Acta Universitatis Matthiae Belii ser. Mathematics, **14**(2007), 51–59.

A GEOMETRICAL APPROACH TO AGGREGATION

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Abstract. Considering the family F of contour curves $F = \{h(x,y) = k)$ of an (idempotent) aggregation operator h in two variables as a one-parametric family of curves, the differential equation $y' = f(x,y)$ having F as general solution is associated to h. Properties of h have then be translated to properties of its differential equation. Reciprocally, for a differential equation fulfilling some easy properties its general solution can be seen as the contour curves of an idempotent aggregation operator so that properties of the equation can have their counterpart in the ones of the aggregation operator.

1. Introduction

It is well known that the orthogonal projection of a point $P = (a,b)$ of the plane on the line l with equation $y = x$ is the point $(\frac{a+b}{2}, \frac{a+b}{2})$, so that the coordinates of the projection of P are the arithmetic mean of the coordinates of P. This gives a nice geometrical interpretation to the arithmetic mean of two numbers. In a similar way, if we project the point P following the direction given by the vector $(-q,p)$ (with $p,q > 0$ and $p+q = 1$) the projection on l is the point with coordinates $(pa+qb, pa+qb)$ obtaining the weighted arithmetic mean in, again, a geometrical way.

Going back to the arithmetic mean, two points lying in a line perpendicular to l will have the same orthogonal projection onto l and therefore their coordinates will have the same arithmetic mean. In this way, we have a one-parametric family of lines $F = \{x + y = k\}$ with the property that all points of a given line of F have the same arithmetic mean.

A similar situation occurs for weighted arithmetic means where, given the weights $p,q > 0$, $p+q = 1$, the one-parametric family $\{px + qy = k\}$ plays the same role as F.

It seems therefore interesting to study what happens if we permit points to move toward l without the constraint of following a straight line, but allowing more general curves. This paper is devoted to this study.

2000 *Mathematics Subject Classification.* 04A72, 34A26, 03E72.

Key words and phrases. aggregation operators, contour curves, differential equations.

Research partially supported by project number TIC2003-04564 and TIN2006-14311.

Next Section will determine which conditions a family of curves must fulfill to be associated to an (idempotent) aggregation operator and which ones are related to desirable properties of these operators.

Of course, if an aggregation operator should be obtained from a family of curves, they must satisfy at least the following two conditions:

- There must pass a curve of the family through every point of the plane (or some restricted domain).
- No two curves can pass through the same point.

These are necessary conditions for the existence of an ordinary differential equation having the family as solution. If the curves are "smooth enough", then the aggregation operator will have associated this differential equation as well.

Two very easy examples are the arithmetic mean and weighted arithmetic means, the family associated to the first one being the solution of the differential equation $y' = -1$ while the latter to the equation $y' = -\frac{p}{q}$.

Reciprocally, a differential equation fulfilling some conditions will generate an aggregation operator and the properties of this operator can be translated to the equation.

Section 3 will study the relation with aggregation operators, one-parametric families of curves and differential equations.

As particular cases, the one-parametric family of curves and differential equations of the most popular aggregation operators will be given; namely: means, quasi-arithmetic means and OWA operators.

2. Contour curves

Definition 2.1. *An aggregation operator (in two variables) is a map $h : X \times X \to X$ where X is some subset of the real line satisfying*

(1) $Min(x, y) \leq h(x, y) \leq Max(x, y) \ \forall x, y \in X$

(2) $h(x_1, y_1) \leq h(x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ $\forall x_1, x_2, y_1, y_2 \in X$ *(monotonicity)*

h is symmetric if and only if

$$h(x, y) = h(y, x) \ \forall x, y \in X$$

It is straightforward to prove that aggregation operators are *idempotent*, i.e.: they satisfy

$$h(x, x) = x \ \forall x \in X$$

Throughout the paper we will assume that all aggregation operators are continuous.

Given an aggregation operator h, we can consider its contour curves, i.e. the sets of points (x, y) in the domain of h with $h(x, y) = k$ where k is a given constant. In this way we associate a continuous one-parametric family F of curves to h with the particularity that the coordinates of all points P of the same curve have the same aggregation with h, which geometrically are the coordinates of the intersecting point (p, p) of this curve with l. (We will write in short that the point (p, p) is the aggregation of P). Some of the properties of h can be translated to F and visualized by its behavior.

Reciprocally, a continuous one-parametric family F of curves of the plane satisfying certain conditions can be seen as the contour curves of h.

Let us consider a family F of continuous parameterized curves $c_k(t) = (x_k(t), y_k(t))$, $k \in X \subset R$ of the plane such that all of them intersect the line l of equation $y = x$ in a single point and let (p_k, p_k) be the intersection of the curve c_k with l. If we want that (p_k, p_k) could be considered an aggregation of any point (a, b) of $c_k(t)$, some restrictions should be imposed to $c_k(t)$.

First, p_k must be between a and b for any point $(a, b) \in c_k(t)$ $(Min(a, b) \leq p_k \leq Max(a, b))$ which means that (p_k, p_k) must lay between the intersections of the lines $x = a$ and $y = b$ with l.

Proposition 2.2. *Let $F = \{c_k\}$ be a one-parametric family of continuous curves such that each curve c_k of F intersects the line l in a point (p_k, p_k). If F is the family of contour curves of an aggregation operator, then for all points (a_k, b_k) of the curve c_k (p_k, p_k) must lay between the intersections of the lines $x = a_k$ and $y = b_k$ with l.*

For example, the curve partly represented in Figure 1 could be a member of a family F generated by an aggregation operator, while the one of Figure 2 could not.

Next proposition provides a geometric translation of monotonicity.

Proposition 2.3. *Let $F = \{c_k\}$ be the contour family of a map h satisfying 2.1.1. h is non-monotonic if and only if there exists a curve c of F with $P = (x_0, y_0)$ and $Q = (x_1, y_1)$ two points of c with $x_0 \leq x_1$ and $y_0 < y_1$.*

Proof. $\Leftarrow$)

In this case, there would be a point $R = (x_2, y_1)$ with $x_2 < x_1$ in the region of points of the plane above c belonging to another curve c' of the family. Since $c \cap c' = \emptyset$, the aggregation of (x_2, y_1) is greater than the aggregation of (x_1, y_1) and therefore the aggregation operator is non-monotonic. (See Figure 3).

$\Rightarrow$)

Trivial.

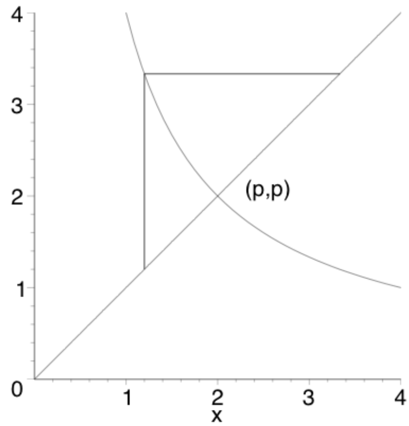

FIGURE 1

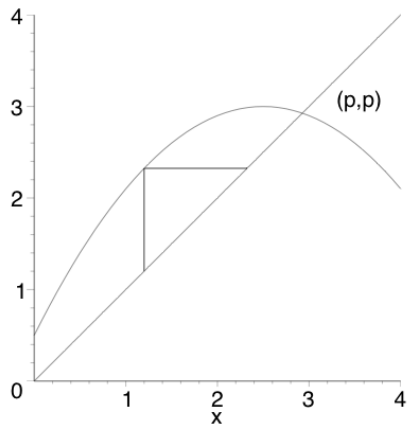

FIGURE 2

In the case that the curves c_k of F are functional, i.e. we can describe c_k with a map $y = f_k(x)$, Proposition 2.3 simply means that the associated map h is monotonic if and only if f_k are non-increasing monotonic maps. $\square$

Symmetry of an aggregation operator can also be easily translated to the behavior of their contour curves.

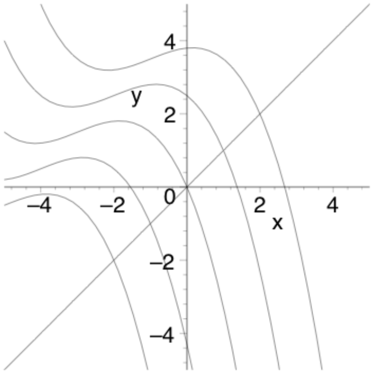

FIGURE 3

Proposition 2.4. *Let $F = \{c_k\}$ be the contour family of an aggregation operator h. h is symmetric if and only if all curves of F are symmetric with respect to the line l (i.e., if (a, b) is a point of curve c_k, then (b, a) is a point of c_k as well).*

If the curves of F are functional ($y = f_k(x)$) then the associated aggregation operator is symmetric if and only if the maps f_k are strictly decreasing with $f_k = f_k^{-1}$ $\forall k$.

3. Differential equations

In the previous Section we have seen the relation between the properties of an aggregation operator h and its family of contour curves $F = \{h(x, y) = k\}$. Let us now suppose that the aggregation has nice differential properties, namely that in some region of the plane there exist $h_x = \frac{\partial h}{\partial x}$ and $h_y = \frac{\partial h}{\partial x} \neq 0$. In this situation, the family F is determined by the differential equation $y' = -\frac{h_x}{h_y}$.

For example, the family F of contour curves of the arithmetic mean $h(x, y) = \frac{x+y}{2}$ is $F = \{x+y = k\}$ that are the solutions of the differential equation $y' = -1$.

Reciprocally, a differential equation $y' = f(x, y)$ satisfying certain conditions will have as solution the family of contour curves of an aggregation operator h. We will say in this case that $y' = f(x, y)$ is the differential equation associated to h.

This Section will study which conditions a differential equation must satisfy for having as solution a family of contour curves of an aggregation operator h and how the properties of h are transfered to the differential equation.

Proposition 3.1. *Let $y' = f(x,y)$ be a differential equation. If $f(x,y) \leq 0\ \forall x, y \in R$ then it is associated to an aggregation operator.*

Proof. If $y' \leq 0$, the curves solution of the equation will not satisfy the hypothesis of Proposition 2.3. □

Proposition 3.2. *The differential equation $y' = f(x,y)$ represents a symmetric aggregation operator iff $f(x,y) \cdot f(y,x) = 1$.*

Proof. This implies that the curves are symmetric with respect to the line $y = x$. □

Example 3.3. Table 1 *shows the most popular symmetric aggregation operators with their respective families of one-parametrized curves F and the corresponding differential equation.*

TABLE 1

	h	F	y'
arithmetic mean	$\frac{x+y}{2}$	$x + y = k$	$y' = -1$
geometric mean	$\sqrt{xy}$	$xy = k$	$y' = -\frac{y}{x}$
harmonic mean	$\frac{xy}{x+y}$	$2xy = k(x+y)$	$y' = -\frac{y^2}{x^2}$
generalized means	$(\frac{x^\alpha+y^\alpha}{2})^{\frac{1}{\alpha}}$	$x^\alpha + y^\alpha = k$	$y' = -\frac{x^{\alpha-1}}{y^{\alpha-1}}$
quasi − arithmetic means	$f^{-1}(\frac{f(x)+f(y)}{2})$	$f(x) + f(y) = k$	$y' = -\frac{f'(x)}{f'(y)}$
OWA operators	$pMax(x,y)$ $+qMin(x,y)$	$pMax(x,y)$ $+qMin(x,y) = k$	$y' = \begin{cases} -\frac{p}{q} & \text{if } x < y \\ -\frac{q}{p} & \text{if } x > y \end{cases}$

Example 3.4. Table 2 *displays the most popular non-symmetric aggregation operators with their respective families of one-parametrized curves F and the corresponding differential equation.*

If $F = \{y = f(x,k)\}$ is a one-parametric family of curves such that for all k $\frac{\partial f}{\partial x}$ is between -1 and 1, then rotating F -45^o with respect to the origin $(0,0)$, we obtain a family of contour curves of an aggregation operator.

Table 2

	h	F	y'
weighted arithmeticmean	$px+qy$	$px+qy=k$	$y'=-\frac{p}{q}$
weighted geometric mean	x^py^q	$x^py^q=k$	$y'=-\frac{py}{qx}$
weighted harmonic mean	$\frac{xy}{qx+py}$	$xy=k(qx+py)$	$y'=-\frac{qx^2}{py^2}$
weighted generalized means	$(px^\alpha+qy^\alpha)^{\frac{1}{\alpha}}$	$px^\alpha+qy^\alpha=k$	$y'=-\frac{px^{\alpha-1}}{qy^{\alpha-1}}$
weighted quasi arithmetic means	$f^{-1}(pf(x)+qf(y))$	$pf(x)+qf(y)=k$	$y'=-\frac{pf'(x)}{qf'(y)}$

A nice easy way to generate aggregation operators is therefore to start with a map $y=f(x)$ with $-1\le f'(x)\le 1$ $\forall x$ and rotate -45^o the family $F=\{y=f(x)+k\}$ with respect to the origin $(0,0)$.

Definition 3.5. *Let $y=f(x)$ be a map with $-1\le f'(x)\le 1$ $\forall x$. m_f will be the aggregation operator whose contour curves are the family obtained form the curves $y=f(x)+k$ rotated -45^o with respect to the origin $(0,0)$.*

The differential equation fulfilled by this family is

$$y'=\frac{f'(\frac{x-y}{\sqrt{2}})-1}{f'(\frac{x-y}{\sqrt{2}})+1}$$

The aggregation m_f is

$$m_f(x,y)=\frac{x+y-\sqrt{2}f(\frac{x-y}{\sqrt{2}})+\sqrt{2}f(0)}{2}$$

Example 3.6. (1) *From $y=\alpha cos(x)$ with $0\le\alpha\le 1$ we get the family solution of*

$$y'=\frac{\alpha sin(\frac{x-y}{\sqrt{2}})+1}{\alpha sin(\frac{x-y}{\sqrt{2}})-1}$$

and with aggregation operator

$$m(x,y)=\frac{x+y-\sqrt{2}\alpha cos(\frac{x-y}{\sqrt{2}})+\sqrt{2}\alpha}{2}$$

(See Figure 4).

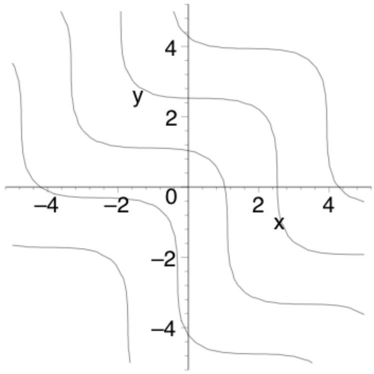

FIGURE 4

(2) *From* $y = \alpha e^{-x^2}$ *with* $0 \leq \alpha \leq \sqrt{\frac{e}{2}}$ *we get the family solution of*

$$y' = \frac{-a\sqrt{2}(x-y)e^{-\frac{(x-y)^2}{2}}) - 1}{-a\sqrt{2}(x-y)e^{-\frac{(x-y)^2}{2}}) + 1}$$

and with aggregation operator

$$m(x,y) = \frac{x + y - \sqrt{2}\alpha e^{-\frac{(x-y)^2}{2}} + \sqrt{2}\alpha}{2}$$

(See Figure 5).

The following result is straightforward.

Proposition 3.7. m_f *is symmetric if and only if* f *is an even function (i.e.: f(x)=f(-x)).*

4. Concluding Remarks

This paper has provided a first attempt to relate aggregation operators with differential equations. This have been achieved assigning to every (idempotent) aggregation operator in two variables the differential equation which has its contour curves as general solution.

Some properties of the aggregation operator have been translated to properties of the associated differential equation and vice versa.

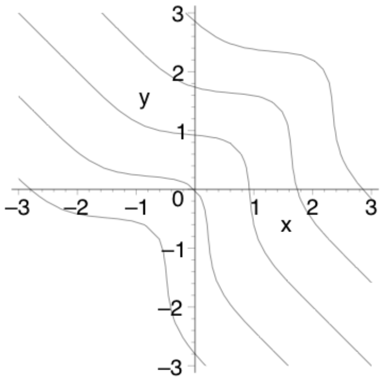

FIGURE 5

The author will try to extend the results of this paper to more than two variables in forthcoming works.

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