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SYMMETRIC CUBIC GRAPHS OF GIRTH AT MOST 7

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Abstract. By a symmetric graph we mean a graph X which automorphism group acts transitively on the arcs of X. A graph is *s-regular* if its automorphism group acts regularly on the set of its s-arcs. Tutte [31, 32] showed that every finite symmetric cubic graph is s-regular for some $s \leq 5$. It is well-known that there are precisely five symmetric cubic graphs of girth less than 6. All these graphs can be represented as one-skeletons of regular polyhedra in the plane, projective plane or in torus. With the exception of $K_{3,3}$, we can find an associated regular polyhedron such that the girth of the graph coincide with the face-size.

In this paper we show that with three more exceptions the symmetric cubic graphs of girth $g \leq 7$ are one-skeletons of trivalent regular maps with face-size g. All the symmetric cubic graphs of girth 6 except the generalised Petersen graphs $GP(8,3)$ and $GP(10,3)$ are one-skeletons of toroidal regular maps of type $\{6,3\}$. We give a simple numerical criterium to determine the degree s of s-regularity of these graphs and derive the presentations of the automorphism groups. As concerns girth 7, the only exceptional graph is the well-known Coxeter graph on 28 vertices. We prove that all the other symmetric cubic graphs of girth 7 are underlying graphs of regular maps of type $\{7,3\}$ which are known as Hurwitz maps. Some more results on symmetric cubic graphs with exactly two girth cycles passing through an edge are proved

1 Introduction

Throughout this paper a graph means an undirected finite graph, without loops or multiple edges. For a graph X, we denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its automorphism group, respectively. For further group- and graph-theoretic notation and terminology, we refer the reader to [15] and [17].

An *s-arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for every $1 \leq i < s$; in other words, a directed walk of length s which never includes the reverse of an arc just crossed. A graph X is said to be *s-arc-transitive* if $\mathrm{Aut}(X)$ is transitive on the set of all s-arcs in X. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. An arc-transitive graph X is said to be *s-regular* if for any two s-arcs in X, there is a unique automorphism of X mapping one to the other. An s-regular graph $(s \geq 1)$

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is a union of isomorphic s-regular connected graphs and isolated vertices. Thus, in what follows we consider only non-trivial connected graphs. Tutte [31, 32] showed that every finite symmetric cubic graph is s-regular for some $s \leq 5$. Depending $s = 1, 2, 3, 4, 5$ the vertex-stabisers of the respective groups acting s-regularly on a (connected) cubic graph are respectively isomorphic to the cyclic group Z_3, to the symmetric group S_3, to the dihedral group D_{12} of order 12, to the symmetric group S_4 and to the direct product $S_4 \times Z_2$. For $s = 2$ and $s = 4$ there are two different possibilities for the edge-stabilisers. Taking into the account possible vertex- and edge-stabilisers we have 7 sorts of arc-transitive actions of a group onto a cubic graph. These 7 sorts of actions give rise to 7 universal groups acting arc-transitively on the infinite cubic tree (see [12, 14]). Presentations of the seven groups were found by Conder and Lorimer in [6]. It follows that the automorphism group of a symmetric cubic graph is an epimorphic image of one of the 7 groups. The corresponding seven families of graphs were proved to be infinite.

In the present paper we consider symmetric cubic graphs with girth constraints. In particular, we will be interested in symmetric cubic graphs of girth at most 7. It is well-known that there are five connected symmetric cubic graphs with girth less than 6, namely the tetrahedral graph, the complete bipartite graph $K_{3,3}$, the 3-dimensional cube, the Petersen graph and the dodecahedral graph. This can easily be shown by case to case analysis with respect to girth 3, 4 or 5. Three of the graphs are one-skeletons of the 3-valent Platonic solids. The Petersen graph has a highly symmetric 5-gonal embedding into the Projective plane while $K_{3,3}$ has a symmetrical 6-gonal embedding into the torus. In all these geometrical representations girth of the graph is equal to the face size, except for the embedding of $K_{3,3}$ into torus. Automorphism groups of symmetric cubic graphs of girth 6 were studied by Miller in [24]. He proved that all but finitely many of them can be defined as double coset graphs from a family of 2-generator groups

$$G(s,t,k) = \langle x, y | x^3 = y^2 = (xy)^6 = [x,y]^{sk} = (xyx^{-1}y)^{st}(x^{-1}yxy)^{-s} = 1 \rangle,$$

where $k > 0$, $0 < 2t \leq k+1$ and $t^2 - t + 1 \equiv 0 (\mathrm{mod}\ k)$. Further, Morton [25] characterised 4-arc-transitive cubic graphs up to girth 13. It follows that the automorphism group of such a graph is an epimorphic image of the triangle group $\Delta^+(12,3,2) = \langle x, y | x^3 = y^2 = (xy)^{12} = 1 \rangle$ or it is one of the nine exceptional graphs. Conder in [5] constructed an infinite family of 4-arc-transitive cubic graphs of girth 12.

In the present paper we deal with the family of symmetric cubic graphs of girth at most 7 in detail. In Section 5 we prove a classification theorem (Theorem 6.2) showing that with two exceptions all the symmetric cubic graphs of girth 6 are one skeletons of toroidal regular maps of type $\{6,3\}$, a popular family of 3-valent hexagonal maps (see Coxeter-Moser [9]). The two exceptional graphs are well-known, these are the generalised Petersen graphs $GP(8,3)$ and $GP(10,3)$. Using this geometric characterisation we describe the automorphism groups of the symmetric graphs of girth 6 by means of group presentations.

Similarly, we prove that except the Coxeter graph all the symmetric cubic graphs of girth 7 are underlying graphs of regular, or orientably regular maps of type $\{7,3\}$ (see Theorem 5.2). These maps are in a correspondence with compact Riemann surfaces of Euler characteristic χ with the maximum possible number of symmetries reaching the Hurwitz bound -84χ. Thus the maps of type $\{7,3\}$ are sometimes called Hurwitz maps. As concerns girth 7, to give a list of presentations for the corresponding groups is a difficult task. It follows from our characterisation that this problem is equivalent with the problem of classification of normal subgroups of finite index of the triangle group $\Delta^+(7,3,2)$ and of the extended triangle group $\Delta(7,3,2)$. Even if we are restricted to simple non-abelian quotients of $\Delta(7,3,2)$ or of $\Delta^+(7,3,2)$, a complete list of such groups is not known (see Section 6 for more details). Some general results on the family of symmetric cubic graphs such that there are precisely two girth cycles passing through an edge are proved in Sections 4 and 6. In particular, Propositions 7.1 and 7.2 give existence results of 2- and 1-regular cubic graphs belonging to the family.

2 Maps and groups acting on maps

The aim of this section is to survey some known facts on regular maps with the emphasis to trivalent regular maps. The proofs of the results mentioned here one can find in [3, 13, 19]. A *map* is a cellular decomposition of a closed surface into 0-cells called *vertices*, 1-cells called *edges* and 2-cells called *faces*. The vertices and edges of a map form its *underlying graph*. A map is said to be *orientable* if the supporting surface is orientable, and is *oriented* if one of two possible orientations of the surface has been specified; otherwise, a map is *unoriented*. Every map can be described in a purely combinatorial way as follows: Let F be the set of mutually incident triples of the form vertex-edge-face which we shall call flags of a map $\mathcal{M}$. There are three fixed-point-free permutations ρ, λ and τ associated with $\mathcal{M}$, ρ interchanges flags sharing the same vertex and face, λ interchanges flags sharing the same face and edge. Finally, τ interchanges flags sharing the same vertex and edge. It follows that $(\lambda\tau)^2 = 1$. We shall write $\mathcal{M} = (F, \rho, \lambda, \tau)$. On the other hand, given set F of (abstract) flags and three involutions acting freely and transitively on F such that two of them commute we can reconstruct the associated topological map. The vertices, edges and faces are in correspondence with the orbits of $\langle\rho, \tau\rangle$, $\langle\lambda, \tau\rangle$ and $\langle\rho, \lambda\rangle$, respectively. The incidence relation between vertices, edges and faces is determined by the (non-empty) intersection of the respective orbits. Given map $\mathcal{M} = (F, \rho, \lambda, \tau)$ the map $(F, \rho, \lambda\tau, \tau)$ will be called the Petrie dual of $\mathcal{M}$. The underlying graph of the Petrie dual and of the original map are the same. The following well-known result determines the topological structure of the surface associated with a map (F, ρ, λ, τ).

Lemma 2.1. Let $M = (F, \rho, \lambda, \tau)$ be a combinatorial map. Denote by $G^+ = \langle\rho\tau, \lambda\tau\rangle$, and by v, e and f the respective numbers of orbits of $\langle\rho, \tau\rangle$, $\langle\lambda, \tau\rangle$ and $\langle\rho, \lambda\rangle$ in the action of $G = \langle\rho, \lambda, \tau\rangle$ on F.

Then the Euler characteristic of the underlying surface S is $v - e + f$ and S is orientable if and only if $G^+ < G$ is index two subgroup of G.

A permutation φ of flags is an automorphism of $\mathcal{M} = (F, \rho, \lambda, \tau)$ if it commutes with ρ, λ, τ. Every map automorphism act on vertices of the underlying graph and preserve the incidence relation between edges and vertices. If the graph is simple and the valency of every vertex is at least 3 we have $\mathrm{Aut}(\mathcal{M}) \leq \mathrm{Aut}(X)$. Thus every map automorphism can be viewed as a graph automorphism. Generally, the action of $\mathrm{Aut}(\mathcal{M})$ on flags is semi-regular so $|\mathrm{Aut}(\mathcal{M})| \leq |F| = 4|E(X)|$. If the equality holds, the action is regular on flags and the map itself will be called *regular*. For regular maps we have the following.

Proposition 2.2. [3] Let $\mathcal{M} = (F, \rho, \lambda, \tau)$ be a map. The following three statements are equivalent:
(1) $\mathcal{M}$ is regular,
(2) $\mathrm{Aut}(\mathcal{M}) \cong \langle \rho, \lambda, \tau \rangle$,
(3) $\mathrm{Aut}(\mathcal{M})$ contains three involutory automorphisms mapping a fixed flag x respectively, onto $\rho(x)$, $\lambda(x)$ and $\tau(x)$.

Hence if M is regular the action of $\mathrm{Aut}(\mathcal{M})$ is arc-transitive with dihedral vertex-stabiliser and with edge-stabiliser isomorphic to the Klein's group $Z_2 \times Z_2$. The backward implication (see [13]) holds true as well. Whenever we have a group G of automorphisms of a graph X satisfying the above assumptions then we can construct a regular map $\mathcal{M}$ with the underlying graph X such that $\mathrm{Aut}(\mathcal{M}) = G$. In particular, if X is a cubic graph we have

Proposition 2.3. [13] Let X be a (simple) cubic graph. Then there is a regular map with the underlying graph X and automorphism group G if and only if $G \leq \mathrm{Aut}(X)$ acts 2-arc-transitively with edge-stabiliser $Z_2 \times Z_2$.
Moreover, the map $\mathcal{M}$ is uniquely determined by G up to Petrie duality.

Assume the underlying surface is orientable, i.e. there exists a subgroup $G^+ = \langle \rho\tau, \lambda\tau \rangle = \langle R, L \rangle \leq$ of G with index 2. A permutation of arcs of the map will be called an orientation preserving automorphism of M if it commutes with R and L. The group of orientation preserving automorphisms $\mathrm{Aut}^+(M)$ acts semi-regularly on arcs of M and if the action is regular then M is called *orientably regular*. If the underlying graph X of a map is simple of valency at least 3, we have a faithful action of both groups on vertices so that $\mathrm{Aut}^+(M) \leq \mathrm{Aut}(M) \leq \mathrm{Aut}(X)$.

If a surface S is orientable we can fix one of the two global rotations. In such case we can describe a map on S by means of rotation and arc- reversing involution acting on the set of arcs of the map. More precisely, by an *oriented map* we mean a triple $\mathcal{M} = (D; R, L)$, where D is the set of arcs, $\langle R, L \rangle$ is a transitive group of permutations of D with L being involutory and R being the rotation. A permutation ψ of D is called a map automorphisms if it commutes with both R and L. The map $\mathcal{M}^{-1} = (D; R^{-1}, L)$ is called the mirror image of $\mathcal{M}$. An oriented map $\mathcal{M}$ is called regular if $\mathrm{Aut}(\mathcal{M})$ acts regularly on D. Similarly as in the non-oriented case we have the following characterisation of oriented regular maps.

Proposition 2.4. [19] Let $\mathcal{M} = (D; R, L)$ be an oriented map. The following three statements are equivalent:
(1) $\mathcal{M}$ is (oriented) regular,
(2) $\mathrm{Aut}(\mathcal{M}) \cong \langle R, L \rangle$,
(3) given edge $e = uv$ the automorphism group $\mathrm{Aut}(\mathcal{M})$ contains two automorphisms, one fixes v and cyclically permutes the incident arcs with v following the local action of R at v, the other rotates the map round the center of uv by 180 degrees interchanging the two arcs associated with uv.

In particular we have

Proposition 2.5. [13] Let X be a (simple) cubic graph. Then there is an oriented regular map with the underlying graph X and with $\mathrm{Aut}(\mathcal{M}) = G$ if and only if $G \leq \mathrm{Aut}(X)$ acts regularly on arcs.
Moreover, the oriented map $\mathcal{M}$ is uniquely determined by G up to mirror image.

Assume that each vertex of $\mathcal{M} = (F, \rho, \lambda, \tau)$ has the same valency k and each face of $\mathcal{M}$ is m-gonal, for some integers k, $m \geq 3$. Then $(\rho\tau)^k = (\rho\lambda)^m = 1$. In this case we say that $\mathcal{M}$ is a map of type $\{m, k\}$. It follows that G is a quotient of the *extended triangle group* with presentation

$$\Delta(m, k, 2) = \langle x, y, z | x^2 = y^2 = z^2 = (xy)^k = (xz)^m = (yz)^2 = 1\rangle.$$

The kernel N of canonical epimorphism $x \mapsto \rho$, $y \mapsto \tau$ and $z \mapsto \lambda$ is a normal torsion free subgroup of $\Delta(m, k, 2)$. The group $\Delta(m, k, 2)$ is the automorphism group of a *k-valent m-gonal tesselation* $\mathcal{U}(m, k)$ of a hyperbolic plane. Hence every regular map arises as a quotient $\mathcal{U}(m, k)/N$, where N is a normal torsion free subgroup of $\Delta(m, k, 2)$ of finite index. The vertices of $\mathcal{U}(m, k)/N$ are the orbits of the action of N on the vertices and two orbits A, B are adjacent if there are vertices $u \in A$ and $v \in B$ such that uv is an edge in $\mathcal{U}(m, k)$. Namely, we have the following statement.

Proposition 2.6. [13] Let X be a cubic graph. Let G be a 2-regular group of automorphisms of X such that an edge stabiliser is isomorphic to $Z_2 \times Z_2$. Then there exists m and a torsion free normal subgroup $N \trianglelefteq \Delta(m, 3, 2)$ such that $G \cong \Delta(m, 3, 2)/N$ and $X \cong \mathcal{U}(m, 3)/N$.

Similarly, setting

$$\Delta^+(m, k, 2) = \langle x, y | x^k = (xy)^m = y^2 = 1\rangle,$$

one can establish a correspondence between the normal torsion free subgroups of finite index of the *triangle group* $\Delta^+(m, k, 2)$ and maps of type $\{m, k\}$. In particular, we have

Proposition 2.7. [13] Let X be a cubic graph. Let G be a 1-regular group of automorphisms of X. Then there exists m and a torsion free normal subgroup $N \trianglelefteq \Delta^+(m, 3, 2)$ such that $G = \Delta^+(m, 3, 2)/N$ and $X \cong \mathcal{U}(m, 3)/N$.

The universal graph $\mathcal{U}(m, 3)$ is 2-regular of girth m. A question arises under what condition a finite quotient $\mathcal{U}(m, 3)/N$ by some normal torsion free subgroup shares the same properties. The following proposition suggests that a combinatorial condition on the number of girth cycles passing through an edge is important.

Proposition 2.8. Let X be a symmetric cubic graph. Then the number of girth cycles c passing through an edge is even. If $c < 2^t$ then X is at most t-regular, for $t = 2, 3, 4, 5$. In particular, if $c = 2$ then either X is 1-regular or it is 2-regular.

Proof. Let m be the number of girth cycles through a vertex. Then $3c = 2m$. Assuming that X is $t+1$-arc-transitive, and considering the automorphisms permuting the $t+1$-arcs based at a fixed arc x we create at least 2^t different girth cycles passing through x. □

3 Five exceptional graphs

In this section we define five exceptional cubic graphs which will play a key role in the following text and mention some of their properties. The first four are of girth 6, the last one is the Coxeter graph, the smallest symmetric cubic graph of girth 7.

Let $n \geq 3$ and $k \in \mathbb{Z}_n \backslash \{0\}$. The *generalised Petersen graph* $GP(n, k)$ is a graph with vertex set $\{x_i, y_i | \, i \in \mathbb{Z}_n\}$ and edge set $\{x_i x_{i+1},\ x_i y_i,\ y_i y_{i+k}; i \in \mathbb{Z}_n\}$.

The generalised Petersen graph $GP(8,3)$**.** The graph is a double cover of $GP(4,3)$ which is the 3-dimensional cube. Hence its group is isomorphic to a semidirect product $(S_4 \times Z_2) : Z_2$.

$$\mathrm{Aut}(GP(8,3)) = \langle h, a, p | h^3 = a^2 = p^2 = (ap)^2 = 1, php = h^{-1}, (ha)^3(h^{-1}a)^3 = 1\rangle$$

The graph is 2-regular, the action of the group determines an octagonal embedding of the graph into the double torus giving rise to a regular map of genus 2 (see [9 (p. 29, Fig. 3.6c)]). One can easily check that girth of the graph is 6 and there are six 6-cycles passing through an edge. However, the graph is only 2-regular showing that the implication in Proposition 2.8 cannot be reversed. More information on this graph one can find in [22].

The generalised Petersen graph $GP(10,3)$**.** Since $GP(10,3)$ is the canonical double cover of the Petersen graph its automorphism group is $\mathrm{Aut}(GP(5,2)) \times Z_2 = S_5 \times Z_2$. By [7] it has presentation

$$\mathrm{Aut}(GP(10,3)) = \langle h, a, p, q | h^3 = a^2 = p^2 = q^2 = 1, qp = pq,$$
$$h^{-1}ph = p, qhq = h^{-1}, apa = q, pq(h^{-1}a)^2(ha)^2(h^{-1}a)^2 = 1\rangle.$$

The automorphism group has 240 elements, and consequently, the graph is 3-regular. Since $\mathrm{Aut}(GP(10,3))$ contains no 1-regular subgroup, $GP(10,3)$ has no regular embedding into an orientable surface. Since it admits a subgroup acting 2-regularly with an edge stabiliser $Z_2 \times Z_2$, it is the underlying graph of a non-orientable regular map. There are two such maps both of type $\{10,3\}$. The maps are Petrie duals of each other.

The Pappus graph 9_3 is the incidence graph of Pappus configuration

$$\{123, 456, 789, 147, 258, 369, 158, 348, 267\},$$

which is a union of the three parallel classes of lines in the affine geometry $AG(2,3)$ (exactly one set of three parallel lines is missing). Consequently, the automorphism group is a semidirect product $(Z_3 \times Z_3) : Z_2$ of a group consisting of 9 translations extended by a point-line duality. The vertex-stabiliser is isomorphic to the dihedral group D_{12}. A presentation of the automorphism group reads by [7] as follows:

$$\mathrm{Aut}(9_3) = \langle h, a, p, q | h^3 = a^2 = p^2 = q^2 = 1, qp = pq,$$
$$h^{-1}ph = p, qhq = h^{-1}, apa = q, (h^{-1}a)^6 = 1\rangle$$

Consequently, the graph 9_3 is 3-regular. Another remarkable property of 9_3 is that it has a hexagonal embedding in the torus giving rise to a self-Petri regular map, the map $\{6,3\}_{3,0}$ in the notation of Coxeter and Moser (see Figure 11).

The Heawood graph is the incidence graph of the Fano plane

$$\mathcal{P} = \{123, 345, 156, 147, 257, 367, 246\}.$$

It follows that the automorphism group of the Heawood graph is $PSL(3,2).2 \cong PGL(2,7)$. In [7] it has presentation:

$$\mathrm{Aut}(He) = \langle h, a, p, q, r | h^3 = a^2 = p^2 = q^2 = r^2 = 1, pq = qp, pr = rp,$$

$$rq = pqr, h^{-1}ph = q, h^{-1}qh = pq, rhr = h^{-1}, apa = p, aqa = r, p(ha)^3(h^{-1}a)^3 = 1\rangle$$

The graph is 4-regular and it admits a 1-regular action. There is a well-known hexagonal embedding of the Heawood graph giving rise to an irreflexible oriented regular map, the map $\{6, 3\}_{2,1}$ in the Coxeter-Moser notation, see Figure 12.

The Coxeter graph. Vertices are antiflags of the Fano plane $\mathcal{P}$, i.e. $\gamma \in V$ if and only if $\gamma = (p, \ell)$ for some line ℓ and a point not incident to ℓ. Two vertices $\gamma = (p, \ell)$ and $\delta = (q, m)$ are adjacent if $\mathcal{P} = \ell \cup m \cup \{p, q\}$. The group $PGL(2, 7)$ has a natural action on the 28 vertices of the Coxeter graph, indeed by [4 (Theorem 12.3.1)] the automorphism group has 336 elements and it is isomorphic to $PGL(3, 2).2 \cong PGL(2, 7)$. A presentation of the group given in [7] reads as follows

$$\mathrm{Aut}(Cox) = \langle h, a, p, q | h^3 = a^2 = p^2 = q^2 = 1, qp = pq,$$

$$h^{-1}ph = p, qhq = h^{-1}, apa = q, pha(h^{-1}a)^2(ha)^2(h^{-1}a)^2 = 1\rangle.$$

Consequently, the Coxeter graph is 3-regular. The automorphism group of the Coxeter graph contains no 1-regular subgroup. It contains 2-regular subgroups but the edge-stabiliser of the respective action is isomorphic to $\mathbb{Z}_4$. Hence the Coxeter graph has no regular embedding into a surface and it is the smallest symmetric cubic graph with this property. There are some other remarkable properties of this graph, see [10, 33, 11] for more information.

4 Graphs with more than two girth cycles passing through an edge

In this section we classify symmetric cubic graphs of girth 6 and 7 such that the number of girth cycles passing through an edge is greater than 2 (Lemma 4.2 and 4.3). It transpires that there are exactly five such graphs. The following Proposition proved in [26] will be useful.

Proposition 4.1. Let X be an arc-transitive cubic graph of girth g. Then X has no cycle separating edge-cut of size $< g$. In particular, X has no edge-cut consisting of $< g$ independent edges.

Let X be a symmetric cubic graph of girth g. Fix a vertex $v \in V(X)$ and denote by $V_i = V_i(v) = \{u \in V(X) | \, d(u, v) = i\}$ the set of vertices at distance i from u. Denote by E_i^{i+1} the edge-set formed by the edges xy, $x \in V_i$ and $y \in V_{i+1}$. Note that E_i^{i+1} is an edge-cut provided it is non-empty. Furthermore, for $j \geq i$ we denote by $V_i^j = V_i \cup V_{i+1} \cup \dots V_j$, and by $[V_i^j]$ the subgraph induced by V_i^j. In what follows we denote by α a fixed element of order 3 in the vertex-stabiliser G_v of v in G, where G is the automorphism group $\mathrm{Aut}(X)$ of X. By the *quotient* $\bar{X} = X/\langle\alpha\rangle$ we mean a graph which vertices are orbits of the action of α on the vertex set of X, two orbits $[v]$ and $[u]$ being adjacent if there exist vertices $v' \in [v]$ and $u' \in [u]$ such that $v'u'$ is an edge in X. We say that an edge xy in X is of *type* AB, if $x \in A$ and $y \in B$, where A, B are orbits of α. The notion of the type of an edge naturally extends to walks in X. In particular, every walk in X projects to a walk in $\bar{X}$ and this fact is expressed by saying that it has some type. There may be double adjacency between two 3-orbits meaning that the corresponding induced subgraph is a 6-cycle. Orbits of length three will be denoted by capital letters while we use the same small letter for a fixed point as well as for the respective orbit of length 1. Since we assume $g \geq 6$ there are no edges joining vertices belonging to the same 3-orbit. Hence the mapping $X \to \bar{X}$ taking $v \mapsto [v]$ restricted to the union of 3-orbits defines a true regular covering between subgraphs.

Lemma 4.2. Let X be a symmetric connected cubic graph with girth 6 and let c be the number of 6-cycles passing through an edge in X. Then, $c = 2, 4, 6$, or 8. If $c > 2$ then X is isomorphic to one of the following four graphs: Heawood graph, Pappus graph 9_3, generalised Petersen graph $GP(8,3)$, and generalised Petersen graph $GP(10,3)$.

Proof. By Proposition 2.8, c is even. There are 8 vertices at distance 2 from a given edge e and edges joining these 8 vertices are in 1-1 correspondence with 6-cycles going through e. Since X has valency 3, we have $c = 2, 4, 6$, or 8.

The proof of the statement is done by a case-to-case analysis. Firstly observe that α acts freely on the 6 elements of V_2 and it fixes at most two vertices in V_3. Indeed, a fixed vertex u in V_2 in the action of α implies an existence of a 4-cycle going through v and u contradicting the assumption. Hence V_2 splits into two α-orbits of length 3, say B and C. If a vertex u in V_3 is fixed by α then either it is adjacent to all the vertices in B, or to all the vertices in C. Assume there are three vertices in V_3 fixed by α. Then two of them share the same vertices in their neighbourhood which gives rise to a 4-cycle, a contradiction.

Note that there are at least six different 6-cycles passing through v and there is no 4-cycle in X. Then one of the following cases happens:

(1) there are two vertices fixed by α in V_3, and they are adjacent to different 3-orbits in V_2;
(2) there is one vertex fixed by α in V_3 and at least one 3-orbit of which each vertex is joined to at least two vertices in V_2;
(3) the action of α is free on V_3, there are exactly two 3-orbits in V_3 and E_2^3 is a union of cycles.

In what follows we denote by A the unique 3-orbit in V_1 and by B and C the two 3-orbits in V_2.

Case 1. Denote by u the fixed vertex adjacent to B and by w the fixed vertex adjacent to C. Let E, F be the 3-orbits adjacent to B and C in V_3, respectively. Assume $E \neq F$. Consider the set of 6-cycles passing through an edge e of type AC. The subgraph $[V_0^3]$ contains exactly two 6-cycles passing through e. They are both of type $(vACwCA)$. The only possibility to create another 6-cycle passing through e is to extend the unique path of type $EBACF$ onto a 6-cycle. But there can be at most one such 6-cycle. Hence we have at most three 6-cycles passing through e, a contradiction. Hence $E = F$. Note that $[V_0^3]$ is uniquely determined. By Proposition 4.1 X cannot contain an independent 3-edge-cut, hence the 3-orbit $E = F$ is adjacent to a fixed vertex in V_4. In this way, we have constructed a unique graph on 16 vertices, namely $GP(8,3)$ (see Fig. 1).

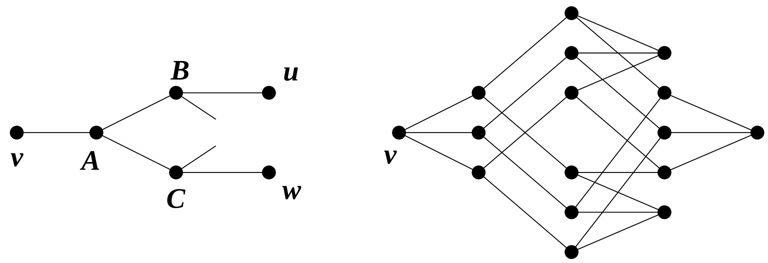

FIGURE 1. A quotient in Case 1 and the generalized Petersen graph $GP(8,3)$

Case 2. Let there be exactly one vertex u fixed by α in V_3 and one 3-orbit D which vertices are joined to at least two vertices in V_2. We may assume that u is adjacent to B.

Subcase 2.1: D is adjacent to both B and C (see Fig. 2). Denote by $E \subseteq V_3$ the third 3-orbit adjacent to C. Clearly, the orbit E is not formed by a fixed point, otherwise we are in Case 1. Assume $E \neq D$. Let e be an edge of type AC. By the assumption there are at least four 6-cycles passing through e. There are two 6-cycles passing through e in the subgraph $[V_0^3]$, one of type $(vABDCA)$, the other one of type $(uBDCAB)$. The only possibility to create another 6-cycle passing through e is to extend the unique path of type $DBACE$ containing e onto a 6-cycle. In this way only one more 6-cycle passing through e can be constructed, a contradiction. Hence $E = D$ and there is a double adjacency between C and D. The graph has 14 vertices and it is isomorphic to the Heawood graph.

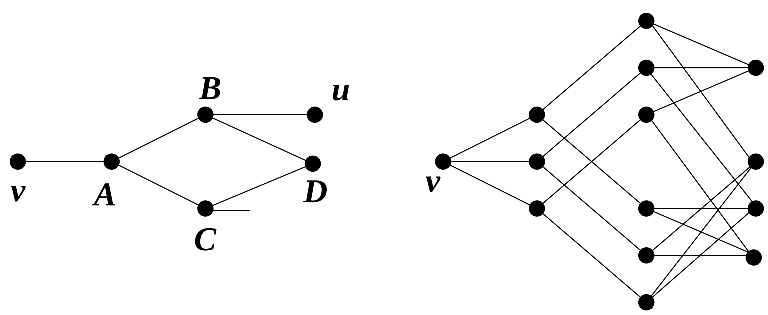

FIGURE 2. A quotient in Case 2.1 and the Heawood graph

Subcase 2.2: D is doubly adjacent to C (see Fig. 3). There is a 3-orbit E in V_3 adjacent to B. We may assume that $E \neq D$, otherwise we get the previously discussed subcase. As above take an edge e of type AC. There are exactly two 6-cycles in $[V_0^3]$ both of type $(vACDCA)$. To create another 6-cycle we need to extend the unique path of length 3 containing e which is of type $EBAC$. Since no 4-cycle exists, at most one such 6-cycle is constructed, a contradiction.

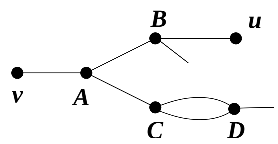

FIGURE 3. A quotient in Case 2.2

Case 3. We distinguish two subcases.

Subcase 3.1: There are two orbits D, E in V_3, both adjacent to both B and C (see Fig. 4).

Note that every 3-arc based at v extents to a 6-cycle. By the vertex-transitivity it holds true for any 3-arc. Consider a 3-arc of type $ECDB$. Then a 6-cycle passing through it contains an edge of type BE, as well. Hence, two vertices in E are connected by a path of length 2. This is possible only if the orbit adjacent to E in V_4 is a fixed vertex. Similarly, there is a fixed vertex adjacent to D. In this way a unique graph on 18 vertices is constructed, namely the Pappus graph 9_3.

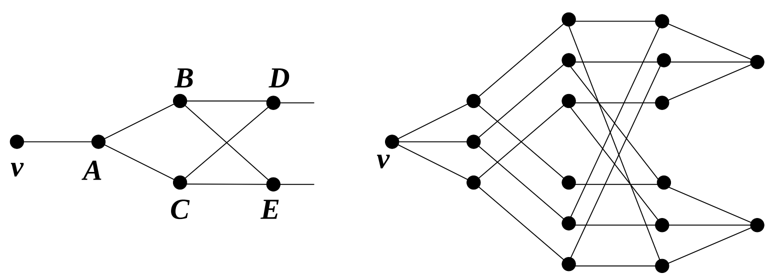

FIGURE 4. A quotient in Subcase 3.1 and the Pappus graph 9_3

Subcase 3.2: There are two orbits D, E in V_3, doubly adjacent to B and C, respectively (see Fig. 5).

As above every 3-arc extends to a 6-cycle. Consider a 3-arc of type $ECAB$. There is no 6-cycle in $[V_0^3]$ passing through such a 3-arc. Hence there is a 3-orbit F in V_4 adjacent to both D and E. Since we cannot have an independent 3-edge-cut, the orbit adjacent to F in V_5 is a fixed point. In this way a unique cubic graph with 20 vertices is constructed, namely $GP(10, 3)$.

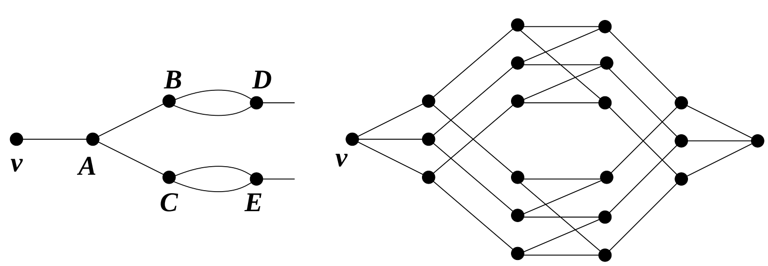

FIGURE 5. A quotient in Subcase 3.2 and the generalized Petersen graph $GP(10, 3)$

Lemma 4.3. Let X be a symmetric cubic graph of girth 7 such that there are more than two 7-cycles passing through an edge of X. Then X is the Coxeter graph.

Proof. Since X has girth 7, there are two 3-orbits in V_2 denoted by B and C, and four 3-orbits in V_3 denoted by D, E, F and G. We may assume that D and E, F and G are adjacent to B, C respectively. Let A be the 3-orbit in V_1. Then A is adjacent to B, C and to v. Clearly, α acts freely on V_i for $i = 1, 2$ or 3. Recall that we use c to denote the number of 7-cycles passing through an edge in X. The proof splits into two claims.

Claim 1. Let $c \geq 4$. Then $c = 4$ and $[V_3]$ is a perfect matching consisting of 6 edges. Furthermore, every 3-arc is included in precisely one 7-cycle.

Let m be the number of edges in V_3. Since α acts freely on V_3 and $c \geq 4$, $m = 6$, 9 or 12. If $m = 12$ then X has 22 vertices. But there is no symmetric cubic graph of girth 7 with 22 vertices. If $m = 9$ we have only 6 edges in E_3^4. By Proposition 4.1 E_3^4 cannot separate cycles. It follows that the complement $\bar{V}_0^3$ has at most 4 vertices, and consequently, the whole graph has at most 26 vertices. However, the least symmetric cubic graph of girth 7 is the Coxeter graph having 28 vertices [7], a contradiction. Thus $m = 6$. Since each edge in V_3 corresponds a 7-cycle passing through v, we have $c = 4$. Since the girth is 7, two 3-orbits in V_3 cannot be doubly adjacent.

Suppose that there is a 3-orbit in $[V_3]$, say E, adjacent to two 3-orbits in V_3. We may assume that E is adjacent to F and one of D and G (see Fig. 6). Since X is vertex-transitive, for any arc v_1v_2 there is a 3-arc $v_1v_2v_3v_4$ and there are two 7-cycles passing through the 3-arc such that the number of the common vertices of the two 7-cycles is 4 because it holds true for each arc of type vA. But this is not true for an arc of type AC. In fact, for a 3-arc of type $ACFE$ or $ACGE$ there is only one 7-cycle passing through the 3-arc, which is of type $(ACFEBAv)$ or $(ACGEBAv)$. For a 3-arc of type $ACFH$ or $ACGH$, where H is an orbit in V_4, any two 7-cycles passing through such a 3-arc have at least 5 vertices in common, because the two 7-cycles pass through a 4-arc of type $BACFH$ or $BACGH$ as well.

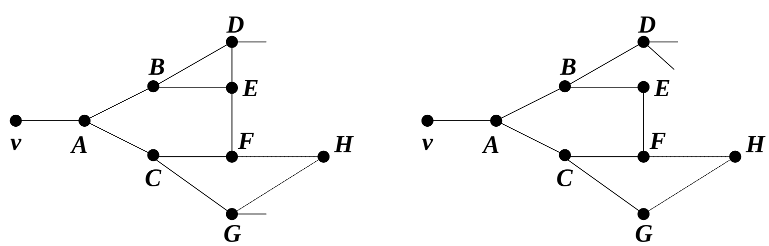

FIGURE 6. Quotients in Claim 1

Thus, $[V_3]$ consists of a perfect matching of 6 edges, implying that each 3-arc based at v extends to a unique 7-cycle and by the transitivity of the action on vertices this property holds true for any 3-arc in X. As concerns the matching between the four 3-orbits in V_3, there are two cases to consider: we have either edges of type DE and FG, or DF and EG.

Claim 2 The graph X has at most 44 vertices.

We shall distinguish two cases.

Case 1. There are some fixed points in V_4.

We may assume that a fixed point u is adjacent to D. If there is a matching of type DE, FG in V_3 then we get a 5-cycle of type $uDEBD$ passing through u, a contradiction. Thus, we assume that there are edges of type DF, EG in V_3. If there is another fixed point, say w, in V_4 then $[V_0^3 \cup \{u, w\}]$ is separated by a 6-edge cut, which implies that X has at most 28 vertices. Thus, we may assume that there is only one fixed point in V_4, that is u. Since E_3^4 has 12 edges, there are one, two or three 3-orbits in V_4. If there is one 3-orbit in V_4 then X has 26 vertices. If there are two 3-orbits in V_4 then one of them, say H, is adjacent to two 3-orbits in V_3, then $[V_0^3 \cup H \cup \{u\}]$ is separated by a 6-edge cut forcing that X has at most 30 vertices. Thus, we may assume that there are three 3-orbits in V_4, say H, I and J with adjacences to E, F and G respectively (see Fig. 7). By the existence of 7-cycle passing through a 3-arc of type $ABEH$, $ACFI$ or $ACGJ$ (Claim 1), there are at least 6 edges in the induced subgraph $[H \cup I \cup J]$. It follows that X has 32 vertices or $[V_0^4]$ is separated by a 6-edge cut. In the latter case, X has at most 36 vertices.

Case 2. The action of α is free on V_4.

We distinguish three subcases.

Subcase 2.1. There are four 3-orbits H, I, J, K in V_4, say we have edges of type DH, EI, FJ and GK (see Fig. 7.

By Claim 1, every 3-arc is included in precisely one 7-cycle. Considering the 3-arcs of type $ABDH$, $ABEI$, $ACFJ$ and $ACGK$, we derive that each 3-orbit in V_4 is adjacent to at least one 3-orbit in V_4. Furthermore, H and I are adjacent to one of J or K. It follows that $[V_4]$ has 6, 9 or 12 edges. If $[V_4]$ has 12 or 9 edges then X has 34 vertices or X has a 6-edge cut. For the later, X has at most 38 vertices. Thus, we may assume that $[V_4]$ has 6 edges, that is $[V_4]$ consists of a perfect matching of 6-edges, which are of type HJ and IK or of type HK and IJ.

If there is a matching of type DE, FG in V_3 then the existence of 7-cycle passing through a 3-arc of type $HDEB$ implies that there is a fixed vertex in V_5 that is adjacent to H. Similarly, there are another three fixed vertices in V_5 that are adjacent to I, J and K, respectively. It follows that X has 38 vertices. Thus, we assume the matching in V_3 is of type DF, EG (Fig. 7). Consider the 3-arcs of type $HDFC$ and $IEGC$, and the existence of 7-cycles passing through these 3-arcs implies that there are two 3-orbit L and M in V_5 such that L is adjacent to H and K, and M is adjacent to I and J. It follows that either M is adjacent to L, or $[V_0^5]$ is separated by a 6-edge-cut, implying that X has 40 vertices or at most 44 vertices.

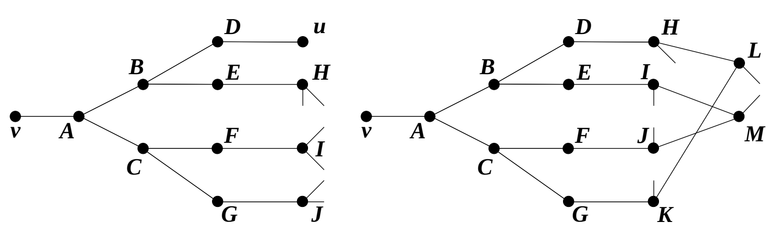

Figure 7. Two quotients in Case 1 and Subcase 2.1

Subcase 2.2. There are three 3-orbits in V_4, say H, I and J (see Fig. 8).

Since E_3^4 has 12 edges, one of 3-orbits in V_4, say I, is adjacent to two 3-orbits in V_3 and so H and J are adjacent to one 3-orbit in V_3. As above considering a 3-arc starting at A and terminating at H or J, we derive that H and J is adjacent to some orbits in V_4. Thus, H and J must be adjacent. If $[V_4]$ has 6 edges then the induced subgraph $[V_0^4]$ is separated by a 3-edge cut and hence X has 32 vertices. We may assume that $[V_4]$ has 3 edges and so E_4^5 has 9 edges. If there is a fixed point in V_5, say u, then the induced subgraph $[V_0^4 \cup \{u\}]$ is separated by a 6-edge cut and so X has most 36 vertices. Thus, we assume that there is one, two or three 3-orbits in V_5. If there is one 3-orbit in V_5 then the graph X has 34 vertices. If there are two 3-orbits in V_5 then a 3-orbit in V_5, say K, is adjacent to two 3-orbits in V_4. This implies that the induced subgraph $[V_0^4 \cup K]$ is separated by a 6-edge cut and so X has at most 38 vertices. Now, we assume that there are three 3-orbits in V_5, say K, L and M that are adjacent to H, I and J respectively (Fig. 8). Considering a 3-arc starting at V_2 and terminating at V_5, Claim 1 implies that $[V_5]$ has at least 6 edges. It follows that $[V_5]$ has 6 or 9 edges. For the later, X has 40 vertices. Let $[V_5]$ have 6 edges. Then $[V_0^5]$ is separated by a 6-edge cut and so X has at most 44 vertices.

Subcase 2.3. There are two 3-orbits in V_4, say H and I (see Fig. 8).

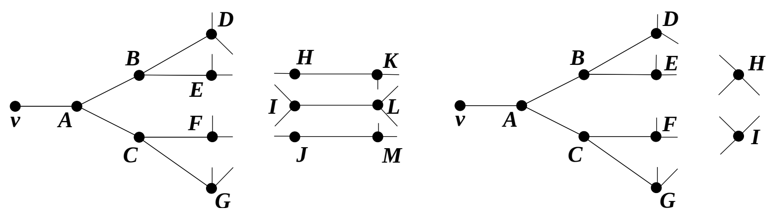

FIGURE 8. Two quotients in Subcases 2.2 and 2.3

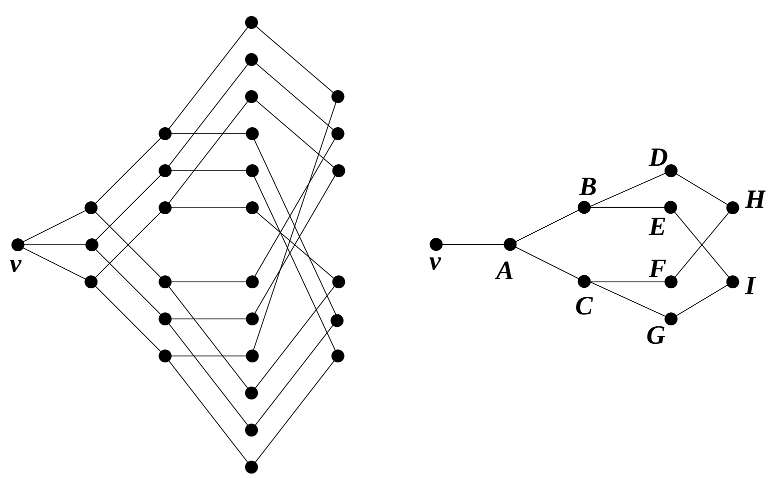

FIGURE 9. The Coxeter graph and its quotient

If one 3-orbit in V_4, say H, is adjacent to three 3-orbits in V_3 then $[V_0^3 \cup H]$ is separated by a 3-edge cut. This implies that there is a fixed point in V_4, which is discussed in Case 1. Thus, we may assume that both H and I are adjacent to two 3-orbits in V_3. If H and I are not adjacent then $[V_0^4]$ is separated by a 6-edge cut. Hence X has at most 32 vertices. If H and I are adjacent then X has 28 vertices. In this case, the Coxeter graph appears (see Fig. 9). The proof of Claim 2 is complete.

Checking the list of arc-transitive cubic graphs in [7] we see that there are only 17 arc-transitive graphs with at most 44 vertices. Out of these 17 graphs only one, the Coxeter graph has girth 7 and it satisfies the property that there are exactly four 7-cycles passing through an edge in it. $\square$

5 Symmetric cubic graphs with exactly two girth cycles passing through an edge

Theorem 5.1. Let X be a symmetric connected cubic graph of girth g such that there are exactly two girth cycles passing through an edge. Then one of the following three cases happen:

(1) X is 2-regular, and $\mathrm{Aut}(X)$ is a quotient of $\Delta(g,3,2) = \langle x,y,z|x^2 = y^2 = z^2 = (xy)^g = (yz)^3 = (xz)^2 = 1\rangle$ by some normal torsion free subgroup,

(2) X is 1-regular, and $\mathrm{Aut}(X)$ is a quotient of $\Delta^+(g,3,2) = \langle x,y|x^3 = y^2 = (xy)^g = 1\rangle$ by some normal torsion free subgroup,

(3) X is 1-regular and g is even, there exists $m > g$ such that $\mathrm{Aut}(X)$ is a quotient of $\Delta^+(m,3,2;g/2) = \langle x,y|x^3 = y^2 = (xy)^m = [x,y]^{g/2} = 1\rangle$ by some normal torsion free subgroup.

Proof. By Proposition 2.8 X is either 1-regular or 2-regular. Let $G = \mathrm{Aut}(X)$ and let $e = vu$ be an edge in X. Let $\{2,4,u\}$ and $\{1,3,v\}$ be the neighbors of v and u, respectively. Since there are exactly two girth cycles passing through an edge there is at most one girth cycle passing through a 2-arc. Thus, we may assume that two girth cycles, say C and C', pass through the 3-arcs $1uv2$ and $3uv4$, respectively. Then, C and C' are the only two girth cycles passing through e.

Assume X is 2-regular. Since $G_v \cong S_3$, there are two involutions y and z in G_v such that y interchanges 2 and u, and z interchanges 2 and 4. Then, $G_v = \langle y,z\rangle$ and yz has order 3. Let $x \in G$ interchange u and v. We claim that x is an involution. Otherwise, x permutes the neighbors of e as (1234) or (1432) and hence x sends the 3-arc $1uv2$ onto $2vu3$ or $4vu1$. This is impossible because there is no girth cycle passing through $2vu3$ or $4vu1$. Thus, x is an involution and $G_e = \langle x,z\rangle \cong Z_2 \times Z_2$. One can easily check that either x or xz preserve the cycle C, say it is x. Thus, x interchanges u and v, 1 and 2, and 3 and 4. Furthermore, xy fixes C and maps the 2-arc $1uv$ onto $uv2$. By the 2-regularity of X, $|xy| = g$ and by the symmetry and connectivity, $G = \langle G_v, x\rangle = \langle x,y,z\rangle$.

Assume that X is 1-regular. Let $x \in G_v$ permute the neighbors of v as $(4u2)$ and let y be the involution in G interchanging u and v. Then, x takes C' onto C and y permutes the neighbors of e as (12)(34) or (14)(23), which corresponds y fixing C and C', or interchanging C and C'. If y interchanges C and C' then yx fixes C and takes $1u$ onto uv. Thus, $|yx| = |xy| = g$ and we have the Case (2). If y fixes C and C' then $[x,y] = xyx^{-1}y^{-1}$ fixes C' and takes $4v$ onto $u3$, that is $[x,y]$ rotates C' with step two. If g is odd then both $[x,y]^{\frac{g+1}{2}}$ and xy take $4v$ onto vu. By the 1-regularity, $[x,y]^{\frac{g+1}{2}} = xy$ and hence $[x,y]^{\frac{g-1}{2}} = yx$. It follows that $yx[x,y] = xy$ and so $x^2y = yx^2$. This implies x and y commute because x has order 3. Clearly, it is impossible. Thus, g must be even and $[x,y]^{\frac{g}{2}} = 1$. Now, we show that if $(xy)^m = 1$ then $m > g$. In fact, xy sends $4v$ onto vu, vu onto $u1$. By considering the action of xy on the arc $4v$, we can get a cycle containing the 3-arc $4vu1$. Since no girth cycle passing through $4vu1$, xy has order more than g. $\square$

Theorem 5.2. Let X be a symmetric cubic graph of girth 7. Then the following statements hold true:

(0) there is no 4-arc-transitive cubic graph of girth 7,
(1) X is 3-regular if and only if it is the Coxeter graph,
(2) X is 2-regular if and only if $X = \mathcal{U}(7,3)/N$, where $N \triangleleft \Delta(7,3,2)$ is a proper normal subgroup of finite index, $\mathrm{Aut}(X) \cong \Delta(7,3,2)/N$.
(3) X is 1-regular if and only if $X = \mathcal{U}(7,3)/N$, where $N \triangleleft \Delta^+(7,3,2)$ is a normal subgroup of finite index but $N \ntriangleleft \Delta(7,3,2)$, $\mathrm{Aut}(X) \cong \Delta^+(7,3,2)/N$.

Moreover, $\mathcal{U}(7,3)/N$ is of girth 7 for any non-trivial normal subgroup $N \triangleleft \Delta^+(7,3,2)$ of finite index.

Proof. By Lemma 4.3, if X is not the Coxeter graph it has exactly two girth cycles passing through an edge. Consequently, X is either 1- or 2-regular. Applying Theorem 5.1 (1) we get that if X is 2-regular then $X \cong \mathcal{U}(7,3)/N$ for some normal subgroup $N \triangleleft \Delta(7,3,2)$ of finite index. Let X be 1-regular. Since the girth is odd Case (2) of Theorem 5.1 applies proving $X = \mathcal{U}(7,3)/N$ for some $N \triangleleft \Delta^+(7,3,2)$ but $N \ntriangleleft \Delta(7,3,2)$. To see the opposite direction observe that $\Delta(7,3,2)/N$ acts 2-regularly on $X = \mathcal{U}(7,3)/N$ provided $N \triangleleft \Delta(7,3,2)$. Similarly, $\Delta^+(7,3,2)/N$ acts 1-regularly on $X = \mathcal{U}(7,3)/N$ if $N \triangleleft \Delta^+(7,3,2)$. Finally, observe that a nontrivial $N \triangleleft \Delta^+(7,3,2)$ is torsion free. Hence, the respective quotient $X = \mathcal{U}(7,3)/N$ is a symmetric cubic graph of girth at most 7. Since 7 divides the number of vertices of $X = \mathcal{U}(7,3)/N$, if X is exceptional of girth ≤ 6 then X is the Heawood graph, which cannot be since X has cycles of length 7. Thus assuming that the girth of X is at most 6 we get that $\mathrm{Aut}(X)$ is either isomorphic to $\Delta(6,3,2)/K$ or $\Delta^+(6,3,2)/K$ for some normal subgroup $K \triangleleft \Delta^+(6,3,2)$. Since $\Delta^+(6,3,2) \cong (Z \times Z) : Z_6$ is solvable (see the next section) $\mathrm{Aut}(X)$ is solvable as well. However, a non-trivial finite quotient of $\Delta(7,3,2)$, or of $\Delta^+(7,3,2)$ is insolvable. Hence the girth of $X = \mathcal{U}(7,3)/N$ is 7. $\square$

6 Quotients of the tesselation $\mathcal{U}(6,3)$ and graphs of girth 6

Let us consider the hexagonal infinite tessellation $\mathcal{U}(6,3)$ of the Euclidean plane $\mathbb{E}_2$. Assume $\mathbb{E}_2$ is endowed with the standard Cartesian coordinate system. Let $\vec{i} = (1,0)$ and $\vec{j} = (1/2, \sqrt{3}/2)$ be two vectors based at $(0,0)$. Note that $\vec{j}$ arises by counterclockwise rotation of $\vec{i}$ by 60 degrees. Without loss of generality we may identify the centers of the hexagons of the tesselation with points of the plane with coordinates $m\vec{i} + n\vec{j}$, where m and n are integers. Hence we can identify the center of every hexagon with a couple (m,n) of integers. Let us denote by $\psi_{m,n} = m\vec{i} + n\vec{j}$ the translation of $\mathbb{E}_2$ shifting the points by the vector $m\vec{i} + n\vec{j}$, hence $\vec{x}^{\psi_{m,n}} = \vec{x} + m\vec{i} + n\vec{j}$. Let ρ be the counterclockwise rotation of the plane by 60 degrees around the point $(0,0)$. See Figure 10.

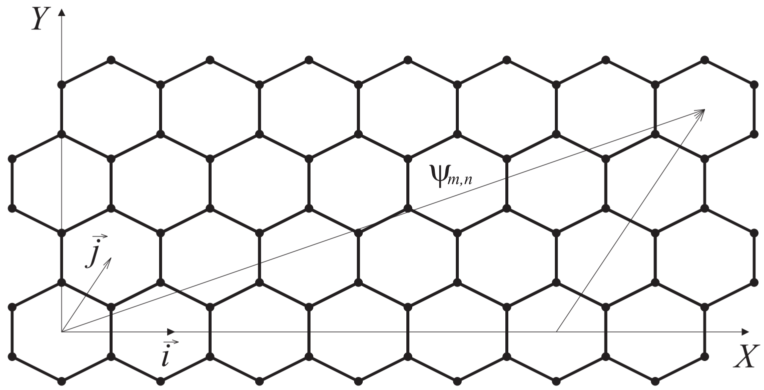

FIGURE 10. Construction of the $\mathcal{G}(m,n)$

Let $N = N(m,n) = \langle \psi_{m,n}, \rho^{-1}\psi_{m,n}\rho \rangle = \langle \psi_{m,n}, \psi_{-n,m+n} \rangle$. It follows that the group of translations $N(1,0) = \langle \psi_{1,0}, \psi_{0,1} \rangle$ acts regularly on the set of centers of all hexagons and it forms a semidirect product $G = N(1,0) \rtimes \langle \rho \rangle$. Observe that G acts 1-regularly on arcs of $\mathcal{U}(6,3)$, hence it is isomorphic to $\Delta^+(6,3,2)$. Indeed, it is easy to check that the assignment $x \mapsto \rho^2\psi_{1,0}$ and $y \mapsto \rho^3\psi_{1,0}$ extends to an isomorphism $\Delta^+(6,3,2) = \langle x, y | x^3 = y^2 = (xy)^6 = 1 \rangle \to G$. This shows that $\Delta^+(6,3,2) \cong N(1,0) \rtimes \langle \rho \rangle$. Since $\Delta(6,3,2)$ is an 2-extension of $\Delta^+(6,3,2)$ by an element of order 2 taking $x \mapsto x^{-1}$ and $y \mapsto y$ we have $\Delta(6,3,2) \cong \Delta^+(6,3,2) \rtimes Z_2$. Using the above interpretation of the triangle group $\Delta^+(6,3,2)$ in the group of isometries of the Euclidean plane one can prove (see [9]) that $N(m,n) \triangleleft \Delta^+(6,3,2)$ is a torsion-free normal subgroup of $\Delta^+(6,3,2)$ for any two non-negative integers m, n, $m+n \neq 0$ and any torsion free normal subgroup of the triangle group $\Delta^+(6,3,2)$ is of this form. Moreover, every torsion free normal subgroup of the extended triangle group $\Delta(6,3,2)$ is $N(m,n)$ for some m, n satisfying $mn = 0$ or $m = n$. It follows that $\mathcal{U}(6,3)/N(m,n)$ is an (oriented) regular map of type $(6,3)$ in the torus, and the respective arc-transitive graph will be denoted by $\mathcal{G}(m,n)$. It is easy to get a picture of the graph $\mathcal{G}(m,n)$ by identifying the parallel sides of the fundamental region (a connected sector of the plane containing representatives of the orbits in the action of $N(m,n)$). In this particular case, the fundamental region forms a parallelogram which corners coincide with the centers of four hexagons with coordinates $(0,0)$, (m,n), $(-n, m+n)$, $(m-n, m+2n)$ in the coordinate system defined by the unit vectors $\vec{i}$, $\vec{j}$.

The automorphism group of $\mathcal{G}(m,n)$ contains a 1-regular subgroup isomorphic to $\Delta^+(6,3,2)/N(m,n)$ and it contains a 2-regular subgroup of the form $\Delta(6,3,2)/N(m,n)$ if and only if $mn(m-n) = 0$ (see [9] (p. 107)). Since $\mathcal{G}(m,n) \cong \mathcal{G}(n,m)$ in what follows we will assume $m \leq n$.

Lemma 6.1. Let $G = \langle x, t | \ x^3 = t^2 = [x,t]^3 = 1 \rangle$. Then $G \leq \Delta(6,3,2)$ is an index 2 subgroup and every normal torsion free subgroup $N \triangleleft G$ of finite index is a normal torsion free subgroup of $\Delta(6,3,2)$ as well. In particular, $N \cong N(m,m)$ or $N \cong N(0,m)$ for some positive integer m.

Proof. Identifying x with an element of order 3 in the stabiliser of a vertex v of $\mathcal{U}(6,3)$ and t with a reflection taking v onto one of its neighbours we get a 1-regular action of G on $\mathcal{U}(6,3)$. Since $\Delta(6,3,2) \cong \mathrm{Aut}(\mathcal{U}(6,3))$ and it acts 2-regularly,

$G \leq \Delta(6,3,2)$ and it is an index 2 subgroup. Moreover, $G^+ = \Delta^+(6,3,2) \cap G$ consists of elements expressed as words in terms of the generators containing even number of appearances of t. Hence G^+ is an index 2 subgroup of G. If $N \triangleleft G$ is a normal torsion free subgroup of finite index, then $N^+ = N \cap G^+ \leq \Delta^+(6,3,2)$ is such a group as well. Hence $N^+ = N(m,n)$ for some integers m, n. Since it is normal in G, N^+ is invariant under the conjugation by the reflection t. Hence $N^+ \triangleleft \Delta(6,3,2)$ as well. Either $N = N^+$, or $N = \langle N^+, t\rangle$. However, the latter case is excluded since we assume that N is torsion free. Hence $N = N^+$ and we are done. □

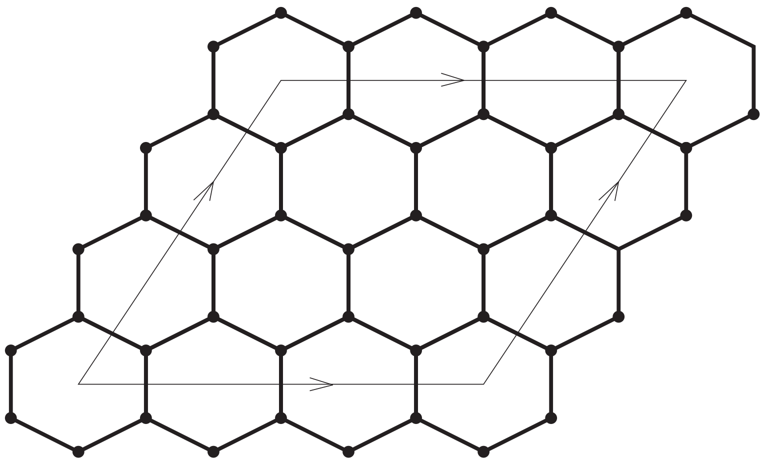

FIGURE 11. Pappus graph represented as $\mathcal{G}(3,0)$

Now we are ready to prove the following classification theorem.

Theorem 6.2. Let X be a symmetric cubic graph of girth 6. Then the following statements hold:

(0) there is no 5-regular cubic graph of girth 6,
(1) X is 4-regular if and only if it is the Heawood graph,
(2) X is 3-regular if and only if it is the Pappus graph, or the generalized Petersen graph $GP(10,3)$,
(3) X is 2-regular if and only if X is the generalized Petersen graph $GP(8,3)$ or $X \cong \mathcal{G}(m,n) = \mathcal{U}(6,3)/N(m,n)$ with $\mathrm{Aut}(X) \cong \Delta(6,3,2)/N(m,n)$, where $0 < m = n$ or $0 = m < n$, and (m,n) is different from $(0,1)$, $(1,1)$, $(0,2)$, $(0,3)$,
(4) X is 1-regular if and only if $X \cong \mathcal{G}(m,n) = \mathcal{U}(6,3)/N(m,n)$, with $\mathrm{Aut}(X) \cong \Delta^+(6,3,2)/N(m,n)$ for some integers m,n satisfying $0 < m < n$, and $(m,n) \neq (1,2)$.

Proof. Assume the number of 6-cycles passing through an edge is grater than 2. By Lemma 4.2 X is one of the four exceptional graphs. Note that 3-arc-transitivity of X implies that X has more than two 6-cycles passing through an edge and thus if X is 3-arc-transitive it is one of the exceptional graphs. Checking the symmetries of the exceptional graphs we get the first three items of the statement. The graph $GP(8,3)$ is 2-regular although it has more than two girth cycles passing through an edge. In what follows we assume that there are exactly 2 girth cycles passing through an edge in which case X is at most 2-regular.

If X is 2-regular Theorem 5.1 (1) applies. Since every normal torsion free subgroup of $\Delta(6,3,2)$ of finite index is either $N(m,m)$ or $N(0,m)$ for some positive integer m we have proved item (3). The exceptional groups $N(0,1)$, $N(1,1)$, $N(0,2)$ and $N(0,3)$ give rise, respectively, to a graph with multiple edges, to the complete bipartite graph $K_{3,3}$ of girth 4, to the cube Q_3 of girth 4 and to the Pappus graph which is known to be 3-regular, see Fig. 11. From the remaining cubic graphs which are 2-regular of girth at most 5 the graphs K_4, $GP(5,2)$ and the dodecahedron are not isomorphic to $\mathcal{G}(m,m)$, or to $\mathcal{G}(0,m)$. An easy argument to see it comes from the fact that the number of vertices of $\mathcal{G}(m,m)$ is $6m^2$, while the number of vertices of $\mathcal{G}(0,m)$ is $2m^2$ (see [9 (p. 107)]).

If X is 1-regular then by Lemma 6.1 case (3) of Theorem 5.1 cannot happen. Hence Theorem 5.1 (2) applies. Consequently, X is one of $\mathcal{G}(m,n) = \mathcal{U}(6,3)/N(m,n)$ for some integers $0 < m < n$. The exceptional group $N(2,1)$ gives rise to the Heawood graph which is known to be 4-regular, see Fig. 12. □

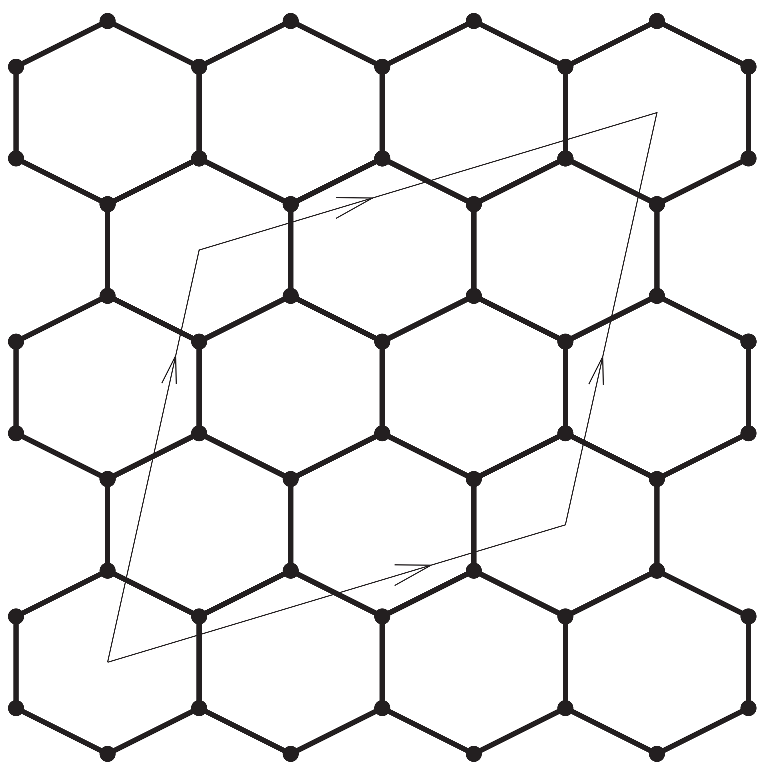

FIGURE 12. The Heawood graph as $\mathcal{G}(2,1)$

Corollary 6.3. Let X be a symmetric cubic graph of girth 6. Then $X \cong \mathcal{G}(m,n)$ for some integers $m \leq n$, $m+n > 1$, except X is the generalized Petersen graph $GP(8,3)$ or $GP(10,3)$. In particular, all symmetric cubic graphs of girth 6 are bipartite.

Proof. First part follows from Theorem 6.2 The biparticity of $GP(8,3)$ and $GP(10,3)$

can be verified directly from the definition. As concerns the graphs $\mathcal{G}(m,n)$, observe that the group of translations $N(1,0)$ acts on the vertices of $\mathcal{U}(6,3)$ with two orbits forming a bipartition of the vertex set. It follows that the vertex-orbits of $N(1,0)/N(m,n) \le \mathrm{Aut}(\mathcal{G}(m,n))$ form a bipartition of $\mathcal{G}(m,n) = \mathcal{U}(6,3)/N(m,n)$. □

Theorem 6.4. Let $X = \mathcal{G}(m,n)$, where $0 \le m \le n$ and

$$(m,n) \notin \{(0,0),(0,1),(1,1),(0,2),(1,2),(0,3)\}.$$

Then either

(1) $0 < m < n$ and $\mathrm{Aut}(X) = \langle x,y |\ x^3 = y^2 = (xy)^6 = 1, (x^{-1}yxy)^m(xyx^{-1}y)^n = 1\rangle$;
  or
(2) $\mathrm{Aut}(X) = \langle t,u,z |\ t^2 = u^2 = z^2 = (tu)^3 = (uz)^2 = (tz)^6 = 1, (utuztz)^m(tzutuz)^n = 1\rangle$, where $mn(m-n) = 0$.

Proof. By the assumptions X is 1-regular or 2-regular. By Theorem 6.2 $\mathrm{Aut}(X) \cong \Delta^+(6,3,2)/N(m,n)$ provided $0 < m < n$. Hence $\mathrm{Aut}X = \langle x,y |\ x^3 = y^2 = (xy)^6 = 1, \dots\rangle$ is a quotient by $N(m,n) = \langle \psi_{m,n}, \psi_{-n,m+n}\rangle \cong Z \times Z$. Without loss of generality we may identify x with the 120 degree counterclockwise rotation of $\mathcal{U}(6,3)$ around the point $(0,0)$ and y with the 180 degree turn round the center of the common edge of the hexagons with centers $(0,0)$ and $(1,0)$. This identification establishes an embedding of $\Delta^+(6,3,2) = \langle x,y |\ x^3 = y^2 = (xy)^6 = 1\rangle$ in the group of isometries of the Euclidean plane. Direct computation of images of the points $(0,0)$ and $(1,0)$ shows that $\psi_{2,-1} = x^{-1}yxy$ and $\psi_{1,1} = xyx^{-1}y$. Hence the relation $(x^{-1}yxy)^m(xyx^{-1}y)^n = 1$ transforms to $\psi_{2,-1}^m\psi_{1,1}^n = \psi_{2m+n,n-m} = 1$. Since $\psi_{2m+n,n-m} \in N(m,n)$ and $N(n,m) = Cl\langle\psi_{2m+n,n-m}\rangle$ is the normal closure of $\langle\psi_{2m+n,n-m}\rangle$ in $\Delta^+(6,3,2)$, we are done.

Let $mn(m-n) = 0$. In this case X is 2-regular and the automorphism group has presentation of the form $\mathrm{Aut}(X) = \langle t,u,z |\ t^2 = u^2 = z^2 = (tu)^3 = (uz)^2 = (tz)^6 = 1, \dots\rangle$. In this case t, u generate the vertex stabiliser and we may assume that $x = tu$ and $y = uz = zu$. The automorphism group contains as an index two subgroup a 1-regular subgroup generated by x and y and satisfying $(x^{-1}yxy)^m(xyx^{-1}y)^n = 1$. Putting $x = tu$ and $y = uz = zu$ we get the required relation. The equality $mn(m-n) = 0$ guarantees then $N(m,n) = Cl\langle\psi_{2m+n,n-m}\rangle \triangleleft \Delta^+(6,3,2)$ is normal in the extended triangle group $\Delta(6,3,2)$ as well. □

7 Graphs with two girth cycles passing through an edge, existence problems

It follows from Theorem 5.1 that a symmetric cubic graph X of girth g such that there are two girth cycles passing through an edge is either 2-regular, or 1-regular and in the 1-regular case there are two sorts of the action on the compact surface S arising by gluing 2-cells to each girth cycle. It is clear, that every graph automorphism extends to a self-homeomorphism of S and hence the embedding of X into S gives rise to a regular map, or to an oriented regular map which automorphism group coincide with the full automorphism group of the graph, or it is the Petrie dual of an oriented (chiral) regular map. In this section, we shall discuss the existence of such graphs and maps for $g \ge 7$. In a correspondence with Theorem 5.1 we distinguish the above mentioned three cases.

Case 1: X is 2-regular.

The following statement holds true.

Proposition 7.1. For every $g \geq 6$ there are infinitely many 2-regular cubic graphs X with girth g. Moreover, each such X is a quotient $X \cong \mathcal{U}(g,3)/N$, where N is an appropriate normal torsion-free subgroup of $\Delta(g,3,2)$.

The history of the proof of this statement is quite long. In the context of permutation groups it was proved in 1902 in [23], later rediscovered, reproved and improved by many other authors (see [27] for more information). A general argument to see the existence of infinitely many finite quotients of $\Delta(g,3,2)$ uses the residual finiteness of $\Delta(g,3,2)$. A group G is called *residually finite* if for any finite set $A \subseteq G$ of elements of G and for any $x \in A$, $x \neq 1$ there exists a normal subgroup $N \trianglelefteq G$ of finite index such that $x \notin N$. The idea of the proof of Proposition 7.1 is to take A to be the set of all elements in $\Delta(g,3,2)$ expressible in terms of the three involutory generators by all words of length at most d for some d. By residual finiteness of $\Delta(g,3,2)$, for every d there is a normal subgroup $N \trianglelefteq \Delta(g,3,2)$ of finite index such that a part of the universal graph $\mathcal{U}(g,3)$ formed by the images of a particularly chosen arc under the elements of A is mapped isomorphically into $\mathcal{U}(g,3)/N$. Thus $\mathcal{U}(g,3)/N$ is a 2-regular cubic graph of girth g. Taking different d we can construct an infinite family of non-isomorphic graphs satisfying the required properties.

A standard argument to see the residual finiteness of triangle groups is by using a deep theorem of Malcev [20] saying that any finitely generated matrix group is residually finite. Concrete matrix representations of $\Delta(g,3,2)$ one can find in [29, 30]. Proofs based on permutation representation of some quotients of $\Delta(g,3,2)$ can be found for instance in [18, 27].

Case 2. X is 1-regular, and $\mathrm{Aut}(X)$ is a quotient of $\Delta^+(g,3,2) = \langle x,y | x^3 = y^2 = (xy)^g = 1\rangle$ by some normal torsion free subgroup.

To prove that there are infinitely many 1-regular cubic graphs of girth $g \geq 7$ of this sort it is not enough to argue by the residual finiteness of the triangle group $\Delta^+(g,3,2)$. We need an additional argument to guarantee that an appropriate normal subgroup $N \triangleleft \Delta^+(g,3,2)$ of finite index used to produce the graph $\mathcal{U}(g,3)/N$ is not normal in $\Delta(g,3,2)$. In general, this is not easy. In what follows we show that it is true for any g divisible by 6.

Proposition 7.2. For every g divisible by 6 there are infinitely many 1-regular cubic graphs X of girth g such that $X \cong \mathcal{U}(g,3)/N$, for some normal torsion free subgroup N of $\Delta^+(g,3,2)$.

Proof. By Theorem 6.2 the graphs $\mathcal{G}(m,1)$ are for any $m \geq 3$ one-regular quotients of $\mathcal{U}(6,3)$ of girth 6. The least 2-regular cubic graph of girth 6 covering $X = \mathcal{G}(m,1)$ is of the form $Y = \mathcal{U}(6,3)/K$ for some normal subgroup $K \triangleleft \Delta(6,3,2)$. It follows that $K \leq N(m,1) \cap N(1,m)$. Since $N(m,1) \cap N(1,m) \triangleleft \Delta(6,3,2)$ we get $K = N(m,1) \cap N(1,m)$. Let κ be the index of the covering $Y \to X$. By the isomorphism theorem κ is equal to the index of the covering $X \to Z = \mathcal{U}(6,3)/(N(m,1)N(1,m))$. An easy calculation shows that the product $N(m,1)N(1,m) = N(1,0)$. Hence Z is a 2-vertex graph. Since X has $2(m^2+m+1)$ vertices we have $\kappa = m^2+m+1$.

Let $X(g) = \mathcal{U}(g,3)/N$ be a symmetric cubic graph of girth g for some normal subgroup $N \triangleleft \Delta^+(g,3,2)$. Since $6|g$ we have a natural epimorphism $\phi : \Delta^+(g,3,2) \to$

$\Delta^+(6,3,2)$. Choose m such that $m > |\mathrm{Aut}^+(X)| = |\Delta^+(g,3,2)/N|$. We claim that the graph $W = \mathcal{U}(g,3)/N \cap \phi^{-1}(N(m,1))$ is 1-regular of girth g.

Firstly, since W contains a 1-regular subgroup $\Delta^+(g,3,2)/N \cap \phi^{-1}(N(m,1))$ isomorphic to a subgroup of $\Delta^+(g,3,2)/N \times \Delta^+(g,3,2)/\phi^{-1}(N(m,1)) \cong \mathrm{Aut}^+(X(g)) \times \mathrm{Aut}(\mathcal{G}(m,1))$ (see [34,1] for more details) the number of arcs of W cannot exceed $6m(m^2+m+1)$. On the other hand, assuming that W is 2-regular we get a graph $\phi(W) = \mathcal{U}(6,3)/\phi(N) \cap N(m,1)$, where $\phi(N) \cap N(m,1) \triangleleft \Delta(6,3,2)$. Hence, it is a 2-regular graph of girth 6 covering X. Consequently, $\phi(W)$ has at least $\kappa 6(m^2+m+1) = 6(m^2+m+1)^2$ arcs, a contradiction. Since W covers $X(g)$, its girth is at least g, but since it is a quotient of $\mathcal{U}(g,3)$ its girth is at most g. Choosing different values for m we create an infinite family of graphs satisfying the required properties. □

Let us note that the above proof employs some general ideas on the 'chirality' of oriented regular maps and hypermaps developed in [1] and later generalised in [2].

Problem 1. Prove that for every $g \geq 6$ there are infinitely many 1-regular cubic graphs with girth g of the form $\mathcal{U}(g,3)/N$ for some $N \triangleleft \Delta^+(g,3,2)$.

Case 3. As concerns the existence of 1-regular cubic graphs of girth g for some even $g \geq 8$ such that the girth cycles come from the relation $[x,y]^{g/2} = 1$, see Theorem 5.1 (3) we cannot say too much. The following example shows that for girth $g = 8$ this case happens.

Example. Checking the list of all symmetric cubic graphs up to 768 vertices [7] we see that there exists a 1-regular graph X of girth 8 on 400 vertices which group has presentation

$$G = < h, a | h^3 = a^2 = [h,a]^4 = (ha)^{12} = 1, (ha)^5 h^{-1} a (ha)^2 (h^{-1}a)^2 ha (h^{-1}a)^5 = 1 > .$$

It follows X admits a 1-regular action of G of the third type.

As concerns the existence of some other examples of this sort, the following statement proved in [16] supports a conjecture that most probably there are infinitely many such graphs for any even $g \geq 8$.

Theorem 7.3. [16] With possible exceptions of $(m,q) = (13,4)$ and $(7,11)$ the group $\Delta^+(m,3,2;q) = \langle x,y | x^3 = y^2 = (xy)^m = [x,y]^q = 1 \rangle$ is infinite if and only if m and q satisfy one of the following conditions: $m = 7$, $q \geq 9$; $m = 8$ or $m = 9$ and $q \geq 6$; $m = 10$ or $m = 11$ and $q \geq 5$; $m \geq 12$ and $q \geq 4$.

Inspired by the above result we give the following problem.

Problem 2. Prove that for every even $g \geq 8$ there are infinitely many 1-regular cubic graphs of girth g which automorphism group is an epimorfic image of $\langle x,y | x^3 = y^2 = [x,y]^{g/2} = 1 \rangle$.

Girth 7. A particular instance of Proposition 7.1 for $g = 7$ establishing the existence of infinitely many 2-regular cubic graphs of girth 7 (Case (2) of Theorem 5.2) is a consequence the theorem of McBeath [21] showing that there are infinitely many Hurwitz maps. A regular, or an oriented regular map of type $\{7,3\}$ is called a *Hurwitz map*. It follows from Theorem 5.2 that the family of symmetric cubic graphs of girth 7 coincide with the exception of the Coxeter graph with the family of underlying graphs of Hurwitz maps.

By Theorem 5.2 a 1-regular cubic graph of girth 7 is a quotient $\mathcal{U}(7,3)/N$ by some nontrivial subgroup $N \triangleleft \Delta^+(7,3,2) = \langle x, y | x^3 = y^2 = (xy)^7 = 1\rangle$ of finite index such that the mapping $x \mapsto x^{-1}$, $y \mapsto y$ does not extend to a group automorphism. The residual finiteness of $\Delta^+(7,3,2)$ is not sufficient to see that there are infinitely many such normal subgroups N. This can be done by presenting some finite groups by means of two generators x, y satisfying $x^3 = (xy)^7 = y^2 = 1$ and using an argument to show a non-existence of a group automorphism taking $x \mapsto x^{-1}$ and $y \mapsto y$. In particular, the Ree group $G = Re(3^f)$, for odd $f > 1$, is a simple epimorphic image of the triangle group $\Delta^+(2,3,7)$, with generators x and y of orders 3 and 2. As shown in [28], x is not inverted by any automorphism of G.

Proposition 7.4. There are infinitely many 1-regular cubic graphs of girth 7.

As concerns the problem to classify symmetric cubic graphs of girth 7 it seems to be difficult if possible at all. The core of the problem consists in the fact that the structure of normal subgroups of the triangle groups $\Delta^+(7,3,2)$ $(\Delta(7,3,2))$ of finite index is too complex. Infinite families of simple nonabelian groups appear as epimorphic images of these groups. In fact, one can show that the automorphism group of every symmetric cubic graph of girth 7 is insolvable, which is in a clear contrast to the situation for the graphs of girth 6 in which case the automorphism groups are solvable with a well-understandable structure, or the graph is one of the exceptional graphs discussed in Section 3.

Remark Recently main results of this paper were generalized by Conder and Nedela [8] by proving that a symmetric cubic graphs of girth at most 9 is either 1-regular or $2'$-regular or it belongs to a small family of 15 exceptional graphs. In contrast to the approach used in this paper, the proof is computer-assisted.

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