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EQUALITY OF SPECIAL MIXTURE OPERATORS AND QUASI-ARITHMETIC MEANS

JANA ŠPIRKOVÁ

Abstract. In our paper we introduce the problem of equality of special mixture operators and quasi-arithmetic means. From equality of mixture operators and quasi-arithmetic means we get as solutions the arithmetic, harmonic, geometric means and more special types of aggregation operators belonging simultaneously to both discussed classes.

1. Introduction

Let $I \subset R$ be any interval. Let $\varphi : I \to R$ be a continuous strictly increasing function. For any weighting function $f : I \to]0, \infty[$, φ and f induces a quasi-mixture operator $M_\varphi^f : \bigcup\limits_{n \in N} I^n \to I$,

$$M_f^\varphi (x_1, x_2, \dots, x_n) = \varphi^{-1} \left(\frac{\sum\limits_{i=1}^{n} \varphi(x_i) \cdot f(x_i)}{\sum\limits_{i=1}^{n} f(x_i)} \right).$$

For details see [4], [5].

In special case, if transformation function is $\varphi(x) = x$, the quasi-mixture operator induces the mixture operator

$$M_f (x_1, x_2, \dots, x_n) = \frac{\sum\limits_{i=1}^{n} f(x_i) \cdot x_i}{\sum\limits_{i=1}^{n} f(x_i)}.$$

More informations about mixture operators can be found in [3], [6], [8].

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If weighting function $f(x) = const$, quasi-mixture operator goes to the quasi-arithmetic mean

$$M^{\varphi}(x_1, x_2 \ldots, x_n) = \varphi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\varphi(x_i)\right).$$

See e.g. [1], Section 5.3 in [3].

In our paper we recall the problem of the equality of two quasi-mixture operators, which was solved in [2], [4], [5]. We modify these solutions to solve a related problem of the equality of a mixture operator M_g and a quasi-arithmetic mean M^{η}.

The paper is organized as follows. In the next section we summarize the solutions of the equality of two quasi-mixture operators from [2], [4], [5] described in a transformed way as an equality of a quasi-mixture operator and the arithmetic mean. In the third section we solve the equality problem $M_g = M^{\eta}$ based separately on all introduced solutions from Section 2. Finally, some conclusions are given.

2. Equality of mixture and quasi-mixture operators

In paper [2] Bajraktarević solved the functional equation

$$(1) \qquad \Phi^{-1}\left(\frac{\sum_{i=1}^{n}\Phi(x_i)F(x_i)}{\sum_{i=1}^{n}F(x_i)}\right) = \Psi^{-1}\left(\frac{\sum_{i=1}^{n}\Psi(x_i)G(x_i)}{\sum_{i=1}^{n}G(x_i)}\right), (x_1, \ldots, x_n \in I)$$

where $\Phi, \Psi : I \to R$ are the strictly monotonic and continuous functions and $F, G : I \to]0, \infty[$ are weighting functions. He supposed for a fixed $n \geq 3$ and that the functions Φ, Ψ, F and G are twice differentiable and proved that there are constants a, b, c, $d \in R$ such that

$$(c^2 + d^2) \cdot (ad - bc) \neq 0$$

and

$$\Psi(x) = \frac{a\Phi(x) + b}{c\Phi(x) + d} \qquad G(x) = F(x) \cdot (c\Phi(x) + d),$$

what attends to the arithmetic mean.

In paper [5] Losonczi solved the two - variable equality problem of the quasi-mixture operators

$$(2) \qquad \Phi^{-1}\left(\frac{\Phi(X)F(X) + \Phi(Y)F(Y)}{F(X) + F(Y)}\right) = \Psi^{-1}\left(\frac{\Psi(X)G(X) + \Psi(Y)G(Y)}{G(X) + G(Y)}\right)$$

that holds all $X, Y \in I$. He supposed six times differentiability of the functions involved and got 32 new families of solutions.

Daróczy et al. in [4] solved the equality of two quasi-mixture operators $M_F^\Phi = M_G^\Psi$ without differentiability conditions. Authors used the substitutions $x = \Psi(X)$, $y = \Psi(Y)$ and with the definitions $J = \Psi(I)$, $g = G \circ \Psi^{-1}$, $f = F \circ \Psi^{-1}$, $\varphi = \Phi \circ \Psi^{-1}$ the equation (2) was rewritten into

$$\frac{\varphi(x)f(x) + \varphi(y)f(y)}{f(x) + f(y)} = \varphi\left(\frac{g(x)x + g(y)y}{g(x) + g(y)}\right),$$

where $x, y \in J$. They supposed that G is a constant, thus g is a constant too, and they got

$$\varphi^{-1}\left(\frac{\varphi(x)f(x) + \varphi(y)f(y)}{f(x) + f(y)}\right) = \frac{x + y}{2}, \qquad (x, y \in J). \tag{3}$$

The solution of the equality (3) is writen in a regularity theorem in [4], where the pair (φ, f) is a solution on J if and only if it has one of the following forms

	$\varphi(x)$	$f(x)$
(1)	$Ax + D$	E
(2)	$\frac{A}{x + C} + D$	$E(x + C)$
(3)	$Atanh(Bx + C) + D$	$Ecosh(Bx + C)$
(4)	$Acoth(Bx + C) + D$	$Esinh(Bx + C)$
(5)	$Atan(Bx + C) + D$	$Ecos(Bx + C)$
(6)	$Aexp(-2Bx) + D$	$Eexp(Bx)$

for all $x \in J$ and for some constants $A, B, C, D \in R$ such that $ABE \neq 0$ and $f(x) > 0$.

Daróczy et al. in [4] for arbitrary g and for recalled couples (φ, f) got the solution of equality of quasi-mixture and mixture operators

$$\varphi^{-1}\left(\frac{\sum_{i=1}^{n} \varphi(x_i) \cdot f(x_i)}{\sum_{i=1}^{n} f(x_i)}\right) = \frac{\sum_{i=1}^{n} g(x_i) \cdot x_i}{\sum_{i=1}^{n} g(x_i)}. \tag{4}$$

In the next we remark and analyze separately the equality of quasi-mixture and mixture operators for the couples 1.-6. (φ, f) and arbitrary g.

(1) Function $\varphi(x) = Ax + D$ from the first couple (f, φ) acts the same as $id \equiv x$. For $g(x) = const$ left side and right side of equation (4) give us the arithmetic mean.

(2) The function $\varphi(x) = \frac{A}{x+C} + D$ has the inverse function $\varphi^{-1}(x) = \frac{A}{x-D} - C$ and weighting function is $f(x) = E(x + C)$. The equality (4) can be rewrite as

$$\frac{A}{\frac{\sum_{i=1}^{n} \varphi(x_i) \cdot f(x_i)}{\sum_{i=1}^{n} f(x_i)} - D} - C = \frac{\sum_{i=1}^{n} g(x_i) \cdot x_i}{\sum_{i=1}^{n} g(x_i)}.$$

For $A = 1$ and $D = 0$ we get

$$\frac{\sum_{i=1}^{n} f(x_i)}{\sum_{i=1}^{n} \varphi(x_i) \cdot f(x_i)} = \frac{\sum_{i=1}^{n} g(x_i) \cdot (x_i + C)}{\sum_{i=1}^{n} g(x_i)}.$$

From the last equality we see for arbitrary f weighting function g is given by

$$g(x) = f(x) \cdot \varphi(x).$$

For arbitrary g we get strictly monotonic function

$$f(x) = g(x) \cdot (x + C) = \frac{g(x)}{\varphi(x)}.$$

For $\varphi(x) = \frac{1}{x+C}$ is satisfied the identity $M_f^{\varphi} = M_{f \cdot \varphi}$.

- Specially for $C = 0$, $f = const(= 1)$ we get $\varphi(x) = \dfrac{1}{x}$, $g(x) = \dfrac{1}{x}$. We get quasi-mixture and mixture operator as a harmonic mean:

$$\varphi^{-1}\left(\frac{\sum_{i=1}^{n} \varphi(x_i) \cdot f(x_i)}{\sum_{i=1}^{n} f(x_i)}\right) = \frac{\sum_{i=1}^{n} f(x_i)}{\sum_{i=1}^{n} f(x_i) \cdot \varphi(x_i)} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$$

and

$$\frac{\sum_{i=1}^{n} g(x_i) \cdot x_i}{\sum_{i=1}^{n} g(x_i)} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}.$$

- For $\varphi(x) = \dfrac{1}{x+C}$ we have

$$M^{\varphi} = \varphi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\varphi(x_i)\right) = \frac{n}{\sum\limits_{i=1}^{n}\varphi(x_i)} - C.$$

When $n = 2$ and $C = 0$ we get harmonic mean $M^{\varphi} = \dfrac{2}{\frac{1}{x}+\frac{1}{y}}$. Similarly mixture operator for $n = 2$ is the same harmonic mean $M_{\varphi} = \dfrac{2}{\frac{1}{x}+\frac{1}{y}}$, so $M^{\varphi} = M_{\varphi}$. So quasi-mixture operator and mixture operator are the same for $\varphi(x) = \dfrac{1}{x+C}$.

(3) For $\varphi(x) = Atanh(Bx + C)$ the inverse function is given by $\varphi^{-1}(x) = \dfrac{arctanh\frac{x}{A} - C}{B}$ and weighting function $f(x) = Ecosh(Bx+C)$. If $A = 1$, $E \neq 0$, $B = 1$, $C = 0$ we get quasi-mixture operator as arithmetic mean $M_f^{\varphi} = \dfrac{x+y}{2}$. For $g(x) = const$, $n = 2$ we get mixture operator as arithmetic mean too $M_f^{\varphi} = \dfrac{x+y}{2}$.

(4) In fourth case the function $\varphi(x) = Acotanh(Bx + C)$ has the inverse function $\varphi^{-1}(x) = \dfrac{arccotanh\frac{x}{A} - C}{B}$ and $f(x) = Esinh(Bx + C)$. If $A = 1$, $E \neq 0$, $B = 1$, $C = 0$ we get quasi-mixture operator as arithmetic mean $M_f^{\varphi} = \dfrac{x+y}{2}$ and for $g(x) = const$, $n = 2$ we get mixture operator again as arithmetic mean $M_g = \dfrac{x+y}{2}$.

(5) For the couple $\varphi(x) = Atan(Bx + C)$,$f(x) = Ecos(Bx + C)$ we have $\varphi^{-1}(x) = \dfrac{arctan\frac{x}{A} - C}{B}$. The equality (4) for this pair and for $A = 1$, $E \neq 0$ is given by

$$\frac{arctan\left(\frac{\sum\limits_{i=1}^{n}\varphi(x_i)\cdot f(x_i)}{\sum\limits_{i=1}^{n}f(x_i)}\right) - C}{B} = \frac{\sum\limits_{i=1}^{n}g(x_i)\cdot x_i}{\sum\limits_{i=1}^{n}g(x_i)}.$$

After some processing we get

$$arctan\left(\frac{\sum\limits_{i=1}^{n}\varphi(x_i)\cdot f(x_i)}{\sum\limits_{i=1}^{n}f(x_i)}\right)=\frac{\sum\limits_{i=1}^{n}g(x_i)\cdot(Bx_i+C)}{\sum\limits_{i=1}^{n}g(x_i)}.$$

For $n=2$, $B=1$, $C=0$ we get arithmetic mean $M_f^{\varphi}=\frac{x+y}{2}$ and once more for $g(x)=const$ again we get $M_g=\frac{x+y}{2}$.

(6) For the last couple $\varphi(x)=Aexp(-2Bx)$, $f(x)=Eexp(Bx)$, $\varphi^{-1}(x)=-\frac{1}{2B}ln\frac{x}{A}$ the equation (4) for $A=1$, $E\neq 0$ we rewrite as

$$-\frac{1}{2B}ln\left(\frac{\sum\limits_{i=1}^{n}\varphi(x_i)f(x_i)}{\sum\limits_{i=1}^{n}f(x_i)}\right)=\frac{\sum\limits_{i=1}^{n}g(x_i)\cdot x_i}{\sum\limits_{i=1}^{n}g(x_i)}.$$

For $n=2$ we get the equation

$$\frac{1}{2B}ln\frac{exp(Bx)+exp(By)}{\frac{exp(Bx)+exp(By)}{exp(Bx)\cdot exp(By)}}=\frac{\sum\limits_{i=1}^{2}g(x_i)\cdot x_i}{\sum\limits_{i=1}^{2}g(x_i)}.$$

For $B=const$ we get quasi-mixture operator as the arithmetic mean $M_f^{\varphi}=\frac{x+y}{2}$. And once more if $g(x)=const$, we get mixture operator as the arithmetic mean $M_g=\frac{x+y}{2}$.

Remark 2.1. For the first two couples (φ,f) the equation (3) is satisfied also if we reformulate it for $n\geq 2$, while for the other couples (φ,f) the equation (3) is satisfied only for $n=2$.

Remark 2.2. Note that for all A, D from R, $A\neq 0$, $E>0$, and for all φ, f it holds $M_f^{\varphi}=M_{Ef}^{A\varphi}+D$. Therefore, when solving the problem of equality of different types of quasi-mixture operators, it is enough to assume $A=E=1$ and $D=0$ when ever this is convenient.

3. Equality of special mixture operators and quasi-arithmetic means

Each quasi-mixture operator M_f^φ on an interval I can be identified with the couple (φ, f). The equality of two quasi-mixture operators $M_f^\varphi = M_g^\eta$ alows to introduce an equivalence $(\varphi, f) \approx (\eta, g)$ for the corresponding pairs of generating and weighting functions.

Proposition 3.1. *Let $I \subset R$. Let φ, η: $I \to R$ be continuous strictly monotone functions and f, g: $I \to]0, \infty[$ be weighting functions.*
Let $\tau : J \to I$ be an increasing bijection and $I = \tau(J)$. Then the equivalence $(\varphi, f) \approx (\eta, g)$ (on interval I), holds if and only if the equivalence $(\varphi \circ \tau, f \circ \tau) \approx (\eta \circ \tau, g \circ \tau)$ (on interval J) is true.

Proof. Suppose that $(\varphi, f) \approx (\eta, g)$, i. e., for all $(x_1, \ldots, x_n) \in I^n$ it holds

$$\varphi^{-1}\left(\frac{\sum_{i=1}^{n} \varphi(x_i) \cdot f(x_i)}{\sum_{i=1}^{n} f(x_i)}\right) = \eta^{-1}\left(\frac{\sum_{i=1}^{n} \eta(x_i) \cdot g(x_i)}{\sum_{i=1}^{n} g(x_i)}\right). \tag{5}$$

We have to show the equality

$$\begin{aligned} &(\varphi \circ \tau)^{-1}\left(\frac{\sum_{i=1}^{n} \varphi \circ \tau(u_i) \cdot f \circ \tau(u_i)}{\sum_{i=1}^{n} f \circ \tau(u_i)}\right) = \\ &= (\eta \circ \tau)^{-1}\left(\frac{\sum_{i=1}^{n} \eta \circ \tau(u_i) \cdot g \circ \tau(u_i)}{\sum_{i=1}^{n} g \circ \tau(u_i)}\right) \end{aligned} \tag{6}$$

for all $(u_1, \ldots, u_n) \in J^n$.
Recall that $(\varphi \circ \tau)^{-1} = \tau^{-1} \circ \varphi^{-1}$, and thus the equality (6) can be rewritten into

$$\varphi^{-1}\left(\frac{\sum_{i=1}^{n} \varphi \circ \tau(u_i) \cdot f \circ \tau(u_i)}{\sum_{i=1}^{n} f \circ \tau(u_i)}\right) = \eta^{-1}\left(\frac{\sum_{i=1}^{n} \eta \circ \tau(u_i) \cdot g \circ \tau(u_i)}{\sum_{i=1}^{n} g \circ \tau(u_i)}\right).$$

Now, it is enough to put $\tau(u_i) = x_i$ and apply the equality (6).
The opposite implication is immediate. □

Our aim is to find the solutions of the equivalence problem $(id, g) \approx (\eta, const)$. Recall that in the Section 2 we have summarized the results from [2], [4], [5] solving the equivalence problem $(\varphi, f) \approx (id, const)$. Based on Proposition 3.1.

and putting $\tau = \varphi^{-1}$, we see that we can transform the solutions of $(\varphi, f) \approx (\eta, const)$ into

$$(id, f \circ \varphi^{-1}) \approx (\varphi^{-1}, const). \tag{7}$$

Now, it is enough to put $g = f \circ \varphi^{-1}$ and $\eta = \varphi^{-1}$ to get the desired solutions of the equality of the mixture operators and the quasi-arithmetic means. Now we will analyze all cases 1 - 6 summarized in Section 2.

(1) In this case we have only the trivial solution

$$(id, const) \approx (Aid + D, const)$$

yielding the arithmetic mean M, independently of the interval $I \subset R$ and for the arbitrary $n \in N$.

(2) Due to the Remark 2.2., we can assume $A = E = 1$. Then for φ given by $\varphi(x) = \dfrac{1}{x+C} + D$ (necessarily defined on a subinterval of $]-\infty, C[$ or $]C, \infty[$) we have $\varphi^{-1}(x) = \dfrac{1}{x-D} - C$. Applying the equivalence (6), we can define g by $g(x) = \dfrac{1}{x-D}$, and to ensure the positiveness of g, necessarily it should be defined on a subinterval J of $]D, \infty[$. Moreover, taking into account Remark 2.2., we can put $\eta = g$. Hence the operator $H_D :]D, \infty[^n \to]D, \infty[$ given for any $n \in N$ and any $(x_1, \ldots, x_n) \in]D, \infty[^n$ by

$$H_D(x_1, \ldots, x_n) = \frac{n}{\sum\limits_{i=1}^{n} \dfrac{1}{x_i - D}} + D = \frac{\sum\limits_{i=1}^{n} \dfrac{x_i}{x_i - D}}{\sum\limits_{i=1}^{n} \dfrac{1}{x_i - D}}$$

is both a mixture operator and a quasi-arithmetic mean. Observe that for $D = 0$ we recover the standard harmonic mean H and that

$$H_D(x_1, \ldots, x_n) = H_0(x_1 - D, \ldots, x_n - D) + D.$$

(3) For the third couple (φ, f), similarly as in the previous case, we can assume $A = E = 1$. The function φ is given by $\varphi(x) = tanh(Bx + C) + D$ is defined on R and its inverse function $\varphi^{-1}(x) = \dfrac{arctanh(x-D) - C}{B}$ is defined on the interval $]-1+D, 1+D[$. Recall that the corresponding weighting function f is given by $f(x) = cosh(Bx + C)$. Applying our couple on the equivalence (7) we get the weighting function $g :]-1+D, 1+D[\to]0, \infty[$ given by $g(x) = cosh(arctanh(x - D))$. Denote by $M_D^{(3)}$ the mixture operator M_g which is also a quasi-arithmetic mean M^η,

$\eta = \varphi^{-1}$.This equality is true only for $n = 2$ and $M_D^{(3)}$ is given by

$$M_D^{(3)}(x,y) = \frac{x \cdot cosh(arctanh(x-D) + y \cdot cosh(arctanh(y-D))}{cosh(arctanh(x-D) + cosh(arctanh(y-D))} =$$

$$= tanh\left[\frac{1}{2}\left(arctanh(x-D) + arctanh(y-D)\right)\right] + D$$

is mixture operator and quasi- arithmetic mean. After some proccesing for $D = 0$ we can rewrite our mixture operator and quasi-arithmetic mean by

$$M_0^{(3)}(x,y) = \frac{x+y}{1 + xy + \sqrt{(1-x^2)\cdot(1-y^2)}}.$$

Recall that also in this case $M_D^{(3)}(x,y) = M_0^{(3)}(x-D, y-D) + D$.

(4) The function $\varphi(x) = cotanh(Bx+C)+D$ is defined on the subinterval $]-\infty, -\frac{C}{B}[$ or $]-\frac{C}{B}, \infty[$ and its inverse function $\varphi^{-1}(x) = \dfrac{arccotanhx - D - C}{B}$, necessarily it should be defined on a subinterval J of $]-\infty, B[\cap]-\infty, -1+D[$ or $]B,\infty[\cap]1+D,\infty[$. The corresponding weighting function f is given by $f(x) = sinh(Bx+C)$. To ensure positivness of $f \circ \varphi^{-1}$ in equivalence (6) we get the weighting function $g :]1+D, \infty[\to]0,\infty[$ given by $g(x) = sinh(arccotanh(x-D))$.

Denote by $M_D^{(4)}$ the mixture operator M_g which is also a quasi-arithmetic mean M^η, $\eta = \varphi^{-1}$. This equality is true only for $n = 2$ and $M_D^{(4)}$ is given by

$$M_D^{(4)}(x,y) = \frac{x \cdot sinh(arccotanh(x-D) + y \cdot sinh(arccotanh(y-D))}{sinh(arccotanh(x-D) + sinh(arccotanh(y-D))} =$$

$$= cotanh\left[\frac{1}{2}\left(arccotanh(x-D) + arccotanh(y-D)\right)\right] + D,$$

which is mixture operator and quasi- arithmetic mean. After some proccesing for $D = 0$ we can rewrite our mixture operator and quasi-arithmetic mean by

$$M_0^{(4)}(x,y) = \frac{x+y}{1 + xy - \sqrt{(x^2-1)\cdot(y^2-1)}}.$$

Recall that also in this case $M_D^{(4)}(x,y) = M_0^{(4)}(x-D, y-D) + D$.

(5) The function $\varphi(x) = tan(Bx+C) + D$ is defined on a subinterval of $]\dfrac{\frac{-\pi}{2} - C}{B}, \dfrac{\frac{\pi}{2} - C}{B}[$ and its inverse function $\varphi^{-1}(x) = \dfrac{arctan(x-D) - C}{B}$ necessarily it should be defined on a subinterval J of $]-\infty, B[$ or $]B, \infty[$.

Note, that weighting function g from this couple is given by $g(x) = cos\,(arctan(x-D)) = \dfrac{1}{\sqrt{1+(x-D)^2}}$ and is positive for $x \in R$. Denote by $M_D^{(5)}$ the mixture operator M_g which is also a quasi-arithmetic mean M^η, $\eta = \varphi^{-1}$. This equality is true only for $n = 2$ and $M_D^{(5)}$ is given by

$$M_D^{(5)}(x,y) = \frac{x \cdot cos(arctan(x-D) + y \cdot cos(arctan(y-D))}{cos(arctan(x-D) + cos(arctan(y-D))} =$$

$$= tan\left[\frac{1}{2}\left(arctan(x-D) + arctan(y-D)\right)\right] + D$$

is both a mixture operator and a quasi-arithmetic mean. For $D = 0$ and using some proccesing we get

$$M_0^{(5)}(x,y) = \frac{x+y}{1 - xy + \sqrt{(1+x^2)\cdot(1+y^2)}}.$$

Recall that also in this case $M_D^{(5)}(x,y) = M_0^{(5)}(x-D, y-D) + D$.

(6) For the sixth couple (φ, f) we have $\varphi(x) = exp(-2Bx) + D$ defined on the interval R. Due inverse function $\varphi^{-1}(x) = -\dfrac{1}{2B}ln(x-D)$ is defined on a subinterval J of $]-\infty, B[\,\cap\,]D,\infty[$ or $]B,\infty[\,\cap\,]D,\infty[$. Weighting function $f \circ \varphi^{-1}$ from the equivalence (6) should be positive, so necessarily is defined on the subinterval J of $]D,\infty[$ and we get the weighting function $g :]D,\infty[\to]0,\infty[$ given by $g(x) = \frac{1}{x-D}$. Denote by G_D the mixture operator M_g which is also a quasi-arithmetic mean M^η, $\eta = \varphi^{-1}$. This equality is true only for $n = 2$ and G_D is given by

$$G_D(x,y) = \frac{\dfrac{x}{x-D} + \dfrac{y}{y-D}}{\dfrac{1}{x-D} + \dfrac{1}{y-D}} = \sqrt{(x-D)(y-D)} + D.$$

For $D = 0$ we recover the standard geometric mean G and that

$$G_D(x,y) = G_0(x-D, y-D) + D.$$

4. Concluding remarks

We have discussed special equality of quasi-mixture, mixture operators and quasi-arithmetic means with special stress on their identity. By solving our equation (6) for different pairs (φ, f) we can conclude, that the intersection of special mixture operators and quasi-arithmetic means includes the arithmetic mean, the harmonic mean, the geometric mean and special type of mixture operators.

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Department of Quantitative Methods and Informatics, Faculty of Economics, Matej Bel University, Tajovského 10, 975 90 Banská Bystrica, Slovakia

E-mail address: jana.spirkova@umb.sk