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DIRECT DECOMPOSITIONS OF BASIC ALGEBRAS AND THEIR IDEMPOTENT MODIFICATIONS

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Dedicated to the 70th birthday of Alfonz Haviar

Abstract. We get a necessary and sufficient condition under which a given basic algebra $\mathcal{A}$ is isomorphic to a direct product of non-trivial basic algebras $\mathcal{A}_1, \mathcal{A}_2$ which are in fact interval subalgebras of $\mathcal{A}$. Further, we prove that the idempotent modification of $\mathcal{A}$ is directly indecomposable whenever $\mathcal{A}$ has at least one element which is not boolean.

1. Introduction

It is well-known that a bounded lattice $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ is directly decomposable into lattices $\mathcal{L}_1, \mathcal{L}_2$ isomorphic to the intervals $[a, 1], [b, 1]$ of $\mathcal{L}$ if b is a complement of a and a, b are standard elements. Since every basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ induces a lattice $\mathcal{L}(A) = (A; \vee, \wedge)$ which is bounded by 0 and $1 = \neg 0$, we can ask if also $\mathcal{A}$ is directly decomposable whenever there exists a complemented and standard element of $\mathcal{L}(A)$. In what follows we show that the condition concerning this element must be enlarged due to the fact that the operations $\oplus$ and $\neg$ cannot be derived by means of the lattice operations of $\mathcal{L}(A)$. However, we set up a natural necessary and sufficient condition for the direct decomposability of $\mathcal{A}$.

By a **basic algebra** (see e.g. [1, 2]) is meant an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following four axioms

(BA1) $x \oplus 0 = x$;
(BA2) $\neg\neg x = x$;
(BA3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$;
(BA4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$.

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As usual, we will write 1 instead of $\neg 0$. We say that a basic algebra $\mathcal{A}$ is **non-trivial** if $0 \neq 1$ (i.e. $|A| > 1$).

Having a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$, one can introduce the **induced order** $\leq$ on $\mathcal{A}$ as follows

$$x \leq y \qquad \text{if and only if} \qquad \neg x \oplus y = 1.$$

It is an easy exercise to verify that $\leq$ is really an order on A and $0 \leq x \leq 1$ for each $x \in A$. Moreover, $(A; \leq)$ is a bounded lattice in which

$$x \vee y = \neg(\neg x \oplus y) \oplus y \qquad \text{and} \qquad x \wedge y = \neg(\neg x \vee \neg y).$$

For some details, the reader is referred to [1]. The lattice $\mathcal{L}(A) = (A; \vee, \wedge)$ will be called the **induced lattice** of $\mathcal{A}$. In particular for each $a \in A$ there exists an antitone involution $x \mapsto x^a$ on the interval $[a, 1]$ (called a **section**) where $x^a = \neg x \oplus a$.

It is well-known (see e.g. [1, 3]) that also conversely, if $(A; \vee, \wedge, ({}^a)_{a \in A}, 0, 1)$ is a bounded lattice with section antitone involutions, we are able to construct a basic algebra using the operations

$$x \oplus y = (x^0 \vee y)^y \qquad \text{and} \qquad \neg x = x^0. \tag{1}$$

Lemma 1. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra, $\leq$ the induced order and $a \in A$. Define the polynomial operations $\neg_a$ and $\oplus_a$ on the interval $[a, 1]$ as follows*

$$\neg_a x = \neg x \oplus a \qquad \text{and} \qquad x \oplus_a y = \neg(\neg x \oplus a) \oplus y.$$

Then $([a, 1]; \oplus_a, \neg_a, a)$ is a basic algebra.

Proof. We use the facts that $y \leq x \oplus y$, $0 \oplus x = x$ and $\neg x \oplus x = 1$ hold in each basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ (for more details see e.g. [1]). If $x, y \in [a, 1]$ then $a \leq y \leq \neg(\neg x \oplus a) \oplus y = x \oplus_a y$ thus $\oplus_a$ is really a binary operation on $[a, 1]$. Since $a \leq \neg x \oplus a$, $\neg_a x$ is a unary operation on $[a, 1]$. Moreover, $\neg_a a = \neg a \oplus a = 1$ and $\neg_a 1 = \neg 1 \oplus a = 0 \oplus a = a$. We must check the axioms (BA1)–(BA4).
(BA1) and (BA2): For $x \in [a, 1]$ we have $x \oplus_a a = \neg(\neg x \oplus a) \oplus a = x \vee a = x$ and, analogously, $\neg_a \neg_a x = \neg(\neg x \oplus a) \oplus a = x \vee a = x$.
(BA3): Assume that $x, y \in [a, 1]$. Since $y \leq \neg x \oplus y$, then also $a \leq \neg x \oplus y$. Further, we have $\neg_a x \oplus_a y = \neg x \oplus y$. Hence, we compute

$$\neg_a(\neg_a x \oplus_a y) \oplus_a y = \neg(\neg x \oplus y) \oplus y = x \vee y$$

and, by symmetry, also

$$\neg_a(\neg_a y \oplus_a x) \oplus_a x = y \vee x = x \vee y.$$

(BA4): Let $x, y, z \in [a, 1]$. Since $a \leq x \oplus_a y$, $a \leq y \leq \neg(x \oplus_a y) \oplus y$ and $a \leq z \leq \neg(\neg(x \oplus_a y) \oplus y) \oplus z$, we obtain

$$\begin{aligned}\neg_a(\neg_a(\neg_a(x \oplus_a y) \oplus_a y) \oplus_a z) \oplus_a (x \oplus_a z) = \\ &= \neg_a(\neg(\neg(x \oplus_a y) \oplus y) \oplus z) \oplus_a (x \oplus_a z) = \\ &= \neg(\neg(\neg(x \oplus_a y) \oplus y) \oplus z) \oplus (x \oplus_a z) = \\ &= \neg(\neg(\neg(w \oplus y) \oplus y) \oplus z) \oplus (w \oplus z) = 1,\end{aligned}$$

where $w = \neg(\neg x \oplus a)$. □

The basic algebra $([a, 1]; \oplus_a, \neg_a, a)$ where the operations $\oplus_a$, $\neg_a$ are defined as in Lemma 1 will be called an **interval basic algebra**. Our motivation for introducing the operations $\oplus_a$ and $\neg_a$ in this way is inspired by (1), where we only replace x^0 by x^a due to the fact that a is the bottom element of the section $[a, 1]$. Since $x \in [a, 1]$, by (1) we have

$$\neg_a x = x^a = (x \vee a)^a = \neg x \oplus a$$

and then for $x, y \in [a, 1]$

$$x \oplus_a y = (\neg_a x \vee y)^y = ((\neg x \oplus a) \vee y)^y = \neg(\neg x \oplus a) \oplus y.$$

Due to the fact that every basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is in a one-to-one correspondence with the enriched lattice $\mathcal{L}(A) = (A; \vee, \wedge, (^a)_{a \in A}, 0, 1)$ as mentioned above (see also [1]), the interval basic algebra $([a, 1]; \oplus_a, \neg_a, a)$ is in the same correspondence with the interval enriched lattice $([a, 1]; \vee, \wedge, (^b)_{b \in [a,1]}, a, 1)$ where $\vee, \wedge, ^b$ are the same as in $\mathcal{L}(A)$. Hence, our interval basic algebra is quite a natural "cut" of the original one.

Lemma 2. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra and $a, b, c \in A$. Then*

$$(b \wedge c) \oplus a = (b \oplus a) \wedge (c \oplus a).$$

Proof. We compute by (1)

$$\begin{aligned}(b \wedge c) \oplus a = (\neg(b \wedge c) \vee a)^a = ((\neg b \vee \neg c) \vee a)^a = ((\neg b \vee a) \vee (\neg c \vee a))^a = \\ = (\neg b \vee a)^a \wedge (\neg c \vee a)^a = (b \oplus a) \wedge (c \oplus a)\end{aligned}$$

since $\neg b \vee a \in [a, 1]$ and $\neg c \vee a \in [a, 1]$. □

2. Direct decomposibility of basic algebras

Now, we will set up the conditions under which a basic algebra $\mathcal{A}$ can be directly decomposed. First, we define several concepts.

Definition 1. *An element a of a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is called **strong** if*

(a) $x \oplus a = x \vee a$ *and* $x \oplus \neg a = x \vee \neg a$
for every $x \in A$.

A strong element a of $\mathcal{A}$ is called a ***decomposing element*** *if it moreover satisfies*

(b) $(x \oplus y) \oplus a = x \oplus (y \oplus a)$, $(x \oplus y) \oplus \neg a = x \oplus (y \oplus \neg a)$
and $x \oplus a = a \oplus x$, $x \oplus \neg a = \neg a \oplus x$
for all $x, y \in A$.

Let us note that 0 and 1 are decomposing elements for every basic algebra $\mathcal{A}$.

Recall (see [4]) that the element a of a lattice $(L; \vee, \wedge)$ is called **distributive** if for all $x, y \in L$

$$(x \wedge y) \vee a = (x \vee a) \wedge (y \vee a)$$

and the element a of a lattice $(L; \vee, \wedge)$ is called **standard** if for all $x, y \in L$

$$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y).$$

Further, recall that if $(L; \vee, \wedge)$ is a lattice and $a \in L$ then the following two conditions are equivalent:

(α) a is standard

(β) a is distributive and, for $x, y \in L$,

$$a \wedge x = a \wedge y \quad \text{and} \quad a \vee x = a \vee y \qquad \text{imply that} \qquad x = y$$

(for more details see [4]).

Lemma 3. *Let a be a strong element of a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$. Then*

(i) *a is boolean (i.e. $a \vee \neg a = 1$, $a \wedge \neg a = 0$);*

(ii) *a and $\neg a$ are distributive elements.*

Proof. (i) By Definition 1 we have $a \oplus a = a \vee a = a$ and $\neg a \oplus \neg a = \neg a \vee \neg a = \neg a$ thus both a and $\neg a$ are $\oplus$-idempotents. Then $\neg a \vee a = \neg a \oplus a = 1$ and dually (by De Morgan law) also $\neg a \wedge a = 0$.
(ii) Follows directly by Lemma 2 and the condition (a) of Definition 1. ☐

Lemma 4. *Let a be a strong element of a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ and $\neg a$ be a standard element of the induced lattice $\mathcal{L}(A) = (A; \vee, \wedge)$. Then the mapping $\varphi_a(x) = (x \vee a, x \vee \neg a)$ is a lattice isomorphism of $\mathcal{L}(A)$ onto the direct product of lattices $([a, 1]; \vee, \wedge) \times ([\neg a, 1]; \vee, \wedge)$.*

Proof. The proof is only a slight modification of the proof of Theorem 1.4. (p. 200) in [4]. Namely, if $\varphi_a(x) = \varphi_a(y)$ then $x \vee a = y \vee a$ and $x \vee \neg a = y \vee \neg a$, i.e.

$$\neg x \wedge \neg a = \neg y \wedge \neg a \tag{2}$$

by the first equality and $\neg x \wedge a = \neg y \wedge a$ by the second one, i.e. also

$$(\neg x \wedge a) \vee \neg a = (\neg y \wedge a) \vee \neg a$$

thus, by Lemma 3, also

$$\neg x \vee \neg a = \neg y \vee \neg a. \tag{3}$$

Since $\neg a$ is a standard element of $\mathcal{L}(A) = (A; \vee, \wedge)$, (2) and (3) yields $\neg x = \neg y$, i.e. $x = \neg\neg x = \neg\neg y = y$. Hence, φ_a is injective. If $\langle c, d \rangle \in [a, 1] \times [\neg a, 1]$ then $a \leq c$, $\neg a \leq d$, i.e. $d \vee a \geq \neg a \vee a = 1$, $c \vee \neg a \geq a \vee \neg a = 1$ and for $c \wedge d \in A$ we have

$$\begin{aligned}\varphi_a(c \wedge d) &= ((c \wedge d) \vee a, (c \wedge d) \vee \neg a) = \\ &= ((c \vee a) \wedge (d \vee a), (c \vee \neg a) \wedge (d \vee \neg a)) = \\ &= ((c \vee a) \wedge 1, 1 \wedge (d \vee \neg a)) = (c, d)\end{aligned}$$

hence φ_a is also surjective. Therefore it is a bijection from A to $[a, 1] \times [\neg a, 1]$. Further,

$$\begin{aligned}\varphi_a(x \vee y) &= ((x \vee y) \vee a, (x \vee y) \vee \neg a) = \\ &= (x \vee a, x \vee \neg a) \vee (y \vee a, y \vee \neg a) = \varphi_a(x) \vee \varphi_a(y)\end{aligned}$$

and

$$\begin{aligned}\varphi_a(x \wedge y) &= ((x \wedge y) \vee a, (x \wedge y) \vee \neg a) = \\ &= ((x \vee a) \wedge (y \vee a), (x \vee \neg a) \wedge (y \vee \neg a)) = \varphi_a(x) \wedge \varphi_a(y)\end{aligned}$$

thus φ_a is a lattice isomorphism of $\mathcal{L}(A)$ onto $([a, 1]; \vee, \wedge) \times ([\neg a, 1]; \vee, \wedge)$. □

Theorem 1. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra. Then $\mathcal{A}$ is isomorphic to a direct product of non-trivial basic algebras $\mathcal{B}_1, \mathcal{B}_2$ if and only if there exists a decomposing element $a \in A$, $0 \neq a \neq 1$ such that $\neg a$ is standard in the induced lattice $\mathcal{L}(A) = (A; \vee, \wedge)$. If it is the case then $\mathcal{A}$ is isomorphic to the direct product of interval basic algebras $([a, 1]; \oplus_a, \neg_a, a)$ and $([\neg a, 1]; \oplus_{\neg a}, \neg_{\neg a}, \neg a)$.*

Proof. Due to Lemma 4, we must only show that φ_a preserves the operations $\oplus$ and $\neg$. Denote by $\widehat{\oplus}$ and $\widehat{\neg}$ the corresponding operations on the direct product of the interval algebras. Then, since a and $\neg a$ are strong elements, we have

$$\varphi_a(\neg x) = (\neg x \vee a, \neg x \vee \neg a) = (\neg x \oplus a, \neg x \oplus \neg a) = (\neg_a x, \neg_{\neg a} x) = \widehat{\neg}\varphi_a(x).$$

Since a is a decomposing element, we derive also

$$\varphi_a(x \oplus y) = ((x \oplus y) \oplus a, (x \oplus y) \oplus \neg a)$$

and, by Lemma 2, we compute

$$\begin{aligned}
\varphi_a(x)\widehat{\oplus}\varphi_a(y) &= (x \vee a, x \vee \neg a)\widehat{\oplus}(y \vee a, y \vee \neg a) = \\
&= (x \oplus a, x \oplus \neg a)\widehat{\oplus}(y \oplus a, y \oplus \neg a) = \\
&= ((x \oplus a) \oplus_a (y \oplus a), (x \oplus \neg a) \oplus_{\neg a} (y \oplus \neg a)) = \\
&= (\neg(\neg(x \oplus a) \oplus a) \oplus (y \oplus a), \neg(\neg(x \oplus \neg a) \oplus \neg a) \oplus (y \oplus \neg a)) = \\
&= (\neg(\neg x \vee a) \oplus (y \oplus a), \neg(\neg x \vee \neg a) \oplus (y \oplus \neg a)) = \\
&= ((x \wedge \neg a) \oplus (y \oplus a), (x \wedge a) \oplus (y \oplus \neg a)) = \\
&= ((x \oplus (y \oplus a)) \wedge (\neg a \oplus (y \oplus a)), (x \oplus (y \oplus \neg a)) \wedge (a \oplus (y \oplus \neg a))) = \\
&= ((x \oplus (y \oplus a)) \wedge (\neg a \vee y \vee a), (x \oplus (y \oplus \neg a)) \wedge (a \vee y \vee \neg a)) = \\
&= ((x \oplus (y \oplus a)) \wedge 1, (x \oplus (y \oplus \neg a)) \wedge 1) = \\
&= (x \oplus (y \oplus a), x \oplus (y \oplus \neg a)).
\end{aligned}$$

Due to (b) of Definition 1, we conclude

$$\varphi_a(x \oplus y) = \varphi_a(x)\widehat{\oplus}\varphi_a(y)$$

thus φ_a preserves $\oplus$ and $\neg$ and hence it is an isomorphism of $\mathcal{A}$ onto the direct product $([a, 1]; \oplus_a, \neg_a, a) \times ([\neg a, 1], \oplus_{\neg a}, \neg_{\neg a}, \neg a)$.

Conversely, assume that a basic algebra $\mathcal{A}$ is isomorphic to a direct product $\mathcal{B}_1 \times \mathcal{B}_2$ of non-trivial basic algebras $\mathcal{B}_1 = (B_1; \oplus_1, \neg_1, 0_1)$ and $\mathcal{B}_2 = (B_2; \oplus_2, \neg_2, 0_2)$. It is an easy exercise to show that $(0_1, 1_2)$ (where $1_2 = \neg_2 0_2$) is a decomposing element of $\mathcal{B}_1 \times \mathcal{B}_2$ and hence $h^{-1}((0_1, 1_2))$ is a decomposing element of $\mathcal{A}$ (where h is the isomorphism of $\mathcal{A}$ onto $\mathcal{B}_1 \times \mathcal{B}_2$). □

Example 1. Consider the lattice drawn in Fig. 1.

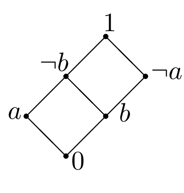

Fig. 1

We can define the operations $\oplus^1$ and $\oplus^2$ as follows

$\oplus^1$	0	a	b	$\neg b$	$\neg a$	1
0	0	a	b	$\neg b$	$\neg a$	1
a	a	a	$\neg a$	$\neg b$	1	1
b	b	$\neg b$	$\neg b$	1	$\neg a$	1
$\neg b$	$\neg b$	$\neg b$	1	1	1	1
$\neg a$	$\neg a$	1	$\neg b$	1	$\neg a$	1
1	1	1	1	1	1	1

$\oplus^2$	0	a	b	$\neg b$	$\neg a$	1
0	0	a	b	$\neg b$	$\neg a$	1
a	a	a	$\neg b$	$\neg b$	1	1
b	b	$\neg b$	$\neg a$	1	$\neg a$	1
$\neg b$	$\neg b$	$\neg b$	1	1	1	1
$\neg a$	$\neg a$	1	$\neg a$	1	$\neg a$	1
1	1	1	1	1	1	1

Then for $A = \{0, a, b, \neg b, \neg a, 1\}$ we have that $\mathcal{A}_1 = (A; \oplus^1, \neg, 0)$ and $\mathcal{A}_2 = (A; \oplus^2, \neg, 0)$ are basic algebras (where $\mathcal{A}_2$ is even an MV-algebra but $\mathcal{A}_1$ is not). In the both cases a is a strong element, but in $\mathcal{A}_1$ a is not a decomposing element, since for $x = b$ we have

$$a \oplus b = \neg a \neq \neg b = b \oplus a$$

which contradicts (b) of Definition 1. On the other hand, one can check by a direct computation that a is a decomposing element of $\mathcal{A}_2$. ♢

3. Idempotent modification of basic algebras

The concept of idempotent modification of an algebra was introduced by J. Ježek [6] as follows.

Definition 2. *An* ***idempotent modification*** *of an algebra $\mathcal{A} = (A; F)$ is an algebra $\mathcal{A}_I = (A; F_I)$ with the same underlying set A, where $|F| = |F_I|$ and for every $f \in F$ the corresponding operation $f_I \in F_I$ is defined as follows*

(i) *if f is at most unary then $f_I = f$;*
(ii) *if f is n-ary with $n > 1$ and $a_1, \dots, a_n \in A$ then*

$$f_I(a_1, \dots, a_n) = \begin{cases} a_1 & \text{if} \quad a_1 = a_2 = \cdots = a_n \\ f(a_1, \dots, a_n) & \text{otherwise.} \end{cases}$$

The main result of [6] is that for any group G its idempotent modification G_I is subdirectly irreducible.

In what follows we will treat direct decomposability of an idempotent modification of a basic algebra.

For this we slightly modify our definition of basic algebra. As mentioned above, every basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ has induced lattice $\mathcal{L}(A) = (A; \vee, \wedge)$ where $\vee$ and $\wedge$ are term operations of $\mathcal{A}$. Hence, inserting $\vee$ and $\wedge$ into the type of $\mathcal{A}$, we obtain an algebra with the same clone of term operations and hence term equivalent to $\mathcal{A}$. From now on, by a basic algebra we will understand an algebra $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ where the term operations $\vee$ and $\wedge$ are defined by $x \vee y = \neg(\neg x \oplus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y)$.

The reason of this insertion is that when an idempotent modification of $(A; \oplus, \neg, 0)$ is considered, the resulting algebra does not have the lattice structure. However, if $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ is treated then the lattice structure for $\mathcal{A}_I$ is preserved because both $\vee$ and $\wedge$ are idempotent operations on A.

Theorem 2. *Let $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ be a basic algebra whose at least one element is not boolean. Then its idempotent modification $\mathcal{A}_I = (A; \oplus_I, \neg, 0, \vee, \wedge)$ is not directly decomposable.*

Proof. At first we show that if $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ is a basic algebra and its idempotent modification $\mathcal{A}_I$ is directly decomposable into non-trivial algebras $\mathcal{B}_1 = (B_1; \oplus_1, \neg_1, 0_1, \vee, \wedge)$ and $\mathcal{B}_2 = (B_2; \oplus_2, \neg_2, 0_2, \vee, \wedge)$ then also $\mathcal{A}$ is directly decomposable. Denote by $1_1 = \neg_1 0_1$ and $1_2 = \neg_2 0_2$. Let φ be an isomorphism of $\mathcal{A}_I$ onto $\mathcal{B}_1 \times \mathcal{B}_2$. For $x \in A$ let $\varphi(x) = (x_1, x_2)$. Define new operations $\oplus^1, \oplus^2$ on B_1, B_2, respectively as follows. If $y_1, z_1 \in B_1$ and $y_1 \neq z_1$ then $y_1 \oplus^1 z_1 = y_1 \oplus_1 z_1$, if $y_2, z_2 \in B_2$ and $y_2 \neq z_2$ then $y_2 \oplus^2 z_2 = y_2 \oplus_2 z_2$. If $x_1 \in B_1$, denote by $\overline{x_1} = \varphi^{-1}((x_1, 1_2))$ and if $x_2 \in B_2$, denote by $\overline{x_2} = \varphi^{-1}((1_1, x_2))$. Now we define

$$x_1 \oplus^1 x_1 = \mathrm{pr}_1(\varphi(\overline{x_1} \oplus \overline{x_1}))$$

and

$$x_2 \oplus^2 x_2 = \mathrm{pr}_2(\varphi(\overline{x_2} \oplus \overline{x_2})).$$

Since $\varphi(x) = (x_1, x_2) = (x_1, 1_2) \wedge (1_1, x_2) = \varphi(\overline{x_1}) \wedge \varphi(\overline{x_2})$ and since φ and also φ^{-1} preserve the lattice operations, we have $x = \overline{x_1} \wedge \overline{x_2}$. This yields that $\oplus^1, \oplus^2$ are correctly defined (i.e. the result $x_1 \oplus^1 x_1$ in the first coordinate does not depend on the second coordinate and vice versa), i.e.

$$\varphi(x \oplus x) = (x_1, x_2) \overline{\oplus} (x_1, x_2) = (x_1 \oplus^1 x_1, x_2 \oplus^2 x_2),$$

where $\overline{\oplus}$ is the binary operation provided coordinatewise on the Cartesian product $B_1 \times B_2$.

It is obvious that $\mathcal{A}^1 = (B_1; \oplus^1, \neg_1, 0_1, \vee, \wedge)$ and $\mathcal{A}^2 = (B_2; \oplus^2, \neg_2, 0_2, \vee, \wedge)$ are basic algebras and φ is also an isomorphism of $\mathcal{A}$ onto $\mathcal{A}_1 \times \mathcal{A}_2$. Moreover, we see that $\mathcal{B}_1 = \mathcal{A}^1_I$ and $\mathcal{B}_2 = \mathcal{A}^2_I$. Hence, if $\mathcal{A}_I$ is directly decomposable then also $\mathcal{A}$ has this property.

Assume now that $x \in A$ is not boolean and that $\mathcal{A}_I$ is directly decomposable. We can apply the reasoning used by J. Jakubík [5]. Let $\varphi(x) = (x_1, x_2)$. Then also $\varphi(x)$ is not boolean, i.e. at least one of x_1, x_2 is not boolean. Without loss of generality, suppose that x_1 is not boolean. Then $x_1 \oplus^1 x_1 \neq x_1$. Since $|B_2| > 1$, there exists $y_2 \in B_2$ such that $x_2 \neq y_2$. Let $y = \varphi^{-1}(x_1, y_2)$. Then $x \neq y$ and $x \oplus y = x \oplus_I y$, hence $\varphi(x \oplus y) = \varphi(x \oplus_I y)$. However, $\varphi(x \oplus y) = (x_1 \oplus^1 x_1, x_2 \oplus^2 y_2)$ and $\varphi(x \oplus_I y) = (x_1 \oplus_1 x_1, x_2 \oplus_2 y_2) = (x_1, x_2 \oplus^2 y_2)$, which is a contradiction. Thus $\mathcal{A}_I$ is not directly decomposable. □

Call a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ **distributive** if the induced lattice $\mathcal{L}(A) = (A; \vee, \wedge)$ is distributive. For example, if $\mathcal{A}$ is commutative then $\mathcal{A}$ is distributive (but not vice versa, see Example 1) see e.g. [1]. For distributive basic algebras, we can modify our result as follows

Corollary. *Let* $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ *be a distributive basic algebra with* $|A| > 2$*. Then its idempotent modification is directly indecomposable if and only if* $\mathcal{A}$ *contains an element which is not boolean.*

Proof. If all elements of $\mathcal{A}$ are boolean then, due to the fact that $\mathcal{L}(A)$ is distributive, its idempotent modification $\mathcal{A}_I$ is in fact a Boolean algebra (where $\oplus$ coincides with $\vee$). Hence, $\mathcal{A}_I$ is directly decomposable since $|A| > 2$.

Conversely, if $\mathcal{A}$ contains an element which is not boolean then $\mathcal{A}_I$ is not directly decomposable by Theorem 2. □

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