_2$ in $\mathbb{IS}^0_\mathrm{c}\mathbb{P}(\underset{\sim}{\mathbf{M}}_2)$ are dual to each other, the result follows. □

We can now state our two-for-one results. The first, a finite-level two-for-one lemma, is very easy.

Lemma 6.3. *Assume that* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are finite compatible structures and define* $\boldsymbol{\mathcal{A}}_i := \mathbb{ISP}(\mathbf{M}_i)$ *and* $\boldsymbol{\mathfrak{X}}_i := \mathbb{IS}^0_\mathrm{c}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_i)$*, for* $i \in \{1,2\}$.

- (i) *If* $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $(\boldsymbol{\mathcal{A}}_1)_{\mathrm{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_2)_{\mathrm{fin}}$*, then* $\underset{\sim}{\mathbf{M}}_1$ *yields a duality between* $(\boldsymbol{\mathcal{A}}_2)_{\mathrm{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_1)_{\mathrm{fin}}$ *that is full provided* $\mathbf{M}_1$ *has named constants.*
- (ii) *Assume that both* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *have named constants. Then* $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $(\boldsymbol{\mathcal{A}}_1)_{\mathrm{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_2)_{\mathrm{fin}}$ *if and only if* $\underset{\sim}{\mathbf{M}}_1$ *yields a full duality between* $(\boldsymbol{\mathcal{A}}_2)_{\mathrm{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_1)_{\mathrm{fin}}$.

Proof. Ignoring the discrete topology, the only difference between $\mathbb{ISP}(\mathbf{M}_i)_{\mathrm{fin}}$ and $\mathbb{IS}^0_\mathrm{c}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_i)_{\mathrm{fin}}$ is the inclusion or exclusion of the empty structure and the complete one-element structure. Assume that $\underset{\sim}{\mathbf{M}}_2$ yields a full duality between $(\boldsymbol{\mathcal{A}}_1)_{\mathrm{fin}}$ and $(\boldsymbol{\mathfrak{X}}_2)_{\mathrm{fin}}$. Then, as in the proof of Lemma 6.1, $\underset{\sim}{\mathbf{M}}_1$ yields a duality on $(\boldsymbol{\mathcal{A}}_2)_{\mathrm{fin}}$. If $\mathbf{M}_1$ has named constants, then by Lemma 6.1(iii), we have $C_1^0 = C_1^1 = K_2$. Thus $\boldsymbol{\varnothing}_1 \in \boldsymbol{\mathfrak{X}}_1$ implies that $\mathbf{M}_2$ has no complete one-element substructures, whence $\mathrm{D}_1\mathrm{E}_1(\boldsymbol{\varnothing}_1) \cong \mathrm{D}_1(\mathbf{1}_2) = \boldsymbol{\varnothing}_1$. It follows that $\underset{\sim}{\mathbf{M}}_1$ yields a full duality between $(\boldsymbol{\mathcal{A}}_2)_{\mathrm{fin}}$ and $(\boldsymbol{\mathfrak{X}}_1)_{\mathrm{fin}}$. Thus (i) holds, and (ii) is an immediate consequence of (i). □

By strengthening the assumptions of this lemma in an asymmetrical way and combining it with the Total Structure Full Duality Theorem 5.3, we obtain the following 'One-and-a-Half-for-a-Half' Full Duality Theorem.

Sesqui Full Duality Theorem 6.4. *Assume that* $\mathbf{M}_1$ *is a finite total structure of finite type that has named constants, let* $\mathbf{M}_2$ *be a structure that is compatible with* $\mathbf{M}_1$ *and define* $\boldsymbol{\mathcal{A}}_i := \mathbb{ISP}(\mathbf{M}_i)$ *and* $\boldsymbol{\mathfrak{X}}_i := \mathbb{IS}^0_\mathrm{c}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_i)$*, for* $i \in \{1,2\}$*. If* $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $(\boldsymbol{\mathcal{A}}_1)_{\mathrm{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_2)_{\mathrm{fin}}$*, then* $\underset{\sim}{\mathbf{M}}_1$ *yields a full duality between* $\boldsymbol{\mathcal{A}}_2$ *and* $\boldsymbol{\mathfrak{X}}_1$.

Proof. Assume that $\underset{\sim}{\mathbf{M}}_2$ yields a full duality between $(\boldsymbol{\mathcal{A}}_1)_{\mathrm{fin}}$ and $(\boldsymbol{\mathfrak{X}}_2)_{\mathrm{fin}}$. As $\mathbf{M}_1$ has named constants, it follows from the previous lemma that $\underset{\sim}{\mathbf{M}}_1$ yields a full duality between $(\boldsymbol{\mathcal{A}}_2)_{\mathrm{fin}}$ and $(\boldsymbol{\mathfrak{X}}_1)_{\mathrm{fin}}$. Since $\mathbf{M}_1$ is of finite type, $\underset{\sim}{\mathbf{M}}_1$ yields a full duality between $\boldsymbol{\mathcal{A}}_2$ and $\boldsymbol{\mathfrak{X}}_1$, by the Total Structure Full Duality Theorem 5.3(ii). □

The following theorem should be compared with the Two-for-One Strong Duality Theorem 3.3.2 in [5]. The theorem here has a stronger assumption, namely that both structures are of finite type, but has the advantage that it separates the fullness of the resulting dualities from considerations of whether they are strong (that is, whether the alter egos are injective in the topological quasivarieties they generate). This is special case of Theorem 2.5 in Hofmann [27].

Two-for-One Full Duality Theorem 6.5. *Assume that* $\mathbf{M}_1 = \langle M; G_1, R_1\rangle$ *and* $\mathbf{M}_2 = \langle M; G_2, R_2\rangle$ *are finite compatible total structures of finite type and that each has named constants. Define* $\boldsymbol{\mathcal{A}}_i := \mathbb{ISP}(\mathbf{M}_i)$ *and* $\boldsymbol{\mathfrak{X}}_i := \mathbb{IS}^0_\mathrm{c}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_i)$*, for* $i \in \{1,2\}$*. Then the following are equivalent:*

(1) $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $(\boldsymbol{\mathcal{A}}_1)_{\mathrm{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_2)_{\mathrm{fin}}$*;*

(2) $\underset{\sim}{\mathbf{M}}_1$ *yields a full duality between* $(\boldsymbol{\mathcal{A}}_2)_{\rm fin}$ *and* $(\boldsymbol{\mathfrak{X}}_1)_{\rm fin}$*;*
(3) $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $\boldsymbol{\mathcal{A}}_1$ *and* $\boldsymbol{\mathfrak{X}}_2$*;*
(4) $\underset{\sim}{\mathbf{M}}_1$ *yields a full duality between* $\boldsymbol{\mathcal{A}}_2$ *and* $\boldsymbol{\mathfrak{X}}_1$*.*

Proof. Apply the previous theorem twice. □

We turn now to two-for-one strong dualities. The injectivity of $\mathbf{M}_1$ in $\boldsymbol{\mathcal{A}}_1$ and $\underset{\sim}{\mathbf{M}}_2$ in $\boldsymbol{\mathfrak{X}}_2$ are closely linked. The following lemma is proved exactly as it is in the case where $\mathbf{M}_1$ is an algebra.

Injectivity Transfer Lemma 6.6. [5, 3.2.10] *Let* $\mathbf{M}_1$ *be a finite structure, let* $\underset{\sim}{\mathbf{M}}_2$ *be an alter ego of* $\mathbf{M}_1$ *and define* $\boldsymbol{\mathcal{A}}_1 := \mathbb{ISP}(\mathbf{M}_1)$ *and* $\boldsymbol{\mathfrak{X}}_2 := \mathbb{IS}^0_{\rm c}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$*. Assume that* $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $\boldsymbol{\mathcal{A}}_1$ *and* $\boldsymbol{\mathfrak{X}}_2$ *[at the finite level].*

(i) *If* $\mathbf{M}_1$ *is a total structure and is injective in* $\boldsymbol{\mathcal{A}}_1$ *[in* $(\boldsymbol{\mathcal{A}}_1)_{\rm fin}$*], then* $\underset{\sim}{\mathbf{M}}_2$ *is injective in* $\boldsymbol{\mathfrak{X}}_2$ *[in* $(\boldsymbol{\mathfrak{X}}_2)_{\rm fin}$*].*
(ii) *If* $\underset{\sim}{\mathbf{M}}_2$ *is a total structure and is injective in* $\boldsymbol{\mathfrak{X}}_2$ *[in* $(\boldsymbol{\mathfrak{X}}_2)_{\rm fin}$*], then* $\mathbf{M}_1$ *is injective in* $\boldsymbol{\mathcal{A}}_1$ *[in* $(\boldsymbol{\mathcal{A}}_1)_{\rm fin}$*].*

An inspection of the proof of 3.2.10 in [5] shows that, in both parts of this lemma, instead of assuming that the structure is a total structure it suffices to assume that, in the appropriate category, the image of every morphism is a substructure. Our first two-for-one strong duality result is a strong-duality version of Lemma 6.3.

Lemma 6.7. *Assume that* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are finite compatible total structures and define* $\boldsymbol{\mathcal{A}}_i := \mathbb{ISP}(\mathbf{M}_i)$ *and* $\boldsymbol{\mathfrak{X}}_i := \mathbb{IS}^0_{\rm c}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_i)$*, for* $i \in \{1, 2\}$*.*

(i) *If* $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality between* $(\boldsymbol{\mathcal{A}}_1)_{\rm fin}$ *and* $(\boldsymbol{\mathfrak{X}}_2)_{\rm fin}$*, then* $\underset{\sim}{\mathbf{M}}_1$ *yields a duality between* $(\boldsymbol{\mathcal{A}}_2)_{\rm fin}$ *and* $(\boldsymbol{\mathfrak{X}}_1)_{\rm fin}$ *that is strong provided* $\mathbf{M}_1$ *has named constants.*
(ii) *Assume that both* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *have named constants. Then* $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality between* $(\boldsymbol{\mathcal{A}}_1)_{\rm fin}$ *and* $(\boldsymbol{\mathfrak{X}}_2)_{\rm fin}$ *if and only if* $\underset{\sim}{\mathbf{M}}_1$ *yields a strong duality between* $(\boldsymbol{\mathcal{A}}_2)_{\rm fin}$ *and* $(\boldsymbol{\mathfrak{X}}_1)_{\rm fin}$*.*

Proof. This follows from Lemma 6.3 and the Injectivity Transfer Lemma 6.6. □

The following is our 'One-and-a-Half-for-a-Half' Strong Duality Theorem.

Sesqui Strong Duality Theorem 6.8. *Let* $\mathbf{M}_1 = \langle M; G_1, R_1\rangle$ *be a finite total structure with* R_1 *finite and assume that* $\mathbf{M}_1$ *has named constants. Let* $\mathbf{M}_2$ *be a structure that is compatible with* $\mathbf{M}_1$ *and, for* $i \in \{1, 2\}$*, define* $\boldsymbol{\mathcal{A}}_i := \mathbb{ISP}(\mathbf{M}_i)$ *and* $\boldsymbol{\mathfrak{X}}_i := \mathbb{IS}^0_{\rm c}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_i)$*.*

(i) *If* $\mathbf{M}_1$ *is injective in* $(\boldsymbol{\mathcal{A}}_1)_{\rm fin}$ *and* $\underset{\sim}{\mathbf{M}}_2$ *yields a full (and therefore strong) duality between* $(\boldsymbol{\mathcal{A}}_1)_{\rm fin}$ *and* $(\boldsymbol{\mathfrak{X}}_2)_{\rm fin}$*, then* $\underset{\sim}{\mathbf{M}}_1$ *yields a strong duality between* $\boldsymbol{\mathcal{A}}_2$ *and* $\boldsymbol{\mathfrak{X}}_1$*.*
(ii) *If* $\mathbf{M}_2$ *is a total structure and* $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality between* $(\boldsymbol{\mathcal{A}}_1)_{\rm fin}$ *and* $(\boldsymbol{\mathfrak{X}}_2)_{\rm fin}$*, then* $\underset{\sim}{\mathbf{M}}_1$ *yields a strong duality between* $\boldsymbol{\mathcal{A}}_2$ *and* $\boldsymbol{\mathfrak{X}}_1$*.*

Proof. Note that the parenthetic remark in (i) follows from the Injectivity Transfer Lemma 6.6(i). Assume that $\mathbf{M}_1$ is injective $(\boldsymbol{\mathcal{A}}_1)_{\rm fin}$ and that $\underset{\sim}{\mathbf{M}}_2$ yields a full duality between $(\boldsymbol{\mathcal{A}}_1)_{\rm fin}$ and $(\boldsymbol{\mathfrak{X}}_2)_{\rm fin}$. By Lemma 6.3(i), $\underset{\sim}{\mathbf{M}}_1$ yields a full duality between $(\boldsymbol{\mathcal{A}}_2)_{\rm fin}$ and $(\boldsymbol{\mathfrak{X}}_1)_{\rm fin}$, and so $\underset{\sim}{\mathbf{M}}_1$ yields a strong duality between $(\boldsymbol{\mathcal{A}}_2)_{\rm fin}$ and $(\boldsymbol{\mathfrak{X}}_1)_{\rm fin}$ since $\mathbf{M}_1$ is injective in $(\boldsymbol{\mathcal{A}}_1)_{\rm fin}$. Thus (i) follows from the Total Structure Strong

Duality Theorem 5.5 with $\mathbf{M}_1$ and $\mathbf{M}_2$ interchanged. Part (ii) follows from part (i) and the Injectivity Transfer Lemma 6.6(ii). □

Combining the Injectivity Lifting Lemma 4.4, the Injectivity Transfer Lemma 6.6 and the Two-for-One Full Duality Theorem 6.5, we see immediately that, under the assumptions of the Two-for-One Full Duality Theorem, if any one of the four dualities listed there is strong, then so are all the others. In fact, by appealing to the Sesqui Strong Duality Theorem 6.8 we can weaken the assumption that $\mathbf{M}_1$ and $\mathbf{M}_2$ are of finite type.

Two-for-One Strong Duality Theorem 6.9. *Assume that* $\mathbf{M}_1 = \langle M; G_1, R_1\rangle$ *and* $\mathbf{M}_2 = \langle M; G_2, R_2\rangle$ *are finite compatible total structures with* R_1 *and* R_2 *finite and that each has named constants. Let* $\boldsymbol{\mathcal{A}}_i := \mathbb{ISP}(\mathbf{M}_i)$ *and* $\boldsymbol{\mathfrak{X}}_i := \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_i)$*, for* $i \in \{1, 2\}$*. Then the following are equivalent:*

(1) $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality between* $\boldsymbol{\mathcal{A}}_1$ *and* $\boldsymbol{\mathfrak{X}}_2$*;*
(2) $\underset{\sim}{\mathbf{M}}_1$ *yields a strong duality between* $\boldsymbol{\mathcal{A}}_2$ *and* $\boldsymbol{\mathfrak{X}}_1$*;*
(3) $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality between* $(\boldsymbol{\mathcal{A}}_1)_{\mathrm{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_2)_{\mathrm{fin}}$*;*
(4) $\underset{\sim}{\mathbf{M}}_1$ *yields a strong duality between* $(\boldsymbol{\mathcal{A}}_2)_{\mathrm{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_1)_{\mathrm{fin}}$*;*
(5) *each of* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *satisfies* (IC) *with respect to the other.*

Proof. The implications (1) ⇒ (3) and (2) ⇒ (4) are trivial. By Lemma 6.7, (3) and (4) are equivalent even without the assumption that R_1 and R_2 are finite. The IC Lemma 4.3 says that (3) and (4) together are equivalent to (5). Finally, the implications (3) ⇒ (2) and (4) ⇒ (1) hold by the previous theorem (or (3) ⇒ (1) and (4) ⇒ (2) hold by the Total Structure Strong Duality Theorem 5.5). □

The Two-for-One Strong Duality Theorem of Clark and Davey [4] ([5, 3.3.2]) is the special case of this theorem in which $R_1 = R_2 = \varnothing$.

7. Examples

The Two-for-One Strong Duality Theorem 6.9 and, more generally, the Sesqui Strong Duality Theorem 6.8 allow us to convert a large number of known strong dualities into new strong dualities, by simply swapping the topology from one side to the other. Assume that $\mathbf{M}_1$ is a finite algebra that is injective in the quasivariety it generates and for which some alter ego yields a strong duality on $\boldsymbol{\mathcal{A}}_1 := \mathbb{ISP}(\mathbf{M}_1)$. Then the Total Structure Theorem (see Clark and Davey [5, 6.1.2]) tells us that there is an alter ego $\underset{\sim}{\mathbf{M}}_2$ of $\mathbf{M}_1$ that is a total structure and yields a strong duality between $\boldsymbol{\mathcal{A}}_1$ and $\boldsymbol{\mathfrak{X}}_2 := \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$. Provided $\mathbf{M}_1$ has named constants, we conclude immediately from the Sesqui Strong Duality Theorem 6.8 that $\underset{\sim}{\mathbf{M}}_1$ yields a strong duality between the categories $\boldsymbol{\mathcal{A}}_2 := \mathbb{ISP}(\mathbf{M}_2)$ and $\boldsymbol{\mathfrak{X}}_1 := \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_1)$. If $\mathbf{M}_1$ is a lattice-based algebra and $\boldsymbol{\mathcal{A}}_1$ is a variety, then an equational description of $\boldsymbol{\mathcal{A}}_1$ yields an equational description of $\boldsymbol{\mathfrak{X}}_1$ via Theorem 3.2. Moreover, if we have a suitable description of $\boldsymbol{\mathfrak{X}}_2$, then we may apply Lemma 3.4 to read off a first-order description of $\boldsymbol{\mathcal{A}}_2$. The first three examples below illustrate these ideas. Our final example is an application of the NU Duality Theorem 4.10 and does not follow by simply swapping the topology on a known duality.

Our first example dates back to Banaschewski [2] in 1976 and is obtained from Priestley duality via the topology-swapping technique described above. It was reproved by Hofmann [27], but with *strong* replaced by *full*.

Example 7.1. *Let $\mathbf{2}_L := \langle\{0,1\}; \vee, \wedge, 0, 1\rangle$ be the two-element bounded lattice and let $\mathbf{2}_O := \langle\{0,1\}; \leqslant\rangle$ be the two-element ordered set with $0 < 1$. Then $\mathcal{P} := \mathbb{IS}^0\mathbb{P}(\mathbf{2}_O)$ is the category of ordered sets, $\mathcal{D}^{01}_{\mathrm{Bt}} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{2}}_L)$ is the category of Boolean topological bounded distributive lattices, and $\underset{\sim}{\mathbf{2}}_L$ yields a strong duality between $\mathcal{P}$ and $\mathcal{D}^{01}_{\mathrm{Bt}}$.*

Proof. Priestley duality [31, 32] (see Clark and Davey [5, 4.3.1 and Exercise 4.5]) tells us that $\underset{\sim}{\mathbf{2}}_O$ yields a strong duality between the category $\mathbb{ISP}(\mathbf{2}_L)$ of bounded distributive lattices and the category $\mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{2}}_O)$ of Priestley spaces. By the Two-for-One Strong Duality Theorem 6.9, $\underset{\sim}{\mathbf{2}}_L$ yields a strong duality between $\mathbb{ISP}(\mathbf{2}_O)$ and $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{2}}_L)$, and therefore $\underset{\sim}{\mathbf{2}}_L$ yields a strong duality between $\mathbb{IS}^0\mathbb{P}(\mathbf{2}_O)$ and $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{2}}_L)$, that is, between $\mathcal{P}$ and $\mathcal{D}^{01}_{\mathrm{Bt}}$, by Lemma 6.2 (taking $\mathbf{M}_1 = \mathbf{2}_O$ and $\mathbf{M}_2 = \mathbf{2}_L$).

It is very easy to show directly that $\mathbb{IS}^0\mathbb{P}(\mathbf{2}_O)$ is the category of ordered sets. (Alternatively, use the fact that the finite Priestley spaces are precisely the finite ordered sets with the discrete topology and apply Lemma 3.4—see Remark 3.5.) Since $\mathbb{ISP}(\mathbf{2}_L)$ is the category of bounded distributive lattices, it follows from Theorem 3.2(ii) that $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{2}}_L)$ is the category of Boolean topological bounded distributive lattices—see Remark 3.3. ☐

Remark 7.2. By redefining $\mathbf{2}_L$ and $\mathbf{2}_O$ in the previous example to be $\mathbf{2}_L := \langle\{0,1\}; \vee, \wedge\rangle$ and $\mathbf{2}_O := \langle\{0,1\}; 0, 1, \leqslant\rangle$, respectively, we obtain a strong duality between the category $\mathcal{P}^{01} := \mathbb{ISP}(\mathbf{2}_O)$ of bounded ordered sets and $\mathcal{D}_{\mathrm{Bt}} := \mathbb{IS}^0_c\mathbb{P}(\underset{\sim}{\mathbf{2}}_L)$ of Boolean topological distributive lattices. An elementary proof of this duality, along with applications to canonical extensions of distributive lattices, may be found in Davey, Haviar and Priestley [16].

Our second example is derived from the author's natural duality for Stone algebras [12, 13] (see also Clark and Davey [5, 4.3.6]), again via a topology swap.

Example 7.3. *Let $\mathbf{3}_S := \langle\{0,a,1\}; \vee, \wedge, {}^*, 0, 1\rangle$, where $\langle\{0,a,1\}; \vee, \wedge, 0, 1\rangle$ is a bounded lattice with $0 < a < 1$ and * is the pseudocomplementation operation, that is, $0^* = 1$ and $a^* = 1^* = 0$. Define $\mathbf{3}_D := \langle\{0,a,1\}; d, \preccurlyeq\rangle$, where $\langle\{0,a,1\}; \preccurlyeq\rangle$ is the ordered set whose only non-trivial relation is $1 \prec a$ and d is the unary operation that maps each element to the unique minimal element below it, that is, $d(0) = 0$ and $d(a) = d(1) = 1$. Define $\mathcal{A} := \mathbb{IS}^0\mathbb{P}(\mathbf{3}_D)$ and $\mathcal{S}_{\mathrm{Bt}} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{3}}_S)$.*

- (i) *A structure $\langle A; d, \preccurlyeq\rangle$ belongs to $\mathcal{A}$ if and only if $\langle A; \preccurlyeq\rangle$ is an ordered set in which each element a is above a unique minimal element, namely $d(a)$.*
- (ii) *$\mathcal{S}_{\mathrm{Bt}}$ consists of all Boolean topological Stone algebras, that is, Boolean topological bounded distributive latttices that are pseudocomplemented and in which the pseudocomplementation operation is continuous and satisfies $x^* \vee x^{**} \approx 1$.*
- (iii) *$\underset{\sim}{\mathbf{3}}_S$ yields a strong duality between $\mathcal{A}$ and $\mathcal{S}_{\mathrm{Bt}}$.*

Proof. It is proved in both Davey [12] and Davey [13] (see also [5, 1.4.7]) that a structure $\langle X; d, \preccurlyeq \mathcal{T}\rangle$ belongs to $\mathbb{IS}^0_c\mathbb{P}(\underset{\sim}{\mathbf{3}}_D)$ if and only if $\langle X; \preccurlyeq, \mathcal{T}\rangle$ is a Priestley space in which each element x is above a unique minimal element, namely $d(x)$, and the map d is continuous. Thus (i) follows by an easy application of Lemma 3.4.

It is well known that $\mathbb{ISP}(\mathbf{3}_S)$ is the variety of Stone algebras (see Grätzer [23]), that is, pseudocomplemented distributive lattices satisfying $x^* \vee x^{**} \approx 1$. An application of Theorem 3.2 yields (ii).

Since $\underset{\sim}{\mathbf{3}}_D$ yields a strong duality between $\mathbb{ISP}(\mathbf{3}_S)$ and $\mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{3}}_D)$ [12, 13], it follows by a combination of the Two-for-One Strong Duality Theorem 6.9 and

Lemma 6.2 (with $\mathbf{M}_1 = \mathbf{3}_D$ and $\mathbf{M}_2 = \mathbf{3}_S$) that $\underset{\sim}{\mathbf{3}}_S$ yields a strong duality between $\mathbb{IS}^0\mathbb{P}(\mathbf{3}_D)$ and $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{3}}_S)$, that is, between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{S}}_{\mathrm{Bt}}$. □

Remark 7.4. A version of this duality is proved from first principles by Haviar and Priestley [25]. They replace the topological category $\boldsymbol{\mathcal{S}}_{\mathrm{Bt}}$ with an isomorphic category consisting of doubly algebraic Stone algebras with morphisms that preserve pseudocomplements and arbitrary joins and meets. They apply their version of the duality to show that, in the language of canonical extensions, *Stone algebras are canonical*. Applications of natural dualities for structures to the canonicity of other classes of algebras will appear in a paper by Davey, Gehrke and Priestley [14].

The simple topology-swapping technique illustrated in the two examples above can be applied to many lattice-based algebras for which we have a well-behaved strong duality. For example, it can be applied to the known strong dualities for

- double Stone algebras (Davey [13] and [5, 4.3.13 and 4.3.14]),
- Kleene algebras (Davey and Werner [20] and [5, 4.3.9 and 4.3.10]), and
- De Morgan algebras (Cornish and Fowler [10] and [5, 4.3.16]).

This technique can also be applied when $\mathbf{M}_1$ is not injective in $\mathbb{ISP}(\mathbf{M}_1)$, but in this case we must use the Sesqui Full Duality Theorem 6.4. For example, consider the Heyting chain $\mathbf{C}_H = \langle\{0, a, b, 1\}; \vee, \wedge, \to, 0, 1\rangle$, with $0 < a < b < 1$. While a natural duality for $\mathbb{ISP}(\mathbf{C}_H)$ dates back to Davey [11] in 1976, a strong duality was discovered only in 1995 by Clark and Davey [4]. Let $\mathbf{C}_D := \langle\{0, a, b, 1\}; g, h\rangle$, where g is the endomorphism of $\mathbf{C}_H$ given by $g(0) = 0$, $g(a) = b$ and $g(b) = g(1) = 1$, and h is the partial endomorphism of $\mathbf{C}_H$ with domain $\{0, b, 1\}$ given by $h(0) = 0$, $h(b) = a$ and $h(1) = 1$. For an explanation of why $\underset{\sim}{\mathbf{C}}_D$ yields a strong duality on $\mathbb{ISP}(\mathbf{C}_H)$ and a proof of the following axiomatization of $\mathbb{IS}_c^0\mathbb{P}(\underset{\sim}{\mathbf{C}}_D)$, see Davey and Talukder [19]. A Boolean topological structure $\mathbf{X} = \langle X; g, h, \mathcal{T}\rangle$, with g a total unary operation and h a partial unary operation, belongs to $\mathbb{IS}_c^0\mathbb{P}(\underset{\sim}{\mathbf{C}}_D)$ if and only if $\mathbf{X}$ satisfies the following axioms:

(S_1) $ggg(x) = gg(x)$,
(S_2) $x \in \mathrm{dom}(h) \iff gg(x) = g(x)$,
(S_3) $g(x) = x \iff \big(x \in \mathrm{dom}(h) \ \&\ h(x) = x\big)$,
(S_4) $x \in \mathrm{dom}(h) \implies gh(x) = x$,
(S_5) $g(x) \in \mathrm{dom}(h)$.

We shall now see that an application of the Sesqui Full Duality Theorem 6.4 and Lemma 6.2 yields a full but not strong duality for the quasivariety generated by the partial algebra $\mathbf{C}_D$. Recall that the *Full versus Strong Problem* [5, 3.2.7] asks whether every full duality for the quasivariety $\mathbb{ISP}(\mathbf{M})$ generated by a finite *total* algebra $\mathbf{M}$ is necessarily strong. The example below shows that if we allow $\mathbf{M}$ to include partial operations in its type, then the answer is 'no'. (Since this paper was written, the problem has been solved in the negative by Clark, Davey and Willard [9].)

Example 7.5. *Let $\mathbf{C}_H$ be the four-element Heyting chain and let $\mathbf{C}_D$ be the four-element partial algebra described above. Let $\boldsymbol{\mathcal{A}} := \mathbb{IS}^0\mathbb{P}(\mathbf{C}_D)$ and $\boldsymbol{\mathcal{C}} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{C}}_H)$.*

(i) *A structure $\langle A; g, h\rangle$ belongs to $\boldsymbol{\mathcal{A}}$ if and only if it satisfies axioms* (S_1)–(S_5).
(ii) *$\boldsymbol{\mathcal{C}}$ consists of all Boolean topological Heyting algebras satisfying the identity $(x_0 \to x_1) \vee (x_1 \to x_2) \vee (x_2 \to x_3) \vee (x_3 \to x_4) \approx 1$.*
(iii) *$\underset{\sim}{\mathbf{C}}_H$ yields a full but not strong duality between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{C}}$.*

Proof. Let $\Sigma := \{(\mathrm{S}_1), \dots, (\mathrm{S}_5)\}$. It is easy to check directly that a 1-generated model of Σ has at most 5 elements. Indeed, if $\mathbf{A}$ is a model of Σ then the substructure generated by $a \in A$ consists of the elements a, $g(a)$, $gg(a)$ and $hg(a)$, and $h(a)$ if it is defined; see Table 1. Hence, as the type is unary, an n-generated model of Σ has at most $5n$ elements. Since Σ describes $\mathbb{IS}^0_{\mathrm{c}}\mathbb{P}(\underset{\sim}{\mathbf{C}}_D)$, part (i) follows from Lemma 3.4.

	a	$g(a)$	$gg(a)$	$hg(a)$	$h(a)$
g	$g(a)$	$gg(a)$	$gg(a)$	$g(a)$	a
h	$h(a)$	$hg(a)$	$gg(a)$	$g(a)$	a

Table 1. The substructure generated by a

Hecht and Katriňák [26] proved that a Heyting algebra belongs to $\mathbb{ISP}(\mathbf{C}_H)$ if and only if it satisfies the identity given in (ii). Thus (ii) follows at once from Theorem 3.2. As $\underset{\sim}{\mathbf{C}}_D$ yields a full duality between $\mathbb{ISP}(\mathbf{C}_H)$ and $\mathbb{IS}^0_{\mathrm{c}}\mathbb{P}(\underset{\sim}{\mathbf{C}}_D)$, the Sesqui Full Duality Theorem 6.4 and Lemma 6.2 (with $\mathbf{M}_1 = \mathbf{C}_H$ and $\mathbf{M}_2 = \mathbf{C}_D$) show that $\underset{\sim}{\mathbf{C}}_H$ yields a full duality between $\boldsymbol{\mathcal{A}}$ and $\mathfrak{C}$. Finally, $\underset{\sim}{\mathbf{C}}_H$ is not injective in $\mathfrak{C}$ as the homomorphism h does not extend to an endomorphism of $\mathbf{C}_H$. Thus (iii) holds. $\square$

We close the paper with an application of the NU Duality Theorem 4.10. Let $\mathbf{2}_b := \langle\{0,1\}; \vee, \wedge, b\rangle$, where $\langle\{0,1\}; \vee, \wedge\rangle$ is the two-element lattice and $\langle\{0,1\}; b\rangle$ is the directed graph with one edge pointing from 0 to 1. Let $\mathbf{2}_{ud} := \langle\{0,1\}; u, d, \leqslant\rangle$, where u (for 'up') is the partial operation with domain $\{0\}$ and $u(0) = 1$, the partial operation d (for 'down') has domain $\{1\}$ and $d(1) = 0$, and $\leqslant$ is the order on $\{0,1\}$ with $0 < 1$. We shall see that, in a certain way, the quasivariety $\mathfrak{D}^b := \mathbb{ISP}(\mathbf{2}_b)$ may be thought of as the disjoint union of the classes $\mathfrak{D}^{01}$ and $\mathfrak{D}$ of bounded distributive lattices and distributive lattices, and likewise, $\mathfrak{X} := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{2}}_{ud})$ may be thought of as the disjoint union of the classes of Priestley spaces and bounded Priestley spaces.

Example 7.6. *Let $\mathbf{2}_b := \langle\{0,1\}; \vee, \wedge, b\rangle$ and $\mathbf{2}_{ud} = \langle\{0,1\}; u, d, \leqslant\rangle$ be as given above, and define $\mathfrak{D}^b := \mathbb{ISP}(\mathbf{2}_b)$ and $\mathfrak{X} := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{2}}_{ud})$.*

(i) *Let $\mathbf{A} = \langle A; \vee, \wedge, b\rangle$ be a structure of the same type as $\mathbf{2}_b$. The following are equivalent:*

  (a) $\mathbf{A}$ *belongs to* $\mathfrak{D}^b$*;*

  (b) $\langle A; \vee, \wedge\rangle$ *is a distributive lattice and* $\mathbf{A}$ *satisfies the quasi-equations* $(x, y) \in b \implies x \leqslant z \leqslant y$*;*

  (c) $\mathbf{A}$ *is either a distributive lattice with bounds* 0 *and* 1 *and with an edge pointing from* 0 *to* 1*, or* $\mathbf{A}$ *is a distributive lattice with no edges.*

(ii) *Let $\mathbf{X} = \langle X; u, d, \leqslant, \mathcal{T}\rangle$ be a Boolean topological structure of the same type as $\mathbf{2}_{ud}$. The following are equivalent:*

  (a) $\mathbf{X}$ *belongs to* $\mathfrak{X}$*;*

  (b) $\langle X; \leqslant, \mathcal{T}\rangle$ *is a Priestley space and* $\mathbf{X}$ *satisfies the quasi-equations*

$$x \in \mathrm{dom}(u) \implies x \leqslant y \;\;\&\;\; u(x) \in \mathrm{dom}(d),$$
$$x \in \mathrm{dom}(d) \implies x \geqslant y \;\;\&\;\; d(x) \in \mathrm{dom}(u);$$

(c) $\mathbf{X}$ *is either a Priestley space with both* $\mathrm{dom}(u)$ *and* $\mathrm{dom}(d)$ *empty, or* $\mathbf{X}$ *is a bounded Priestley space in which u maps the bottom to the top and d maps the top to the bottom.*

(iii) $\underset{\sim}{\mathbf{2}}_{ud}$ *yields a strong duality between* $\mathfrak{D}^b$ *and* $\mathfrak{X}$.

Proof. The equivalences in parts (i) and (ii) follow by standard arguments concerning distributive lattices and Priestley spaces using Lemmas 1.2 and 3.1. Thus, $\mathfrak{D}^b$ is essentially the disjoint union of $\mathfrak{D}^{01}$ and $\mathfrak{D}$. The morphism class of $\mathfrak{D}^b$ is just the disjoint union of the morphisms in $\mathfrak{D}^{01}$, the morphisms in $\mathfrak{D}$ and the class of all lattice homomorphisms from lattices in $\mathfrak{D}$ to lattices in $\mathfrak{D}^{01}$. Similarly, $\mathfrak{X}$ can be thought of as the disjoint union of the categories of Priestley spaces and bounded Priestley spaces with the continuous order-preserving maps from Priestley spaces into bounded Priestley spaces added as additional morphisms.

The set of all subuniverses of $(\mathbf{2}_b)^2$ is

$$\{\, \leqslant, \geqslant, \Delta_{\{0,1\}} \,\} \cup \{\, A \times B \;\big|\; A, B \subseteq \{0,1\} \,\}.$$

Since u and d entail their domains, the structure $\underset{\sim}{\mathbf{2}}_{ud}$ entails every non-empty subuniverse of $(\mathbf{2}_b)^2$ (see [5, 2.4.5]). The NU Duality Theorem 4.10 now tells us that $\underset{\sim}{\mathbf{2}}_{ud}$ yields a duality between $\mathfrak{D}^b$ and $\mathfrak{X}$, and that $\underset{\sim}{\mathbf{2}}_{ud}$ is injective in $\mathfrak{X}$. The fact that the duality is full follows from the fact that both the duality between bounded distributive lattices and Priestley spaces and the duality between (not necessarily bounded) distributive lattices and bounded Priestley spaces are full. □

This duality is also a candidate for a topology swap. By the Sesqui Strong Duality Theorem 6.8(i) and Lemma 6.2, the alter ego $\underset{\sim}{\mathbf{2}}_b = \langle \{0,1\}; \vee, \wedge, b, \mathfrak{T} \rangle$ of the structure $\mathbf{2}_{ud} = \langle \{0,1\}; u, d, \leqslant \rangle$ yields a strong duality between $\boldsymbol{\mathcal{A}} := \mathbb{IS}^0\mathbb{P}(\mathbf{2}_{ud})$ and $\mathfrak{D}^b_{\mathrm{Bt}} := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}(\underset{\sim}{\mathbf{2}}_b)$. By Theorem 3.4, a structure $\mathbf{A} = \langle A; u, d, \leqslant \rangle$ belongs to $\boldsymbol{\mathcal{A}}$ if and only if $\langle A; \leqslant \rangle$ is an ordered set and $\mathbf{A}$ satisfies the quasi-equations in (ii) above. Since $\mathbb{ISP}(\mathbf{2}_b)$ is closed under (finite) quotients, Theorem 3.2 tells us that $\mathbf{X} = \langle X; \vee, \wedge, b, \mathfrak{T} \rangle$ belongs to $\mathfrak{D}^b_{\mathrm{Bt}}$ if and only if $\langle X; \vee, \wedge, \mathfrak{T} \rangle$ is a Boolean topological distributive lattice and $\mathbf{X}$ satisfies satisfies the quasi-equations in (i) above.

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