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A NOTE ON AN EXAMPLE OF USE OF FUZZY PREFERENCE STRUCTURES

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Abstract. In this paper we present a problem of multicriterial optimization and different models to solve it. We illustrate various alternatives on a practical example.

1. Motivation Example

In this paper we illustrate several aspects of a multicriterial problem that we try to approach from different perspectives – deductive, inductive, and different formal models – Choquet integrals, fuzzy preference structures.

Example 1. *(Michel Grabisch, Marc Roubens* [3]*)*
In [3] the authors consider the problem of the evaluation of trainees learning to drive military vehicles. The instructors evalueted the trainees according to 4 criteria:

C1. Firing precision: The percentage of success during the exercise is computed.

C2. Target detection rapidity: The elapsed time between the appearance of the target and the detection is measured in tu (time unit).

C3. Driving: In order to go from one point to another, the trainee has to choose a suitable trajectory, or to follow a given one as precisely as possible. A qualitive score is given by the instructor, ranging from A (excelent) to E (hopeless).

C4. Communication: The trainee is supposed to belong to some unit, and thus he should understand and obey orders, and also report actions. As for the driving criterion, a qualitative score is given by the instructor, ranging from A (perfect) to E (incontrollable).

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Table 1. Performances of the different trainees, cf. [3].

name	*precision (%)*	*rapidity (tu)*	*driving*	*communication*
Arthur	90	2	B	D
Lancelot	80	4	B	B
Yvain	95	5	C	A
Perceval	60	6	B	B
Erec	65	2	C	B

Table 2. Scores on the different criteria, cf. [3]

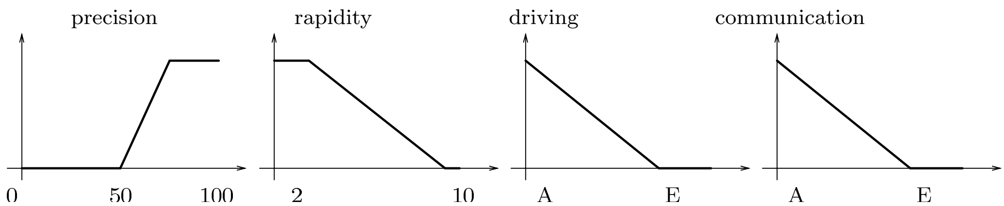

Table 3. Numerical scores on criteria, cf. [3].

name	*precision*	*rapidity*	*driving*	*communication*
Arthur	1.000	1.000	0.750	0.250
Lancelot	0.750	0.750	0.750	0.750
Yvain	1.000	0.625	0.500	1.000
Perceval	0.250	0.500	0.750	0.750
Erec	0.375	1.000	0.500	0.750

In this example, Grabisch and Roubens consider 5 trainees, whose names and performances on each criterion are given in Table 1.

Instructor's comments:

C.1 (precision): over 90% of success is perfect, below 50% is totally unacceptable.

C.2 (rapidity): below 2 tu is perfect, over 10 tu is totally unacceptable.

C.3 and C.4: these criteria are already expressed in the form of an equidistant numerical score.

This permits us to draw utility functions which give the following numerical scores for the trainees in Tables 2, 3.

Looking at the performances of the different trainees, the instructor is able to rank the trainees, as given in Table 4. There are three predetermined classes,

TABLE 4. Ranking of the five trainees, cf. [3].

name	*class*	*rank in the class*
Arthur	bad	2
Lancelot	good	1
Yvain	good	2
Perceval	bad	1
Erec	average	1

TABLE 5. Mapping from class and rank to $[0, 1]$, cf. [3].

class	*interval for the global score*
good	[0.75, 1.0]
average	[0.4, 0.75]
bad	[0.0, 0.4]

called good, average, bad. In each class, a ranking is done, labelling by 1 the best in the class, by 2 the second best, etc.
Inductive task. Now we are in a multicriterial situation. In [3] the authors solve the inductive problem, given a global evaluation, how to learn an objective function which explains global ranking from particular attributes. This is the point where different models have different representations of a utility function.

In [3] an approach is taken, where the global ranking is represented as Choquet integral, and we have to learn the measure. The condition for learning is either;

1. approach by the minimization of the quadratic error,

or

2. approach based on constraint satisfaction.

For this, [3] assignes intervals to classes as in Table 5.

Grabish and Roubens [2] present an algorithm which specifies a measure such that the **Choquet integral** mimics the global evaluation. The idea of the first approach (minimization of squared errors) is to identify the fuzzy measure in a Choquet integral: We suppose that the decision maker is able to assess a numerical score for each act and each criterion, and also a numerical global score for each act. So we want to find the fuzzy measure which minimizes the total squared error of the model.

In the second approach (**constrained satisfaction**) we assume that we have an expert who is able to tell the relative importance of criteria and kind of interaction between them, if any. All this information can be transformed in terms of linear equalities or inequalities linking the unknown weights. These

TABLE 6. Numerical data on criteria and global performance, cf. [3].

name	*precision*	*rapidity*	*driving*	*communication*	global 1st
Arthur	1.000	1.000	0.750	0.250	0.133
Lancelot	0.750	0.750	0.750	0.750	0.917
Yvain	1.000	0.625	0.500	1.000	0.833
Perceval	0.250	0.500	0.750	0.750	0.276
Erec	0.375	1.000	0.500	0.750	0.575

name	*precision*	*rapidity*	*driving*	*communication*	global 2nd
Arthur	1.000	1.000	0.750	0.250	0.3
Lancelot	0.750	0.750	0.750	0.750	0.75
Yvain	1.000	0.625	0.500	1.000	0.7
Perceval	0.250	0.500	0.750	0.750	0.35
Erec	0.375	1.000	0.500	0.750	0.5

methods are in fact not comparable, since they do not take exactly the same input, nor provide the same kind of output.

The results of both aproaches are given in Table 6.

Our approach is based on conenction between fuzzy and annotated logic programs [5] and an inductive logic programming method for learning rules of annotated programs [4]. In this approach we start from an instructors evaluation expressed in a lineary ordered scale, here it can be

$$bad2 < bad1 < average < good2 < good1$$

or any order preserving mapping into $[0, 1]$ (here understood as an ordinal scale). Then the task has a possible input as in the following Table 7.

TABLE 7. Linear ranking of the five trainees

name	*global rank*
Arthur	0.125
Lancelot	0.875
Yvain	0.75
Perceval	0.375
Erec	0.625

This global numerical rank gives a partial function f from $[0, 1]^4$ into $[0, 1]$, as depicted in Table 8. This function can be extended to F on whole $[0, 1]^4$ preserving monotonicity, in the following sense.

Table 8. Function on attributes

name	*precision*	*rapidity*	*driving*	*communication*	*global rank*
Arthur	1.000	1.000	0.750	0.250	0.125
Lancelot	0.750	0.750	0.750	0.750	0.875
Yvain	1.000	0.625	0.500	1.000	0.75
Perceval	0.250	0.500	0.750	0.750	0.375
Erec	0.375	1.000	0.500	0.750	0.625

Denote Lancelot's attribute scores as $x^L = (x_1^L, x_2^L, x_3^L, x_4^L)$ and Perceval's attribute scores as $x^P = (x_1^P, x_2^P, x_3^P, x_4^P)$. Note that $x_i^P \leq x_i^L$ for $i = 1, ..., 4$ and $f(x^P) \leq f(x^L)$. Hence global score of trainees does not violate monotonicity. A straitforward way to prolongate it the whole $[0,1]^4$ is the definition

$$F(y_1, y_2, y_3, y_4) = \max\{f(x^T) : T \in \{A, E, L, P, Y\} \text{ and } (\forall i) x_i^T \leq y_i\}$$

Note that $\max \emptyset = 0$. Our method from [4] is able to learn such a monotonic extension of any function given in a multirelational set of data and different preferences. This si especialy interesting on bigger data and intervals asigned to global score violating colinearity (Choqeut integral is able to represent only colinear functions).

Deductive task. In a similar setting, having trainees and their achievements (same data) we can assume that from previous experiments we already have a utility function. Now the problem is about efficient algorithms to find the best trainee, assuming we have a huge set of data, possibly distributed, and so the question of efficiency becomes crucial.

In this paper we describe the problem setting which is a common starting point for different approaches.

2. Introduction to preference structures and fuzzy preference structures

The preference structure is a basic step of preference modeling. Given two alternatives, decision maker defines three binary relation-preference, indifference and incomparability.

A preference structure is a basic concept of preference modelling. In a classical preference structure (PS) a decision-maker makes three decission for any par (a, b) from the set $\boldsymbol{A}$ of all alternatives. His decision define a triplet P, I, J of a crisp binary relations on $\boldsymbol{A}$:

1) a is prefered to $b \Leftrightarrow (a, b) \in P$ (strict preference).
2) a and b are indifferent $\Leftrightarrow (a, b) \in I$ (indifference).
3) a and b are incomparable $\Leftrightarrow (a, b) \in J$ (incomparability).

A preference structure (PS) on a set $\boldsymbol{A}$ is a triplet (P, I, J) of binary relations on $\boldsymbol{A}$ such that

(ps1) I is reflexive, P and J are irreflexive.
(ps2) P is asymmetric, I and J are symmetric.
(ps3) $P \cap I = P \cap J = I \cap J = \emptyset$.
(ps4) $P \cup I \cup J \cup P^t = A \times A$ where $P^t(x,y) = P(y,x)$.

A preference structure can be characterized by the reflexive relation $R = P \cup I$ called the large preference relation. The relation R can be interpreted as

$$(a,b) \in R \Leftrightarrow a \text{ is prefered to } b \text{ or } a \text{ and } b \text{ are indifferent.}$$

It can be easily proved that

$$co(R) = P^t \cup J$$

where $coR(a,b) = 1 - R(a,b)$ and

$$P = R \cap co(R^t), I = R \cap R^t, J = co(R) \cap co(R^t).$$

It allows us to construct a preference structure (P, I, J) from a reflexive binary operation R only.

Decision-makers are often uncertain even inconsistent in their judgements. The restriction on two-valued relations have been an important drawback to their practical use. A natural demand led researchers to the introduction of a fuzzy preference structure (FPS). The original idea of using numbers between zero and one to describe the strenth of links between two alternatives goes back to Menger. The introduction of fuzzy relations allowed to express degrees of preference, indifference and incomparability. Of course, the attempts simply to replace the notion used in the definition of (PS) by their fuzzy equivalents have met some problems.

To define (FPS) it is necessary to consider some fuzzy connectives. We shall consider a continuous De Morgan triple (T, S, N) consisting of a continuous t-norm T, continuous t-conorm S and a strong negator N such that $T(x,y) = N(S(N(x), N(y)))$. The main problem consists in the fact that the completeness condition (ps4) can be written in many forms, e.g.:

$$co(P \cup P^t) = I \cup J, P = co(P^t \cup I \cup J), P \cup I = co(P^t \cup J).$$

Let (T,S,N) be De Morgan triplet. A fuzzy preference structure (FPS) on a set $\boldsymbol{A}$ is a triplet (P, I, J) of binary fuzzy relations on $\boldsymbol{A}$ such that

(f1) I is reflexive, P and J are irreflexive. $I(a,a) = 1, P(a,a) = J(a,a) = 0$
(f2) P is T-asymmetrical, I and J are symmetrical. $T(P(a,b), P(b,a)) = 0$
(f3) $T(P, I) = T(P, J) = T(I, J) = 0$. for all pair of alternatives
(f4) $(\forall (a,b) \in A^2) S(P, P^t, I, J) = 1$ or $N(S(P, I)) = S(P^t, J)$ or another completeness conditions.

Note that it was proved [1, 7] that reasonable constructions of fuzzy preference structure (FPS) should use a nilpotent t-norm only. Since any nilpotent t-norm (t-conorm) is isomorphic to the Lukasiewicz t-norm (t-conorm), it is enough to restrict our attention to De Morgan triple $(T_L, S_L, 1-x)$.

3. Preference structures and fuzzy preference structures and their applications

Let us turn our attention to motivation example. We denote by $M = \{A, E, L, P, Y\}$ the set of all trainees. We are able to construct the large preference relations R_P, R_R, R_D and R_C derived from orderings in our four criteria (precision, rapidity, driving, communication):

R_P	A	E	L	P	Y
A	1	1	1	1	1
E	0	1	0	1	0
L	0	1	1	1	0
P	0	0	0	1	0
Y	1	1	1	1	1

$R_P = \{[A,A],[E,E],[L,L],[P,P],[Y,Y],[A,Y],[Y,A],[A,L],[A,E],[A,P],[Y,L],[Y,E],[Y,P],[L,E],[L,P],[E,P]\}$

R_R	A	E	L	P	Y
A	1	1	1	1	1
E	1	1	1	1	1
L	0	0	1	1	1
P	0	0	0	1	0
Y	0	0	0	1	1

$R_R = \{[A,A],[E,E],[L,L],[P,P],[Y,Y],[A,E],[E,A],[A,L],[A,Y],[A,P],[E,L],[E,Y],[E,P],[L,Y],[L,P],[Y,P]\}$

R_D	A	E	L	P	Y
A	1	1	1	1	1
E	0	1	0	0	1
L	1	1	1	1	1
P	1	1	1	1	1
Y	0	1	0	0	1

$R_D = \{[A,A],[E,E],[L,L],[P,P],[Y,Y],[A,L],[L,A],[A,P],[P,A],[L,P],[P,L],[A,E],[A,Y],[L,E],[L,Y],[P,E],[P,Y],[E,Y],[Y,E]\}$

R_C	A	E	L	P	Y
A	1	0	0	0	0
E	1	1	1	1	0
L	1	1	1	1	0
P	1	1	1	1	0
Y	1	1	1	1	1

$R_C = \{[A,A],[E,E],[L,L],[P,P],[Y,Y],[Y,L],[Y,P],[Y,E],[Y,A],[L,P],[P,L],[L,E],[E,L],[P,E],[E,P],[L,A],[P,A],[E,A]\}$

And we are able to construct large preference relation R_I which is derived from instructor's global ordering, too:

R_I	A	E	L	P	Y
A	1	0	0	1	0
E	1	1	0	1	0
L	1	1	1	1	1
P	1	0	0	1	0
Y	1	1	1	1	1

$R_I = \{[A,A],[E,E],[L,L],[P,P],[Y,Y],[L,Y],[Y,L],[L,E],[L,A],[L,P],[Y,E],[Y,A],[Y,P],[E,A],[E,P],[A,P],[P,A]\}$

The relation R_I is not linear order set. For global evaluation we will modify this ordering to linear ordering. First, we need order the criteria.

The first idea is; we can pairwise compare the relations R_P, R_R, R_D and R_C with respect to relation R_I by the following rule:

$$X > Y \iff \frac{|R_X \cap R_I|}{|R_X \triangle R_I|} > \frac{|R_Y \cap R_I|}{|R_Y \triangle R_I|}, \tag{1}$$

where $X, Y \in \{P, R, D, C\}$. The idea is; the more R_X is similar to R_I, the more important criterion are X is. This method gives the following ordering of criteria: communication >precision >rapidity > driving. Note that this method is not the only one possible, and investigation of other possibilities is subject of ongoing research.

Generaly speaking, we obtain the ordering of criteria from the relation preference which is given by $P = R \cap co(R^t)$. However, in this example we have got the same ordering of criteria via both the preference relations and the large relations (with respect to previous method (1) for comparing the relations).

P_P	A	E	L	P	Y
A	0	1	1	1	0
E	0	0	0	1	0
L	0	1	0	1	0
P	0	0	0	0	0
Y	0	1	1	1	0

$P_P = \{[A, L], [A, E], [A, P], [Y, L], [Y, E], [Y, P], [L, E], [L, P], [E, P]\}$

P_R	A	E	L	P	Y
A	0	0	1	1	1
E	0	0	1	1	1
L	0	0	0	1	1
P	0	0	0	0	0
Y	0	0	0	1	0

$P_R = \{[A, L], [A, Y], [A, P], [E, L], [E, Y], [E, P], [L, Y], [L, P], [Y, P]\}$

P_D	A	E	L	P	Y
A	0	1	0	0	1
E	0	0	0	0	0
L	0	1	0	0	1
P	0	1	0	0	1
Y	0	0	0	0	0

$P_D = \{[A, E], [A, Y], [L, E], [L, Y], [P, E], [P, Y]\}$

P_C	A	E	L	P	Y
A	0	0	0	0	0
E	1	0	0	0	0
L	1	0	0	0	0
P	1	0	0	0	0
Y	1	1	1	1	0

$P_C = \{[Y, L], [Y, P], [Y, E], [Y, A], [L, A], [P, A], [E, A]\}$

P_I	A	E	L	P	Y
A	0	0	0	0	0
E	1	0	0	1	0
L	1	1	0	1	0
P	0	0	0	0	0
Y	1	1	0	1	0

$P_I = \{[L, E], [L, A], [L, P], [Y, E], [Y, A], [Y, P], [E, A], [E, P]\}$

Fuzzification. For better expresion of reality, we can use fuzzy preference structures. Natural way of fuzzification of preference relations P_P, P_R, P_D and P_C from our motivation example is given as follows:

The value of fuzzy preference in precision (FP_P) for Arthur and Erec, we compute from Table 3 as $FP_P(A, E) = \max\{x_1^A - x_1^E, 0\}$, where x_1^A and x_1^E are Arthur's and Erec's precision score in Table 3, etc. The fuzzification of preference relation P_I is given in the last table and it is derived from Tables 6 and 7.

FP_P	A	E	L	P	Y
A	0	0.625	0.25	0.75	0
E	0	0	0	0.125	0
L	0	0.375	0	0.5	0
P	0	0	0	0	0
Y	0	0.625	0.25	0.75	0

FP_R	A	E	L	P	Y
A	0	0	0.25	0.5	0.375
E	0	0	0.25	0.5	0.375
L	0	0	0	0.25	0.125
P	0	0	0	0	0
Y	0	0	0	0.125	0

FP_D	A	E	L	P	Y
A	0	0.25	0	0	0.25
E	0	0	0	0	0
L	0	0.25	0	0	0.25
P	0	0.25	0	0	0.25
Y	0	0	0	0	0

FP_C	A	E	L	P	Y
A	0	0	0	0	0
E	0.5	0	0	0	0
L	0.5	0	0	0	0
P	0.5	0	0	0	0
Y	0.75	0.25	0.25	0.25	0

FP_I	A	E	L	P	Y
A	0	0	0	0	0
E	0.5	0	0	0.25	0
L	0.75	0.25	0	0.5	0.125
P	0.125	0	0	0	0
Y	0.75	0.25	0	0.5	0

Using the formula for comparing the relations (compare to (1))

$$X \succ Y \iff \frac{\sum_{i,j} |\{fp_{x_{ij}}\} \cap \{fp_{I_{ij}}\}|}{\sum_{i,j} |\{fp_{x_{ij}}\} \triangle \{fp_{I_{ij}}\}|} > \frac{\sum_{i,j} |\{fp_{y_{ij}}\} \cap \{fp_{I_{ij}}\}|}{\sum_{i,j} |\{fp_{y_{ij}}\} \triangle \{fp_{I_{ij}}\}|}, \tag{2}$$

where X, Y are our criteria, m is a number of alternatives, $fp_{x_{ij}}, fp_{y_{ij}}$ are values of fuzzy preference structures of criterion X, Y, $fp_{I_{ij}}$ are values of fuzzy preference relation which is based on expert's global score and $i, j \in \{1, 2 \ldots m\}$, we obtain the following ordering $\succ$ of our criteria:

$$\text{communication} \succ \text{precision} = \text{driving} \succ \text{rapidity}.$$

Note that our intersection ∩ and symmetric difference △ are ordinary. We can see, this ordering is different from ordering, which we obtain via strict preference structure. Note, that another fuzzification leeds to different ordering of criteria and subsequently to different ordering of trainees.
Simple deduction. Let us imagine the next situation: We have ordering $\succ$, and now look at another trainee Bruno with scores given in the Table 9.

TABLE 9. Function on attributes

name	*precision*	*rapidity*	*driving*	*communication*
Arthur	1.000	1.000	0.750	0.250
Lancelot	0.750	0.750	0.750	0.750
Yvain	1.000	0.625	0.500	1.000
Perceval	0.250	0.500	0.750	0.750
Erec	0.375	1.000	0.500	0.750
Bruno	0.400	0.750	0.600	0.750

Our task is to compare Bruno with others. Denote Bruno's attribute scores as $x^B = (x_1^B, x_2^B, x_3^B, x_4^B)$ and Erec's attribute scores as $x^E = (x_1^E, x_2^E, x_3^E, x_4^E)$. We can see that $x_i^E \leq x_i^B$ for $i = 1, 3, 4$. Since there is a tie in the most important criterion (communication) so we decide based on the next criteria (precision and driving) in our ordering $\succ$. This results in Bruno's superiority over Erec. The final ordering of trainees is: Yvain > Lancelot > Bruno > Erec > Perceval > Arthur.

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