Acta Universitatis Matthiae Belii ser. Mathematics, **14**(2007), 5–17.

T_L AND S_L EVALUATORS: AGGREGATION AND MODIFICATION

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Abstract. T_L and S_L evaluators were introduced in [1] and their basic properties were studied. In this paper we discuss which aggregation of T_L (S_L) evaluators yields a T_L (S_L) evaluator and which modification of an evaluator will change it into a T_L (S_L) evaluator. Duality of evaluators is also studied.

1. Evaluators and aggregation operators

We will consider a complete lattice $(L, \leq, \bot, \top)$ with the least and the greatest elements $\bot$ and $\top$, respectively. Normalized scalar evaluators of elements from L were characterized in [3] by a function $\varphi : L \to [0,1]$ satisfying properties

(i) $\varphi(\bot) = 0$, $\varphi(\top) = 1$,
(ii) for all $a, b \in L$, if $a \leq b$ then $\varphi(a) \leq \varphi(b)$.

Evaluator φ is called existentional if for $a \in L$,

$$\varphi(a) = 0 \Rightarrow a = \bot. \tag{1}$$

Evaluator φ is called universal if for $a \in L$,

$$\varphi(a) = 1 \Rightarrow a = \top. \tag{2}$$

In applications, different properties of the same object are evaluated by different evaluators. For comparison of two or more objects, an aggregation of evaluations is needed. An aggregation operator [2, 6, 9] is a function $\boldsymbol{A}\colon \bigcup_{n \in N} [0,1]^n \to [0,1]$ such that

(i) $\boldsymbol{A}\ (x_1, \ldots, x_n) \leq \boldsymbol{A}\ (y_1, \ldots, y_n)$ whenever
$$x_i \leq y_i \text{ for all } i \in \{1, \ldots, n\},$$
(ii) $\boldsymbol{A}\ (x) = x$ for all $x \in [0,1]$,
(iii) $\boldsymbol{A}\ (0, \ldots, 0) = 0$ and $\boldsymbol{A}\ (1, \ldots, 1) = 1$.

2000 *Mathematics Subject Classification.* 08A72, 03G10, 03E72.

Key words and phrases. t-norm, t-conorm, evaluators, aggregation operators, lattice, duality, fuzzy sets.

Each aggregation operator $\boldsymbol{A}$ can be canonically represented by a family $(A_n)_{n\in N}$ of n-ary operations, e.g., functions $A_n : [0,1]^n \to [0,1]$ given by

$$A_n(x_1,\ldots,x_n) = \mathbf{A}(x_1,\ldots,x_n). \tag{3}$$

Function A_n is an evaluator on the lattice
$([0,1]^n, \leq, \perp, \top)$, where $\perp = (0,\ldots,0)$ and $\top = (1,\ldots,1)$. If $\mathbf{A}(x_1,\ldots,x_n) = 0$ implies that $x_i = 0$ for $i = 1,\ldots,n$, we say that aggregation operator $\mathbf{A}$ does not have zero divisors. In this case, function A_n is an existentional evaluator and $\mathbf{A}$ is an existentional aggregator. If $\mathbf{A}(x_1,\ldots,x_n) = 1$ implies that $x_i = 1$ for $i = 1,\ldots,n$, function A_n is a universal evaluator and $\mathbf{A}$ is a universal aggregator.

Proposition 1. *Let $\Phi = \{\varphi_1,\ldots,\varphi_n\}$ be a set of evaluators on a complete lattice $(L,\leq,\perp,\top)$ and let $\mathbf{A}$ be an aggregation operator. Then function $A_\Phi : L \to [0,1]$ defined for all $a \in L$ by*

$$A_\Phi(a) = \mathbf{A}(\varphi_1(a),\ldots,\varphi_n(a)) \tag{4}$$

is an evaluator on L.

Obviously, aggregation of existentional evaluators by an existentional aggregator yields an existentional evaluator and aggregation of universal evaluators by a universal evaluator yields a universal evaluator.

Frequently used aggregation operators are averaging operators. We will consider only arithmetic mean

$$\mathbf{M}(x_1,\ldots,x_n) = (x_1 + \ldots + x_n)/n.$$

Aggregator $\mathbf{M}$ is existentional as well as universal.

Arithmetic mean belongs to the family of ordered weighted averaging operators (OWA operators) introduced in [10].

$$\mathbf{OWA}(x_1,\ldots,x_n) = \sum_{j=1}^{n} w_j y_j,$$

where y_j is the j^{th} largest value of x_i, $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. More about OWA operators can be found in [4, 9].

A special class of aggregation operators is the class of triangular norms (t-norms) and triangular conorms (t-conorms). For more details refer to [7, 8, 9]. The four basic t-norms are:
the minimum $T_M(x,y) = \min(x,y)$,
the product $T_P(x,y) = x.y$,
the Łukasiewicz t-norm $T_L(x,y) = \max(x+y-1,0)$,
and the drastic product

$$T_D(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

The four basic t-conorms are:
the maximum $S_M(x,y) = \max(x,y)$,
the probabilistic sum $S_P(x,y) = x + y - x.y$,
the Łukasiewicz t-conorm $S_L(x,y) = \min(x+y,1)$,
and the drastic sum

$$S_D(x,y) = \begin{cases} 1 & \text{if } (x,y) \in]0,1]^2, \\ \max(x,y) & \text{otherwise.} \end{cases}$$

It can be shown that T_M, T_P, T_L and T_D are universal aggregators while S_M, S_P, S_L and S_D are existentional aggregators.

The relationship between evaluators and Łukasiewicz t-norm and t-conorm was studied in [1], where the notions of T_L and S_L evaluators were introduced.

Definition 1. *Consider a complete lattice $(L, \leq, \bot, \top)$. A normalized evaluator φ on L is called a T_L evaluator if and only if for all $a, b \in L$*

$$T_L(\varphi(a), \varphi(b)) \leq \varphi(a \wedge b), \tag{5}$$

and it is called an S_L evaluator if and only if

$$S_L(\varphi(a), \varphi(b)) \geq \varphi(a \vee b). \tag{6}$$

Proposition 2. *Consider a complete lattice $(L, \leq, \bot, \top)$.*
A normalized evaluator φ is a T_L evaluator if and only if for all $a, b \in L$

$$\varphi(a \wedge b) \geq \varphi(a) + \varphi(b) - 1, \tag{7}$$

and it is an S_L evaluator if and only if

$$\varphi(a \vee b) \leq \varphi(a) + \varphi(b). \tag{8}$$

Example 1. Let $\mathcal{F}(X)$ denote the family of all fuzzy sets on a universal finite set X. For $A, B \in \mathcal{F}(X)$, $A \leq B$ means that $A(x) \leq B(x)$ for all $x \in X$, $(A \vee B)(x) = \max\{A(x), B(x)\}$ and $(A \wedge B)(x) = \min\{A(x), B(x)\}$. Obviously, $(\mathcal{F}(X), \leq)$ is a complete lattice with $\bot = \emptyset$ and $\top = X$. Some normalized scalar evaluators on $\mathcal{F}(X)$ are height ht, plinth pl and relative cardinality RC defined for $A \in \mathcal{F}(X)$ by (9), (10) and (11), respectively.

$$ht(A) = \max_{x \in X} A(x), \tag{9}$$

$$pl(A) = \min_{x \in X} A(x), \tag{10}$$

$$RC(A) = \frac{|A|}{|X|}, \tag{11}$$

where $|.|$ denotes cardinality. Because X is a finite set,

$$|A| = \sum_{x \in X} A(x). \tag{12}$$

Evaluator ht is an S_L evaluator, pl is a T_L evaluator and RC is both S_L and T_L evaluator.

Example 2. All fuzzy measures defined on the power set of a nonempty crisp set X are evaluators on the complete lattice $(2^X, \subseteq, \emptyset, X)$. Well known examples of fuzzy measures are probability measure Pr, possibility measure Pos and necessity measure Nec. Then Pos is anS_L evaluator, Nec is a T_L evaluator, and Pr is an S_L as well as a T_L evaluator.

2. Aggregation of T_L and S_L evaluators

We already know that aggregation of evaluators yields an evaluator (Proposition 1). Now we will focus on aggregation of T_L and S_L evaluators. We will consider as possible aggregation operators some OWA operators (arithmetic mean), some t-norms (T_M and T_L) and some t-conorms (S_M and S_L). We would like to know what aggregation of T_L (S_L) evaluators yields a T_L (S_L) evaluator.

Proposition 3. *Arithmetic mean of T_L (S_L) evaluators is a T_L (S_L) evaluator.*

Proof: Let $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ be a set of T_L evaluators on a complete lattice $(L, \leq, \perp, \top)$. Then for all $a, b \in L$ we have

$$\varphi_i(a \wedge b) \geq \varphi_i(a) + \varphi_i(b) - 1,$$

for all $i \in \{1, \ldots, n\}$. Therefore

$$\sum_{i=1}^{n} \varphi_i(a \wedge b) \geq \sum_{i=1}^{n} \varphi_i(a) + \sum_{i=1}^{n} \varphi_i(b) - n.$$

Then

$$\begin{aligned} M_\Phi(a \wedge b) &= \mathbf{M}(\varphi_1(a \wedge b), \ldots, \varphi_n(a \wedge b)) \\ &= \left(\sum_{i=1}^{n} \varphi_i(a \wedge b) \right) / n \\ &\geq \left(\sum_{i=1}^{n} \varphi_i(a) \right) / n + \left(\sum_{i=1}^{n} \varphi_i(b) \right) / n - 1 \\ &= M_\Phi(a) + M_\Phi(b) - 1, \end{aligned}$$

and therefore M_Φ is a T_L evaluator.
Analogously we can prove that arithmetic mean of S_L evaluators yields an S_L evaluator.

Corollary 1. *Let $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ be a set of $T_L(S_L)$ evaluators on a complete lattice $(L, \leq, \perp, \top)$. Let for all $a \in L$, $\varphi_1(a) \leq \ldots \leq \varphi_n(a)$. Then $OWA_\Phi = \mathbf{OWA}(\varphi_1, \ldots, \varphi_n)$ is a T_L (S_L) evaluator on L.*

Proposition 4. *Aggregation of T_L evaluators by t-norm T_M yields a T_L evaluator. Aggregation of S_L evaluators by t-conorm S_M yields an S_L evaluator.*

Proof. Let $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ be a set of T_L evaluators on a complete lattice $(L, \leq, \perp, \top)$. Then for all $a, b \in L$

$$\varphi_i(a \wedge b) \geq \varphi_i(a) + \varphi_i(b) - 1,$$

for all $i \in \{1, \ldots, n\}$. Let $\min(\varphi_1(a \wedge b), \ldots, \varphi_n(a \wedge b)) = \varphi_r(a \wedge b),\ r \in \{1, \ldots, n\}$. Then

$$\begin{aligned} T_{M_\Phi}(a \wedge b) &= \varphi_r(a \wedge b) \geq \varphi_r(a) + \varphi_r(b) - 1 \\ &\geq \min(\varphi_1(a), \ldots, \varphi_n(a)) + \min(\varphi_1(b, \ldots, \varphi_n(b)) - 1 \\ &= T_{M_\Phi}(a) + T_{M_\Phi}(b) - 1, \end{aligned}$$

and therefore T_{M_Φ} is a T_L evaluator.
Analogously we can prove that aggregation of S_L evaluators by t-conorm S_M yields an S_L evaluator. □

Proposition 5. *Aggregation of T_L evaluators by Łukasiewicz t-norm yields a T_L evaluator. Aggregation of S_L evaluators by Łukasiewicz t-conorm yields an S_L evaluator.*

Proof. Let $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ be a set of T_L evaluators on a complete lattice $(L, \leq, \perp, \top)$. Then for all $a, b \in L$ we have

$$\varphi_i(a \wedge b) \geq \varphi_i(a) + \varphi_i(b) - 1,$$

for all $i \in \{1, \ldots, n\}$. For $a, b \in L$ we obtain

$$\begin{aligned} T_{L_\Phi}(a \wedge b) &= T_L(\varphi_1(a \wedge b), \ldots, \varphi_n(a \wedge b)) \\ &= \max\left(\sum_{i=1}^{n} \varphi_i(a \wedge b) - (n-1), 0\right). \end{aligned}$$

Then, because

$$\sum_{i=1}^{n} \varphi_i(a \wedge b) \geq \sum_{i=1}^{n} \varphi_i(a) + \sum_{i=1}^{n} \varphi_i(b) - n,$$

we have that

$$\begin{aligned} T_{L_\Phi}(a \wedge b) &\geq \sum_{i=1}^{n} \varphi_i(a \wedge b) - (n-1) \geq \\ &\geq \sum_{i=1}^{n} \varphi_i(a) + \sum_{i=1}^{n} \varphi_i(b) - n - (n-1) = \\ &= \sum_{i=1}^{n} \varphi_i(a) + \sum_{i=1}^{n} \varphi_i(b) - n - (n-1) + 1 - 1 = \\ &= \sum_{i=1}^{n} \varphi_i(a) - (n-1) + \sum_{i=1}^{n} \varphi_i(b) - (n-1) - 1. \end{aligned}$$

Because $\max\left(\sum_{i=1}^{n} \varphi_i(a \wedge b) - (n-1), 0\right) \geq 0$, we can rewrite the inequality above as follows:

$$\begin{aligned} T_{L_\Phi}(a \wedge b) &= \max\left(\sum_{i=1}^{n} \varphi_i(a \wedge b) - (n-1), 0\right) \geq \\ &\geq \max\left(\sum_{i=1}^{n} \varphi_i(a) - (n-1), 0\right) + \\ &+ \max\left(\sum_{i=1}^{n} \varphi_i(b) - (n-1), 0\right) - 1 = \\ &= T_{L_\Phi}(a) + T_{L_\Phi}(b) - 1, \end{aligned}$$

which shows that T_{L_Φ} is a T_L evaluator.

Now we will prove that aggregation of S_L evaluators by Łukasiewicz t-conorm results in an S_L evaluator.

Let $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ be a set of S_L evaluators on a complete lattice $(L, \leq, \perp, \top)$. Then for all $a, b \in L$ we have

$$\varphi_i(a \vee b) \leq \varphi_i(a) + \varphi_i(b),$$

for all $i \in \{1, \ldots, n\}$. For $a, b \in L$ we obtain

$$\begin{aligned} S_{L_\Phi}(a \vee b) &= S_L(\varphi_1(a \vee b), \ldots, \varphi_n(a \vee b)) \\ &= \min\left(\sum_{i=1}^{n} \varphi_i(a \vee b), 1\right). \end{aligned}$$

Then, because

$$\sum_{i=1}^{n} \varphi_i(a \vee b) \leq \sum_{i=1}^{n} \varphi_i(a) + \sum_{i=1}^{n} \varphi_i(b),$$

we have that

$S_{L_\Phi}(a \vee b) \le \sum_{i=1}^n \varphi_i(a \vee b) \le \sum_{i=1}^n \varphi_i(a) + \sum_{i=1}^n \varphi_i(b)$.

Because $\min\left(\sum_{i=1}^n \varphi_i(a \vee b), 1\right) \le 1$, we can rewrite the inequality above as follows:

$$S_{L_\Phi}(a \vee b) = \min\left(\sum_{i=1}^n \varphi_i(a \vee b), 1\right) \le$$
$$\le \min\left(\sum_{i=1}^n \varphi_i(a), 1\right) + \min\left(\sum_{i=1}^n \varphi_i(b), 1\right) =$$
$$= S_{L_\Phi}(a) + S_{L_\Phi}(b),$$

which shows that S_{L_Φ} is an S_L evaluator.

In the following example we will show that aggregation of T_L evaluators by Łukasiewicz t-conorm does not need to result in a T_L evaluator. □

Example 3. Let $(\mathcal{F}(X), \le)$ be the lattice from Example 1, where $X = \{x_1, x_2, x_3\}$. Consider $A, B \in \mathcal{F}(X)$ defined by membership functions $A = 0.6/x_1 + 1/x_2 + 0.6/x_3$ and $B = 0.9/x_1 + 0.2/x_2 + 0.1/x_3$, respectively. We will use the following set of T_L evaluators: $\Phi = \{pl, RC\}$, where pl is plinth and RC is relative cardinality. Then $pl(A) = 0.6,\ RC(A) = 0.73$ and $pl(B) = 0.1,\ RC(B) = 0.4$. For $A \wedge B = 0.6/x_1 + 0.2/x_2 + 0.1/x_3$ we have
$pl(A \wedge B) = 0.1,\ RC(A \wedge B) = 0.3)$. Then
$S_{L_\Phi}(A \wedge B) = S_L(pl(A \wedge B), RC(A \wedge B)) = S_L(0.1, 0.3) = \min(0.1 + 0.3, 1) = 0.4$.
On the other hand,
$S_{L_\Phi}(A) = S_L(0.6, 0.73) = \min(0.6 + 0.73, 1) = 1$ and
$S_{L_\Phi}(B) = S_L(0.1, 0.4) = \min(0.1 + 0.4, 1) = 0.5$.
Obviously,
$0.4 = S_{L_\Phi}(A \wedge B) < S_{L_\Phi}(A) + S_{L_\Phi}(B) - 1 = 0.5$,
and therefore S_{L_Φ} is not a T_L evaluator.

Analogously, aggregation of S_L evaluators by Łukasiewicz t-norm does not need to be an S_L evaluator.

Example 4. Let $(\mathcal{F}(X), \le)$ be the lattice from Example 1, where $X = \{x_1, x_2, x_3\}$. Consider $A, B \in \mathcal{F}(X)$ defined by membership functions $A = 0.4/x_1 + 0/x_2 + 0.9/x_3$ and $B = 0.1/x_1 + 0.8/x_2 + 0.2/x_3$, respectively. We will use the following set of S_L evaluators: $\Phi = \{ht, RC\}$, where ht is height and RC is relative cardinality. Then $ht(A) = 0.9,\ RC(A) = 0.433$ and $ht(B) = 0.8,\ RC(B) = 0.367$. For $A \vee B = 0.4/x_1 + 0.8/x_2 + 0.9/x_3$ we have
$ht(A \vee B) = 0.9,\ RC(A \vee B) = 0.7$. Then
$T_{L_\Phi}(A \vee B) = T_L(ht(A \vee B), RC(A \vee B)) = T_L(0.9, 0.7) = \max(0.9 + 0.7 - 1, 0) = 0.6$.
On the other hand,
$T_{L_\Phi}(A) = T_L(0.9, 0.433) = \max(0.9 + 0.433 - 1, 0) = 0.333$ and

$T_{L_\Phi}(B) = T_L(0.8, 0.367) = \max(0.8 + 0.367 - 1, 0) = 0.167$. Obviously,
$0.6 = T_{L_\Phi}(A \vee B) > T_{L_\Phi}(A) + T_{L_\Phi}(B) = 0.5$,
and therefore T_{L_Φ} is not an S_L evaluator.

3. Composition of evaluators

Properties of evaluators can be changed by appropriate modifications of evaluators. We will explore how evaluators on a complete lattice $(L, \leq, \perp, \top)$ can be transformed (modified) into T_L and S_L evaluators. In this section we will discuss modification of evaluators on L by a composition with evaluators on the lattice $([0,1], \leq, 0, 1)$.

Proposition 6. *Consider a lattice $(L, \leq, \perp, \top)$. Function $\varphi : L \to [0,1]$ is an evaluator on L if and only if there exists an evaluator ψ on L and an evaluator f on $([0,1], \leq, 0, 1)$ such that*

$$\varphi = f \circ \psi. \tag{13}$$

If ψ and f are universal (existentional) evaluators then φ is a universal (existentional) evaluator.

Proof. If φ is an evaluator on L, we can choose $\psi = \varphi$ and $f = id$ (identity). Then for all $a \in L$,

$$\varphi(a) = (id \circ \psi)(a) = id(\psi(a)) = id(\varphi(a)) = \varphi(a).$$

Now we will show that $\varphi = f \circ \psi$ is an evaluator on L.
(i) $\varphi(\perp) = (f \circ \psi)(\perp) = f(\psi(\perp)) = f(0) = 0$,
and $\varphi(\top) = (f \circ \psi)(\top) = f(\psi(\top)) = f(1) = 1$.
(ii) For all $a, b \in L$, if $a \leq b$ then $\varphi(a) = (f \circ \psi)(a)$
$= f(\psi(a)) \leq f(\psi(b)) = (f \circ \psi)(b) = \varphi(b)$.
Therefore φ is an evaluator on L.

If f and ψ are universal, then for all $a \in L$,
$\varphi(a) = f(\psi(a)) = 0 \Rightarrow \psi(a) = 0 \Rightarrow a = \perp$, and therefore φ is universal.
If f and ψ are existentional, then for all $a \in L$,
$\varphi(a) = f(\psi(a)) = 1 \Rightarrow \psi(a) = 1 \Rightarrow a = \top$, and therefore φ is existentional.

For an evaluator φ on L we want to find an evaluator f on $[0,1]$ such that $f \circ \varphi$ is an T_L (S_L) evaluator. □

Proposition 7. *Let f be an evaluator on $([0,1], \leq, 0, 1)$ such that for all $x \in [0,1[$, $f(x) \in [0, 0.5]$. Let φ be a universal evaluator on $(L, \leq, \perp, \top)$. Then $\varphi_f = f \circ \varphi$ is a universal T_L evaluator on L.*

Proof. Because f is a universal evaluator, from Proposition 6 it follows that φ_f is a universal evaluator on L.

Now we will show that for all $a, b \in L$,

$$\varphi_f(a \wedge b) \geq \varphi_f(a) + \varphi_f(b) - 1. \tag{14}$$

If $a = \top$ or $b = \top$ then $\varphi_f(a \wedge b) = \min\{\varphi_f(a), \varphi_f(b)\}$, and $\varphi_f(a) + \varphi_f(b) = \min\{\varphi_f(a), \varphi_f(b)\} + 1$. Therefore $\varphi_f(a) + \varphi_f(b) - 1 = \min\{\varphi_f(a), \varphi_f(b)\} = \varphi_f(a \wedge b)$ and inequality (14) holds.
Let $a \neq \top$ and $b \neq \top$. Because φ is universal, $\varphi(a) < 1$, $\varphi(b) < 1$ and $f(\varphi(a)) \leq 0.5$, $f(\varphi(b)) \leq 0.5$. Hence $\varphi_f(a) + \varphi_f(b) = f(\varphi(a)) + f(\varphi(b)) \leq 1$. Therefore $\varphi_f(a) + \varphi_f(b) - 1 \leq 0 \leq \varphi_f(a \wedge b)$, which proves inequality (14). □

Some examples of a function f satisfying properties from Proposition 7 are:

1) For $x \in [0, 1]$ and $\alpha \in [0, 0.5]$,

$$f_1(x) = \begin{cases} 1 & \text{if } x = 1, \\ \alpha & \text{if } \alpha \leq x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Evaluator φ on L modified by composition $f_1 \circ \varphi$ is the alpha-lower levelization of φ discussed in [1]. For $\alpha = 0$, composition $f_1 \circ \varphi$ is the trivial universal evaluator

$$\varphi_U(a) = \begin{cases} 1 & \text{if } a = \top, \\ 0 & \text{otherwise.} \end{cases}$$

2) For $x \in [0, 1]$ and $\alpha \in [0, 0.5]$,

$$f_2(x) = \begin{cases} 1 & \text{if } x = 1, \\ \min(\alpha, x) & \text{otherwise.} \end{cases}$$

For $\alpha = 0$, $f_2 \circ \varphi = \varphi_U$.

3) For $x \in [0, 1]$,

$$f_3(x) = \begin{cases} 1 & \text{if } x = 1, \\ 1 - 0.5^x & \text{otherwise.} \end{cases}$$

Proposition 8. *Let g be an evaluator on $([0, 1], \leq, 0, 1)$ such that for all $x \in]0, 1]$, $f(x) \in [0.5, 1]$. Let φ be an existentional evaluator on $(L, \leq, \bot, \top)$. Then $\varphi_g = g \circ \varphi$ is an existentional S_L evaluator on L.*

Proof. Because g is an existentional evaluator, from Proposition 6 it follows that φ_g is an existentional evaluator on L. Now we will show that for all $a, b \in L$,

$$\varphi_g(a \vee b) \leq \varphi_g(a) + \varphi_g(b). \tag{15}$$

If $a = \bot$ or $b = \bot$ then $\varphi_g(a \vee b) = \max\{\varphi_g(a), \varphi_g(b)\}$, and $\varphi_g(a) + \varphi_g(b) = \max\{\varphi_g(a), \varphi_g(b)\} + 0$. Therefore $\varphi_g(a) + \varphi_g(b) = \max\{\varphi_g(a), \varphi_g(b)\} = \varphi_g(a \vee b)$ and inequality (15) holds.
Let $a \neq \bot$ and $b \neq \bot$. Because φ is existentional, $\varphi(a) > 0$, $\varphi(b) > 0$ and $g(\varphi(a)) \geq 0.5$, $g(\varphi(b)) \geq 0.5$ Hence $\varphi_g(a) + \varphi_g(b) = g(\varphi(a)) + g(\varphi(b)) \geq 1$. Therefore $\varphi_g(a) + \varphi_g(b) \geq \varphi_g(a \vee b)$, which proves inequality (15). □

Some examples of a function g satisfying properties from Proposition 8 are:
1) For $x \in [0,1]$ and $\alpha \in [0.5,1]$,

$$g_1(x) = \begin{cases} 0 & \text{if } x = 0, \\ \alpha & \text{if } 0 < x \leq \alpha, \\ 1 & \text{otherwise.} \end{cases}$$

Evaluator φ on L modified by composition $g_1 \circ \varphi$ is the alpha-upper levelization of φ discussed in [1]. For $\alpha = 1$, composition $g_1 \circ \varphi$ is the trivial existentional evaluator

$$\varphi_E(a) = \begin{cases} 0 & \text{if } a = \bot, \\ 1 & \text{otherwise.} \end{cases}$$

2) For $x \in [0,1]$ and $\alpha \in [0.5,1]$,

$$g_2(x) = \begin{cases} 0 & \text{if } x = 0, \\ \max(\alpha, x) & \text{otherwise.} \end{cases}$$

For $\alpha = 1, g_2 \circ \varphi = \varphi_E$.
3) For $x \in [0,1]$,

$$g_3(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.5^{1-x} & \text{otherwise.} \end{cases}$$

One can recognize duality between Proposition 7 and Proposition 8 and also between functions $(f_i, g_i), i = 1,2,3$. We will discuss duality of evaluators in the next section.

4. Duality of evaluators

We will consider a complemented lattice $(L, \leq, ', \bot, \top)$, where for each $a \in L$, there is complement $a' \in L$ such that $a \wedge a' = \bot$ and $a \vee a' = \top$. For $a, b \in L$,

$$a \wedge b = (a' \vee b')'. \tag{16}$$

The proof of the following proposition is trivial.

Proposition 9. *Let φ be an existentional (universal) evaluator on a complemented lattice $(L, \leq, ', \bot, \top)$. Then function $\bar{\varphi} : L \to [0,1]$ defined for all $a \in L$ by*

$$\bar{\varphi}(a) = 1 - \varphi(a') \tag{17}$$

is a universal (existentional) evaluator on L.

Evaluator $\bar{\varphi}$ given by (17) will be called the dual evaluator to φ.

Assume the lattice of fuzzy sets from Example 1. The standard complement of a fuzzy set A is fuzzy set A', where $A'(x) = 1 - A(x)$ for all $x \in X$. Evaluators height (ht) and plinth (pt) of fuzzy sets are dual to each other. Evaluator relative cardinality (RC) is dual to itself.

Assume the lattice of crisp sets from Example 2. For each crisp set $A \in 2^X$, complement is defined by $A' = X - A$. Fuzzy measures necessity (Nec) and possibility (Pos) are dual evaluators on the lattice $(2^X, \subseteq, ', \emptyset, X)$. Fuzzy measure probability (Pr) is dual to itself.

Proposition 10. *Let $(L, \leq, ', \bot, \top)$ be a complemented lattice and let φ be a T_L and also S_L evaluator on L. Then for all $a \in L$,*

$$\varphi(a) + \varphi(a') = 1. \tag{18}$$

Proof: If φ is a T_L evaluator on L, then $0 = \varphi(a \wedge a') \geq \varphi(a) + \varphi(a') - 1$, and therefore

$$\varphi(a) + \varphi(a') \leq 1. \tag{19}$$

If φ is an S_L evaluator on L, then $1 = \varphi(a \vee a') \leq \varphi(a) + \varphi(a')$, and therefore

$$\varphi(a) + \varphi(a') \geq 1. \tag{20}$$

From (19) and (20) it follows that $\varphi(a) + \varphi(a') = 1$.

Note that one can find an evaluator φ on L such that for all $a \in L$, $\varphi(a) + \varphi(a') = 1$, but φ is neither T_L nor S_L evaluator.

Example 5 Let $X = \{x_1, x_2, x_3, x_4\}$. Consider lattice $(2^X, \subseteq, ', \emptyset, X)$. Let fuzzy measure $m : 2^X \to [0,1]$ be given as follows: $m(\emptyset) = 0$, $m(X) = 1$, $m(x_1) = m(x_2) = m(x_3) = 0.2$, $m(x_4) = 0.5$, $m(x_1, x_2) = m(x_2, x_3) = m(x_1, x_3) = 0.2$, $m(x_3, x_4) = m(x_1, x_4) = m(x_2, x_4) = 0.8$, $m(x_2, x_3, x_4) = m(x_1, x_3, x_4) = m(x_2, x_1, x_4) = 0.8$, $m(x_1, x_2, x_3) = 0.5$. Function m is an evaluator on 2^X such that for each $A \in 2^X$, $m(A) + m(A') = 1$.
However, $m((x_1, x_2) \cup (x_2, x_3)) = m(x_1, x_2, x_3) = 0.5$ is greater than $m(x_1, x_2) + m(x_2, x_3) = 0.2 + 0.2 = 0.4$, and therefore m is not an S_L evaluator. We also obtain that $m((x_3, x_4) \cap (x_1, x_4)) = m(x_1) = 0.5$ is less than $\max\{m(x_3, x_4) + m(x_1, x_4) - 1, 0\} = \max\{0.8 + 0.8 - 1, 0\} = 0.6$, and therefore m is not a T_L evaluator.

Proposition 11. *Let φ be a T_L (S_L) evaluator on a complemented lattice $(L, \leq, ', \bot, \top)$. Then dual evaluator of φ is an S_L (T_L) evaluator on L.*

Proof. Let φ be a T_L evaluator. Then for all $a, b \in L$, $\varphi(a \wedge b) \geq \varphi(a) + \varphi(b) - 1$. Because of (16), $(a \vee b)' = a' \wedge b'$ and we obtain:
$\bar{\varphi}(a \vee b) = 1 - \varphi((a \vee b)') = 1 - \varphi(a' \wedge b') \leq 1 - (\varphi(a') + \varphi(b') - 1) = 1 - \varphi(a') + 1 - \varphi(b') = \bar{\varphi}(a) + \bar{\varphi}(b)$, and therefore $\bar{\varphi}$ is an S_L evaluator.
Let φ be an S_L evaluator. Then for all $a, b \in L$, $\varphi(a \vee b) \leq \varphi(a) + \varphi(b)$. Because of (16), $(a \wedge b)' = a' \vee b'$ and we obtain:
$\bar{\varphi}(a \wedge b) = 1 - \varphi((a \wedge b)') = 1 - \varphi(a' \vee b') \geq 1 - (\varphi(a') + \varphi(b')) = 1 - \varphi(a') + 1 - \varphi(b') - 1 = \bar{\varphi}(a) + \bar{\varphi}(b) - 1$, and therefore $\bar{\varphi}$ is a T_L evaluator. □

Corollary 2. *Assume evaluator* ψ *on* $(L, \leq, ', \perp, \top)$ *and evaluator* f *on*$([0,1], \leq, ', 0, 1)$. *Let* $f \circ \psi$ *be a* T_L (S_L) *evaluator on* L. *Then* $\bar{f} \circ \bar{\psi}$ *is an* S_L (T_L) *evaluator on* L.

Proof: It is enough to show that $\bar{f} \circ \bar{\psi}$ is dual evaluator of $f \circ \psi$. For all $a \in L$ we obtain:
$\overline{f \circ \psi}(a) = 1 - (f \circ \psi)(a') = 1 - f(\psi(a')) = \bar{f}([(\psi(a')]') = \bar{f}(1 - \psi(a')) = \bar{f}(\bar{\psi}(a)) = \bar{f} \circ \bar{\psi}(a)$.

5. Conclusion

We have shown that aggregation of evaluators by an aggregation operator yields an evaluator. Aggregation of T_L evaluators by arithmetic mean, t-norm T_M or by Łukasiewicz t-norm results in a T_L evaluator. Aggregation of S_L evaluators by arithmetic mean, t-conorm S_M or by Łukasiewicz t-conorm results in an S_L evaluator. A normalized evaluator on a complete lattice can be transformed into a T_L or S_L evaluator by composition with an appropriate evaluator on $[0, 1]$. Dual evaluator of a T_L (S_L) evaluator is an S_L (T_L) evaluator. An evaluator which is T_L and also S_L is dual of itself. However, not every evaluator which is dual of itself is a T_L and S_L evaluator. Successful applications of T_L (S_L) evaluators were already reported in [5]. More applications will be presented in our future paper.

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