Acta Univ. M. Belii Math.
No. 13(2006), pp. 29–31

AN EXAMPLE OF A CONGRUENCE DISTRIBUTIVE VARIETY HAVING NO NEAR-UNANIMITY TERM

I. Chajda and R. Halaš

Abstract. By a nearlattice is meant a join-semilattice whose every principal filter is a lattice with respect to the induced order. Every nearlattice can be described as an algebra with one ternary operation satisfying eight simple identities. This algebra is called a nearlattice-algebra. Hence, nearlattice-algebras form a variety $\mathcal{N}$. We shall show that the variety $\mathcal{N}$ is congruence distributive but $\mathcal{N}$ has not a near-unanimity term.

By a *nearlattice* we mean a semilattice $\mathcal{S} = (A; \vee)$ where for each $a \in A$ the principal filter $[a) = \{x \in A; a \leq x\}$ is a lattice with respect to the induced order $\leq$ of $\mathcal{S}$.

Nearlattices were studied by M. Scholander [3,4] under a different name. The term "nearlattice" was firstly used by A. S. Noor and W. H. Cornish [2].

Obviously, the operation $x \wedge y$ (meet) in $\mathcal{S}$ is defined if and only if the elements x, y have a common lower bound. This implies that for $x, y, z \in A$ the operation

$$m(x, y, z) = (x \vee z) \wedge (y \vee z)$$

is everywhere defined. Moreover, $m(x, y, z)$ satisfies the identities (P1) – (P8):

(P1) $m(x, y, x) = x$;
(P2) $m(x, x, y) = m(y, y, x)$;
(P3) $m(m(x, x, y), m(x, x, y), z) = m(x, x, m(y, y, z))$;
(P4) $m(x, y, p) = m(y, x, p)$;
(P5) $m(m(x, y, p), z, p) = m(x, m(y, z, p), p)$;
(P6) $m(x, m(y, y, x), p) = m(x, x, p)$;
(P7) $m(m(x, x, p), m(x, x, p), m(y, x, p)) = m(x, x, p)$;
(P8) $m(m(x, x, z), m(y, y, z), z) = m(x, y, z)$.

It was shown in [1] that there is a one-to-one correspondence between nearlattices and algebras $\mathcal{A} = (A; m)$ of type (3) satisfying the identities (P1) – (P8). Hence, an algebra $\mathcal{A} = (A; m)$ of type (3) satisfying (P1) – (P8) will be called a *nearlattice-algebra*.

One can easily see that for nearlattice-algebras the relation $\leq$ defined by

$$x \leq y \quad \text{iff} \quad m(x, x, y) = y$$

is an order and $(A; \leq)$ is a semilattice where $x \vee y = m(x, x, y)$ and for each $z \in A$ the filter $[z)$ is a lattice with $x \wedge y = m(x, y, z)$. Hence, $m(x, y, z) = (x \vee z) \wedge (y \vee z)$. Now, we can state

2000 Mathematics Subject Classification. 06B05; Secondary: 06A12, 08B10.
Key words and phrases. congruence distributivity, near-unanimity term, nearlattice, ternary operation.
Submitted: April 4, 2006

Theorem 1. The variety $\mathcal{N}$ of all nearlattice-algebras is congruence distributive.

Proof. By Jónsson's Theorem, we only need to verify the existence of ternary Jónsson terms. We take $n = 4$, and

$$\begin{aligned} p_0(x,y,z) &= x; \\ p_1(x,y,z) &= m(z,y,x); \\ p_2(x,y,z) &= m(x,x,z); \\ p_3(x,y,z) &= m(x,y,z); \\ p_4(x,y,z) &= z. \end{aligned}$$

Then we need to show that:

$$p_0(x,y,z) = x; \quad p_4(x,y,z) = z;$$

$$p_0(x,y,x) = p_1(x,y,x) = p_2(x,y,x) = p_3(x,y,x) = p_4(x,y,x) = x;$$

$$p_0(x,x,y) = p_1(x,x,y); \quad p_2(x,x,y) = p_3(x,x,y);$$

$$p_1(x,y,y) = p_2(x,y,y); \quad p_3(x,y,y) = p_4(x,y,y).$$

Evidently $p_0(x,y,z) = x$ and $p_4(x,y,z) = z$ hold.
Clearly

$$m(x,y,x) \overset{(P1)}{=} x, \quad m(x,x,x) \overset{(P1)}{=} x,$$

so we obtain $p_0(x,y,x) = x$, $p_1(x,y,x) = m(x,y,x) = x$, $p_2(x,y,x) = m(x,x,x) = x$, $p_3(x,y,x) = m(x,y,x) = x$, $p_4(x,y,x) = x$.

Further,

$$p_0(x,x,y) = x \overset{(P1)}{=} m(x,y,x) \overset{(P4)}{=} m(y,x,x) = p_1(x,x,y),$$

$$p_2(x,x,y) = m(x,x,y) = p_3(x,x,y).$$

Finally,

$$p_1(x,y,y) = m(y,y,x) \overset{(P2)}{=} m(x,x,y) = p_2(x,y,y)$$

$$p_3(x,y,y) = m(x,y,y) \overset{(P4)}{=} m(y,x,y) \overset{(P1)}{=} y = p_4(x,y,y). \quad \square$$

To prove that $\mathcal{N}$ has not a near-unanimity term, we introduce the following concept.

Let $p(x_1,\dots,x_n)$ be a term of the variety $\mathcal{N}$. By induction of term complexity, we define: a variable x_i is called the *right-most* of p if

(i) $p(x_1,\dots,x_n)$ is a projection and $p(x_1,\dots,x_n) = x_i$ or
(ii) $p(x_1,\dots,x_n) = m(p_1,p_2,p_3)$ where p_1,p_2,p_3 are subterms of p and x_i is the right-most variable of p_3.

Theorem 2. Let $\mathcal{A} = (A; m) \in \mathcal{N}$ has a greatest element 1 and $p(x_1, \dots, x_n)$ be an n-ary term of $\mathcal{N}$. If x_i is the right-most variable of p and $a_1, \dots, a_n \in A$ then $p(a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) = 1$.

Proof. We proceed by induction of complexity of $p(x_1, \dots, x_n)$. If p is projection, the proof is trivial. Hence, suppose $p(x_1, \dots, x_n) = m(p_1, p_2, p_3)$ for some subterms p_1, p_2, p_3 and let x_i be the right-most variable of p. Then x_i is the right-most variable of p_3 and, by the induction hypothesis,

$$p_3(a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) = 1.$$

Thus

$$p(a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) = (p_1 \vee p_3) \wedge (p_2 \vee p_3) = \\ = (p_1 \vee 1) \wedge (p_2 \vee 1) = 1 \wedge 1 = 1. \quad \square$$

Example. Let $\mathcal{I} = (\{0, 1\}, m)$ be a two-element nearlattice-algebra. Then it is the two-element chain but there is no near-unanimity term-function on $\mathcal{I}$ since $u(0, \dots, 0, 1, 0, \dots, 0) = 1$ (i-th position) whenever x_i is the right-most variable of u. On the contrary, there exists a near-unanimity polynomial on $\mathcal{I}$ which is e.g.

$$t(x, y, z) = m(m(x, y, z), m(x, x, y), 0).$$

It is in fact the majority term $(x \vee z) \wedge (y \vee z) \wedge (x \vee y)$. However, this polynomial exists only on nearlattices having the least element 0 which need not be the case.

Corollary 1. The variety $\mathcal{N}$ is congruence distributive but it has not a near-unanimity term.

Acknowledgement

This paper was prepared under the support by Research Project MSM 6198959214.

References

1. Chajda, I. and Kolařík, M., *Nearlattices*, Discrete Math. (sumbitted).
2. Noor, A. S. and Cornish, W. H., *Multipliers on nearlattices*, Comment. Math. Univ. Carol. (CMUC) **27** (1986), 815–827.
3. Scholander, M., *Trees, lattices, order and betweeness*, Proc. Amer. Math. Soc. **3** (1952), 369–381.
3. Scholander, M., *Medians and betweeness*, Proc. Amer. Math. Soc. **5** (1954), 801–807.

Palacký University Olomouc; Department of Algebra and Geometry; Tomkova 40; 779 00 Olomouc; Czech Republic

E-mail address: chajda@inf.upol.cz; halas@inf.upol.cz