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T-FILTERS AND T-IDEALS

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Abstract. This paper is devoted to generalizing of fuzzy filters and fuzzy ideals and to studying the relationship between maximal T-filters (i.e. maximal elements of the lattice of all T-filters) and T-ultrafilters (which are so-called T-and S-evaluators).

1. Introduction and basic definitions

Filters are broadly used in topology and in set-theoretical constructions (ultraproducts). Since a couple of years the notion of filters has been fuzzified (as stated below) to generalized filters and to Łukasiewicz filters. The main importance of Łukasiewicz filters lies in preserving of T_L-transitivity when constructing a fuzzy relation by aggregating some partial T_L-transitive fuzzy relations. More the reader can find in [9].

For the purposes of this paper we will use the following definition of a (proper) filter on a non-empty set X:

Definition 1. *Let $X \neq \emptyset$. A function $F : 2^X \to \{0, 1\}$ is said to be a filter on X iff the following is satisfied:*

- $F(X) = 1,\ F(\emptyset) = 0$
- *for $A, B \subseteq X$ if $A \subset B$, then $F(A) \leq F(B)$*
- *for $A, B \subseteq X$ we have $F(A \cap B) \geq F(A) \cdot F(B)$.*

As a complementary notion to filters we have a (proper) ideal on the set $X \neq \emptyset$ (more precisely, on the Boolean lattice of subsets of X, equipped with union and intersection):

Definition 2. *Let $X \neq \emptyset$. A function $I : 2^X \to \{0, 1\}$ is said to be an ideal on X iff the following is satisfied:*

- $I(X) = 0,\ I(\emptyset) = 1$
- *for $A, B \subseteq X$ if $A \subset B$, then $I(A) \geq I(B)$*
- *for $A, B \subseteq X$ we have $I(A \cup B) \geq I(A) \cdot I(B)$.*

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The relationship between a filter on X and an ideal on X gives the following lemma:

Lemma 1. *Let $X \neq \emptyset$. $F : 2^X \to \{0, 1\}$ is a filter on X if and only if $I : 2^X \to \{0, 1\}$, defined by $I(A) = F(A^c)$ for each $A \in 2^X$, is an ideal on X, where $A^c = X \setminus A$.*

An important notion is that of an ultrafilter on X:

Definition 3. *Let $X \neq \emptyset$. A function $U : 2^X \to \{0, 1\}$ is said to be an ultrafilter on X iff U is a filter on X and moreover if for each $A \subseteq X$ either $U(A) = 1$ or $U(A^c) = 1$.*

The following assertions may be used as alternative definitions of ultrafilters on X:

Proposition 1. *Let us denote $\Psi(X)$ the system of all filters on X. Then $(\Psi(X), \wedge, \vee)$ is a lattice with*

$$F_0(A) = \begin{cases} 1, & \text{if } A = X \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

as its bottom element. Ultrafilters on X are its maximal elements.

Proposition 2. *Let $X \neq \emptyset$ and $F : 2^X \to \{0, 1\}$ be a filter on X. Then F is an ultrafilter on X if and only if $I = 1 - F$ is an ideal on X.*

As Proposition 2 states, we have two possibilities how to define ideals via an ultrafilter U on X: $I_1(A) = U(A^c)$, $I_2(A) = 1 - U(A)$. An easy consideration gives $I_1 = I_2$.

To avoid confusion, filters, ultrafilters and ideals on X will be called crisp filters on X, crisp ultrafilters on X and crisp ideals on X, respectively.

Filters were already fuzzified to so-called generalized filters in [2, 3, 5, 6] in the following way:

Definition 4. *Let $X \neq \emptyset$. A function $G : 2^X \to [0, 1]$ is said to be a generalized filter on X iff the following is satisfied:*

- $G(X) = 1,\ G(\emptyset) = 0$
- *for $A, B \subseteq X$ if $A \subset B$, then $G(A) \leq G(B)$*
- *for $A, B \subseteq X$ we have $G(A \cap B) \geq \min\{G(A), G(B)\}$.*

Before proceeding, we give the definition of a t-norm, which will be a very important notion for us (for details on t-norms an their duals, t-conorms, see [12]):

Definition 5. $T : [0,1] \times [0,1] \to [0,1]$ *is said to be a t-norm iff the following is satisfied:*

- *for each $y \in [0,1]$ $T(1, y) = y$*

- *for all* $x, y_1, y_2 \in [0,1]$ *if* $y_1 \leq y_2$ *then* $T(x, y_1) \leq T(x, y_2)$
- *for all* $x, y \in [0,1]$ $T(x, y) = T(y, x)$
- *for all* $x, y, z \in [0,1]$ $T(x, T(y, z)) = T(T(x, y), z)$.

There are the following four basic t-norms:

(1) minimum t-norm, $T_M(x, y) = \min\{x, y\}$
(2) product t-norm, $T_P(x, y) = x \cdot y$
(3) Łukasiewicz t-norm, $T_L(x, y) = \max\{0, x + y - 1\}$
(4) drastic product,

$$T_D(x, y) = \begin{cases} 0, & \text{if } \max\{x, y\} < 1 \\ \min\{x, y\}, & \text{if } \max\{x, y\} = 1 \end{cases}$$

To each t-norm $T : [0,1] \times [0,1] \to [0,1]$ we may define its dual t-conorm $S : [0,1] \times [0,1] \to [0,1]$ by

$$S(x, y) = 1 - T(1 - x, 1 - y)$$

i.e. to each of the basic four t-norms we have a t-conorm respectively:

(1) maximum t-conorm, $S_M(x, y) = \max\{x, y\}$
(2) probabilistic sum, $S_P(x, y) = x + y - xy$
(3) Łukasiewicz t-conorm, $S_L(x, y) = \min\{1, x + y\}$
(4) drastic sum,

$$S_D(x, y) = \begin{cases} 1, & \text{if } \min\{x, y\} > 0 \\ \max\{x, y\}, & \text{if } \min\{x, y\} = 0 \end{cases}$$

If we replace in Definition 4 min by the Łukasiewicz t-norm T_L, we get the Łukasiewicz filter, which was proposed in [10]. In papers [7, 8, 11] the properties of Łukasiewicz filters were studied.

Definition 6. *Let* $X \neq \emptyset$. *A function* $\mathcal{F} : 2^X \to [0, 1]$ *is said to be a Łukasiewicz filter on* X *iff the following is satisfied:*

- $\mathcal{F}(X) = 1,\ \mathcal{F}(\emptyset) = 0$
- *for* $A, B \subseteq X$ *if* $A \subset B$, *then* $\mathcal{F}(A) \leq \mathcal{F}(B)$
- *for* $A, B \subseteq X$ *we have*

$$\mathcal{F}(A \cap B) \geq T_L\{\mathcal{F}(A), \mathcal{F}(B)\}. \tag{2}$$

Some useful properties of Łukasiewicz filters, when constructing fuzzy preference relations, were shown in [9]. Łukasiewicz ideals were introduced in [8] and their connections to Łukasiewicz ultrafilters and fuzzy preference relations were studied in [9].

Definition 7. *Let* $X \neq \emptyset$. *A function* $\mathcal{I} : 2^X \to [0, 1]$ *is said to be a Łukasiewicz ideal on* X *iff the following is satisfied:*

- $\mathcal{I}(X) = 0,\ \mathcal{I}(\emptyset) = 1$

- *for* $A, B \subseteq X$ *if* $A \subset B$, *then* $\mathcal{I}(A) \geq \mathcal{I}(B)$
- *for* $A, B \subseteq X$ *we have*

(3) $$\mathcal{I}(A \cup B) \geq T_L\{\mathcal{I}(A), \mathcal{I}(B)\}.$$

Similarly to crisp filters, each Łukasiewicz filter $\mathcal{F}$ defines a Łukasiewicz ideal $\mathcal{I}$ by $\mathcal{I}(A) = \mathcal{F}(A^c)$.

2. Łukasiewicz ultrafilters

In the whole paper by X will be denoted a fixed non-empty set.

As it was already stated above, there are at least three possible characterizations of crisp ultrafilters U on X:

- ultrafilters are maximal elements of the lattice $(\Psi(X), \wedge, \vee)$
- ultrafilters are such filters that for each $A \subseteq X$ $U(A) + U(A^c) = 1$
- a filter U is an ultrafilter on X if $1 - U$ is an ideal on X.

In [4] evaluators were characterized. In [1] so-called T_L and S_L evaluators were proposed:

Definition 8. *Let* $(L, \wedge, \vee, \perp, \top)$ *be a lattice with its bottom and top elements* $\perp$ *and* $\top$, *respectively. Then* $\varphi : L \to [0, 1]$ *is a normalized evaluator if*

- $\varphi(\perp) = 0$, $\varphi(\top) = 1$
- *for* $a, b \in L$ $a \leq b$ *implies* $\varphi(a) \leq \varphi(b)$.

A normalized evaluator φ *is said to be a* T_L *evaluator if*

- *for* $a, b \in L$ $\varphi(a \wedge b) \geq T_L(\varphi(a), \varphi(b))$.

A normalized evaluator φ *is said to be an* S_L *evaluator if*

- *for* $a, b \in L$ $\varphi(a \vee b) \leq S_L(\varphi(a), \varphi(b))$.

Theorem 1 ([1])**.** *Let us have the lattice* $(2^X, \cap, \cup, \emptyset, X)$. *Then* $\varphi : 2^X \to [0, 1]$ *is a* T_L *evaluator iff it is a Łukasiewicz filter.* $\psi : 2^X \to [0, 1]$ *is an* S_L *evaluator iff* $1 - \psi$ *is a Łukasiewicz ideal.*

As a direct corollary to the definitions of Łukasiewicz t-norm T_L and t-conorm S_L and to Theorem 1 we get the following

Lemma 2 ([1])**.** $\varphi : 2^X \to [0, 1]$ *is a* T_L *and* S_L *evaluator iff for each* $A \subseteq X$

$$\varphi(A) + \varphi(A^c) = 1$$

Denote $\Phi(X, T_L)$ the system of all Łukasiewicz filters on X. Theorem 1 and Lemma 2 imply

Theorem 2. *Let* $\mathcal{F} \in \Phi(X, T_L)$. *Then the following are equivalent:*

1. *for each* $A \subseteq X$ $\mathcal{F}(A) + \mathcal{F}(A^c) = 1$
2. $1 - \mathcal{F}$ *is a Łukasiewicz ideal*
3. $\mathcal{F}$ *is a maximal element of the lattice* $(\Phi(X, T_L), \wedge, \vee)$.

Since property 2 plays an important role in construction of fuzzy preference relations (particularly, in decision wether there is some incomparability or not, see [9]) we define Łukasiewicz ultrafilters by the following:

Definition 9. *$\mathcal{U} \in \Phi(X, T_L)$ is a Łukasiewicz ultrafilter iff $1-\mathcal{U}$ is a Łukasiewicz ideal.*

As Theorem 2 states, from the algebraic point of view Łukasiewicz ultrafilters behave exactly as crisp ultrafilters.

3. T-filters and T-ideals

If we replace in formulae (2) and (3) the Łukasiewicz t-norm by some other t-norm T, we get the definition of a T-filter and T-ideal, respectively. Let us denote $\Phi(X, T)$ the system of all T-filters on X.

Definition 10. *$\mathcal{U} \in \Phi(X, T)$ is a T-ultrafilter iff $1 - \mathcal{U}$ is a T-ideal.*

Obviously, if $T_1 \geq T_2$ are some t-norms, then $\Phi(X, T_1) \leq \Phi(X, T_2)$, and since each T-filter defines some T-ideal, the same inequality holds also for systems of T-ideals. As a result we get

Lemma 3. *Let $T_1 \geq T_2$ be arbitrary t-norms. Then, if $\mathcal{U}_1$ is a T_1-ultrafilter, then it is also a T_2-ultrafilter.*

The definition of T-ultrafilters implies that each T-ultrafilter $\mathcal{U}$ defines two T-ideals on X:

$$\mathcal{I}_1(A) = \mathcal{U}(A^c), \quad \mathcal{I}_2(A) = 1 - \mathcal{U}(A) \tag{4}$$

As we will see later on, unlike crisp ultrafilters and Łukasiewicz ultrafilters, for a general t-norm T we may get $\mathcal{I}_1 \neq \mathcal{I}_2$.

By definitions of a T-ultrafilter and T-ideal we get the following for each T-ultrafilter $\mathcal{U}$ on X and each $A \subseteq X$:

$$\begin{aligned} \mathcal{U}(A \cap A^c) &\geq T(\mathcal{U}(A), \mathcal{U}(A^c)) \\ 1 - \mathcal{U}(A \cup A^c) &\geq T(1 - \mathcal{U}(A), 1 - \mathcal{U}(A^c)) = 1 - S(\mathcal{U}(A), \mathcal{U}(A^c)) \end{aligned}$$

hence we get the following system of equations:

$$\begin{aligned} T(\mathcal{U}(A), \mathcal{U}(A^c)) = 0 \\ S(\mathcal{U}(A), \mathcal{U}(A^c)) = 1 \end{aligned} \tag{5}$$

Now, we will distinguish a couple of types of t-norms T. For each of the type we will study the structure of the system of T-ultrafilters:

3.1. **T-norms with no 0-divisors.** A t-norm T has no 0-divisors iff

$$T(x,y)=0 \quad \Leftrightarrow \quad \min\{x,y\}=0$$

The above condition gives the following for each $\mathcal{F}\in\Phi(X,T)$:

$$(\forall A\subseteq X)\,\mathcal{F}(A)>0 \;\Rightarrow\; \mathcal{F}(A^c)=0$$

Hence we get that only crisp ultrafilters are T-ultrafilters and moreover crisp ultrafilters are the only maximal elements of $(\Phi(X,T),\wedge,\vee)$.

3.2. **Left-continuous T-norms $T>T_L$ with 0-divisors.** We split this paragraph into two parts:

(1) Let us consider t-norms T such that

$$T(x,y)=0\,\&\,0<x<1 \quad \Rightarrow \quad x+y<1$$

As an example of such a t-norm is the Yager t-norm

$$T_Y(x,y)=\max\left\{0,\,1-\sqrt{(1-x)^2+(1-y)^2}\right\}.$$

Let $T>T_L$ be an arbitrary t-norm with 0 divisors. Then for the dual t-conorm S we get

$$S(x,y)=1\,\&\,0<x<1 \quad \Rightarrow \quad x+y>1$$

Hence we get that only crisp ultrafilters are T-ultrafilters. Since T is left-continuous, there exists

$$z=\max\{x;T(x,x)=0\}.$$

If we put

$$\mathcal{F}(A)=\begin{cases}1, & \text{if } A=X\\ 0, & \text{if } A=\emptyset\\ z, & \text{otherwise,}\end{cases}$$

then $\mathcal{F}$ is a maximal element of the lattice $(\Phi(X,T),\wedge,\vee)$. I.e., in this case the system of T-ultrafilters does not coincide with the system of maximal elements of $(\Phi(X,T),\wedge,\vee)$.

(2) Let T_N be the nilpotent minimum, which means the following t-norm:

$$T_N(x,y)=\begin{cases}0, & \text{if } x+y\le 1\\ \min\{x,\,y\}, & \text{otherwise.}\end{cases}$$

Then the dual t-conorm S_N is the following:

$$S_N(x,y)=\begin{cases}1, & \text{if } x+y\ge 1\\ \max\{x,\,y\}, & \text{otherwise.}\end{cases}$$

The system of equations (5) has the following solution for each T_N-ultrafilter $\mathcal{U}$:

$$\forall\, A\subseteq X \quad \mathcal{U}(A)+\mathcal{U}(A^c)=1$$

We get the following result:

Theorem 3. *Let $\mathcal{F} \in \Phi(X, T_N)$. Then the following are equivalent:*
(a) *for each $A \subseteq X$ $\mathcal{F}(A) + \mathcal{F}(A^c) = 1$*
(b) *$1 - \mathcal{F}$ is a T_N-ideal*
(c) *$\mathcal{F}$ is a maximal element of the lattice $(\Phi(X, T_N), \wedge, \vee)$.*

The following is an example of a T_N-ultrafilter and of a Łukasiewicz ultrafilter, which is not a T_N-ultrafilter:

Example 1. Let $X = \{a, b, c\}$. The following table defines a T_N-ultrafilter on X:

A	X	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$
$\mathcal{U}(A)$	1	0	0.1	0.2	0.8	0.2	0.8	0.9

The next example is that of a Łukasiewicz ultrafilter on X, which is not a T_N-ultrafilter (nor a T_N-filter):

A	X	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$
$\mathcal{U}(A)$	1	0	0.1	0.1	0.8	0.2	0.9	0.9

3.3. **Left-continuous t-norms $T < T_L$.** Left-continuous t-norms $T < T_L$ have the following property:

$$0 < x < 1 \;\&\; z = \max_y \{x, y\} = 0 \quad \Rightarrow \quad x + z > 1.$$

As an example for such t-norms we can take again a Yager t-norm

$$T_Y(x, y) = \max\left\{0,\, x + y - 1 - 2\sqrt{(1-x)(1-y)}\right\}.$$

Evidently, Łukasiewicz ultrafilters are not maximal elements of $(\Phi(X, T), \wedge, \vee)$, where $T < T_L$ is an arbitrary left-continuous t-norm, however they are T-ultrafilters, since the system of T-ultrafilters is antitone with respect to t-norms (as it was already stated above).

If we take the, just defined Yager t-norm T_Y, we get the following example:

Example 2. Let $X \neq \emptyset$. We have the following T_Y-ultrafilter $\mathcal{U}$ on X:

$$\mathcal{U}(A) = \begin{cases} 1, & \text{if } A = X \\ 0, & \text{if } A = \emptyset \\ \frac{3}{4}, & \text{otherwise} \end{cases}$$

The ultrafilter $\mathcal{U}$ defines two different T_Y-ideals:

$$\mathcal{I}_1(A) = \begin{cases} 0, & \text{if } A = X \\ 1, & \text{if } A = \emptyset \\ \frac{3}{4}, & \text{otherwise} \end{cases} \qquad \mathcal{I}_2(A) = \begin{cases} 0, & \text{if } A = X \\ 1, & \text{if } A = \emptyset \\ \frac{1}{4}, & \text{otherwise} \end{cases}$$

where $\mathcal{I}_1(A) = \mathcal{U}(A^c)$, $\mathcal{I}_2(A) = 1 - \mathcal{U}(A)$.

We can formulate the following characterization of T-ultrafilters and T-ideals:

Theorem 4. *Let $T < T_L$ be an arbitrry left-continuous t-norm. Each maximal element of $(\Phi(X,T), \wedge, \vee)$ is a T-ultrafilter on X. There are ultrafilters on X which are not maximal elements of $(\Phi(X,T), \wedge, \vee)$. Let $\mathcal{U}$ be a T-ultrafilter on X. Then T-ideals $\mathcal{I}_1(A) = \mathcal{U}(A^c)$ and $\mathcal{I}_2(A) = 1 - \mathcal{U}(A)$ may be different. $\mathcal{I}_1 = \mathcal{I}_2$ if and only if $\mathcal{U}$ is a Łukasiewicz ultrafilter.*

3.4. **Drastic product t-norm T_D.** This t-norm is not left-continuous. This implies that the only maximal elements of $(\Phi(X,T_D), \wedge, \vee)$ are crisp ultrafilters on X. However, by definition of T_D and S_D we get that a T_D-ultrafilter is each crisp ultrafilter and each monotonic function $\mathcal{F} : 2^X \to [0,1]$ such that

$$\begin{aligned} \mathcal{F}(\emptyset) &= 0, \\ \mathcal{F}(X) &= 1, \\ \mathcal{F}(A) &\in \ ]0,1[\quad \text{for } A \notin \{X, \emptyset\}. \end{aligned}$$

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