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FACIAL NON-REPETITIVE EDGE COLOURING OF SEMIREGULAR POLYHEDRA

STANISLAV JENDROĽ AND ERIKA ŠKRABUĽÁKOVÁ

Dedicated to the 70th birthday of Alfonz Haviar

Abstract. A sequence $r_1, r_2, \ldots, r_{2n}$ such that $r_i = r_{n+i}$ for all $1 \leq i \leq n$, is called a *repetition*. A sequence S is called *non-repetitive* if no subsequence of consecutive terms of S is a repetition. Let G be a graph whose edges are coloured. A trail in G is called *non-repetitive* if the sequence of colours of its edges is non-repetitive. If G is a plane graph, a *facial non-repetitive edge-colouring* of G is an edge-colouring such that any *facial trail* is non-repetitive. We denote $\pi'_f(G)$ the minimum number of colours needed. In this paper we prove that for graphs of Platonic, Archimedean and prismatic polyhedra $\pi'_f(G)$ is either 3 or 4.

1. Introduction

A *polyhedron* P in the three-dimensional Euclidean space is a finite collection of planar convex polygons, called the *faces*, such that every edge of every polygon is an edge of precisely one other polygon. The edge set of a polyhedron is the set of intersections of adjacent faces, and the vertex set is the set of intersections of adjacent edges. A polyhedron P is called *semiregular* if all of its faces are regular polygons and there exists a sequence $\sigma = (p_1, p_2, \ldots, p_q)$ called the *cyclic sequence* of P, such that every vertex of P is surrounded by a p_1-gon, a p_2-gon, …, a p_q-gon, in this order within rotation and reflexion. A semiregular polyhedron P is called the $(p_1, p_2, \ldots, p_q)$-polyhedron if it is determined by the cyclic sequence $(p_1, p_2, \ldots, p_q) = \sigma$ (see [7], [10]). The five polyhedra with equal regular faces that can be inscribed in a sphere (the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron) are known as *Platonic solids*. Thirteen polyhedra, which were discovered by Archimedes and are contained by equilateral and equiangular but not similar polygons are known as *Archimedean solids* (see [3]). The pseudo-Archimedean solid that has congruent solid angles but they are not all equivalent, satisfy the above conditions too and is known as a *Miller*

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solid, a *Ashkinuze polyhedron*, a *pseudo rhomb-cub-octahedron* or a $(3,4,4,4)$-polyhedron. The family of semiregular polyhedra completes a set of *prismatic polyhedra* consists of two infinite families: the *prisms* i.e. $(4,4,n)$-polyhedra for every $n \geq 3$, $n \neq 4$, and the *antiprisms* i.e. $(3,3,3,n)$-polyhedra for every $n \geq 4$ (see [3]).

The study of the semiregular polyhedra began with the abstraction of regularity in Euclide's Book XIII of *Elements*. Since those times they continually treat a lot of attention. Thanks to Steinitz theorem [5] that asserts that the graph is a graph of a convex polyhedron if and only if it is planar and 3-connected, instead of a study of combinatorial properties of convex polyhedra it is enough to study their graphs. Hence we use the same name for a polyhedron and its graph. The family of graphs of semiregular polyhedra is very inspirating and many questions that deal with their graphs were asked.

Maehara asked for the the smallest integer n such that the graph of a semiregular polyhedra can be represented as the intersection graph of a family of unit-diameter spheres in Euclidean n-dimensional space. Such n is called the *sphericity* of the graph and in [12] is determined for graphs of semiregular polyhedra except for a few prisms. The generalized Archimedean solids were studied by Karabáš and Nedela (see [8], [9]). They gave a complete census of Archimedean solids of genera from two to five. But study of properties of semiregular polyhedra does not occur only in mathematics; man can find it also in chemistry (see [11]); architecture, art, cartography (see [3]); ... and so on. A lot of posed questions relate to colouring of semiregular polyhedra and determining some colouring characteristic of it: The rainbowness of semiregular polyhedra, the parameter $rb(P)$, had been studied by Jendrol' and Schrötter in [7]. They found the exact value of $rb(P)$ for all graphs of semiregular polyhedra except of three Archimedean solids for which the parameter is only estimated. A. Kemnitz and P. Wellmann in [10] determined the circular chromatic number $\chi_c(G)$ for Platonic solid graphs, Archimedean solid graphs and regular convex prism graphs. In this paper we determine a variant of non-repetitive edge-colouring for plane graphs of semiregular polyhedra introduced in [6].

A sequence $r_1, r_2, \ldots, r_{2n}$ such that $r_i = r_{n+i}$ for all $1 \leq i \leq n$, is called a *repetition*. A sequence S is called *non-repetitive* if no subsequence of consecutive terms of S is a repetition. Thue [13] states that arbitrarily long non-repetitive sequences can be formed using only three symbols.

An *edge k-colouring* of G is a mapping $\varphi : E(G) \to \{1, 2, \ldots, k\}$. Alon et al. [1] introduced a natural generalization of Thue's sequences for edge-colouring of graphs. An edge-colouring φ of a graph G is *non-repetitive* if the sequence of colours on any path in G is non-repetitive. The minimum numbers of colours $\pi'(G)$ needed in any non-repetitive colouring of G is called the *Thue chromatic index* of G.

For a face f, the *size* (or degree) of f is defined to be the length of the shortest closed facial walk containing all edges from the boundary of f. The face of degree r is known as an r-gonal face.

Let G be a plane graph. A *facial trail* in G is a trail made of consecutive edges of the boundary walk of some face. A *facial non-repetitive edge colouring* of G is an edge colouring of G such that any facial trail is non-repetitive. The *facial Thue chromatic index* of G, denoted $\pi'_f(G)$, is the minimum number of colours of a facial non-repetitive edge colouring of G. Note that the facial Thue chromatic index depends on the embedding of the graph. In the following, all the graphs we will consider come along with an embedding in the plane.

We show the exact value of Thue chromatic index for graphs of all semiregular polyhedra, that is the first step towards the Conjecture 18 setted in [6].

The notation and terminology used but not defined in this paper can be found in [2].

2. Basic preliminaries

Thue's sequences (see [13]) show that the Thue chromatic index of a path is at most 3. Actually, $\pi'(P_n) = 3$, for all $n \geq 5$, as it is easy to see that every sequence of length 4 on two symbols contains a repetition. An immediate corollary is that the Thue chromatic index of a cycle is at most 4. In [4], Currie showed that $\pi(C_n) = 4$ only for $n \in \{5, 7, 9, 10, 14, 17\}$. For other values of $n \geq 3$, $\pi(C_n) = 3$.

From the above remarks it is easy to see that for our less constrained parameter $\pi'_f(G)$ the following holds (see [6]):

Theorem 1. *Let G be a cycle C_n.*

(i) if $n = 2$, then $\pi'_f(G) = 2$;
(ii) if $n \notin \{2, 5, 7, 9, 10, 14, 17\}$, then $\pi'_f(G) = 3$ and
(iii) if $n \in \{5, 7, 9, 10, 14, 17\}$, then $\pi'_f(G) = 4$.

Corollary 2. *Let G be a plane graph and let a facial trail of one of its faces be isomorphic to C_n.*

(i) If $n = 2$, then $\pi'_f(G) \geq 2$;
(ii) if $n \notin \{2, 5, 7, 9, 10, 14, 17\}$, then $\pi'_f(G) \geq 3$ and
(iii) if $n \in \{5, 7, 9, 10, 14, 17\}$, then $\pi'_f(G) \geq 4$.

3. Prismatic polyhedra

An *r-sided antiprism* A_r is defined as follows: The vertex set $V(A_r) = \{u_{r+1} = u_1, u_2, \ldots, u_r, v_{r+1} = v_1, v_2, \ldots, v_r\}$, $r \geq 3$. The edge set $E(A_r) = \{ \{u_i u_{i+1}\} \cup \{v_i v_{i+1}\} \cup \{u_i v_i\} \cup \{u_{i+1} v_i\}, i = 1, \ldots, r \}$. The face set of A_r consists of two r-gonal faces f and h where $f = [u_1, \ldots, u_r]$, $h = [v_1, \ldots, v_r]$ and $2r$ faces $f_i = [u_i, u_{i+1}, v_i]$ and $h_i = [v_i, v_{i+1}, u_{i+1}]$, $i = 1, \ldots, r$, indices taken modulo r.

Theorem 3. *Let A_r be the graph of antiprism. If $r \in \{5, 7, 9, 10, 14, 17\}$ then $\pi'_f(A_r) = 4$; else $\pi'_f(A_r) = 3$.*

Proof. According to Theorem 2 the lower bound is clear.
Upper bound: Colour the edges of the cycle on vertices $u_1, u_2, \ldots, u_r$ nonrepetitively using 4 colours when $r = 5, 7, 9, 10, 14$ or 17; else use only 3 colours. For $i = 1, \ldots, r$, indices modulo r, use the colour of the edge u_iu_{i+1} for colouring the edges $u_{i+1}v_{i+1}$ and $v_{i+1}v_{i+2}$.
Note that in such a case the cycle on vertices $v_1, v_2, \ldots, v_r$ is coloured nonrepetitively too. Noncoloured edges are diagonals of the 4-gonal faces coloured with two colours. Thus there is still at least one colour more that can be used to obtain facial non-repetitive colouring of each 3-gonal face. □

An *r-sided prism* D_r, $r \geq 3$, is defined as follows: The vertex set $V = \{u_{r+1} = u_1, u_2, \ldots, u_r, v_{r+1} = v_1, v_2, \ldots, v_r\}$ and the edge set $E = \{\{u_i, u_{i+1}\} \cup \{v_i, v_{i+1}\} \cup \{u_i, v_i\}$, for $i = 1, \ldots, r$ $\}$. The set of faces of D_r consists of two r-gonal faces: the outer face $f = [u_1, \ldots, u_r]$ and the inner face $h = [v_1, \ldots, v_r]$; and r quadrangles $[u_i, u_{i+1}, v_{i+1}, v_i]$ for any $i = 1, \ldots, r$, indices taken modulo r.

Theorem 4. *Let D_r be a graph of prism. Then for $r \geq 4$ $\pi'_f(D_r) = 4$ and $\pi'_f(D_3) = 3$.*

Proof. It is easy to see that $\pi_f(D_r) \geq 3$ and that $\pi_f(D_3) = 3$.
Now we show the upper bound for D_r; $r > 3$: According to the Theorem of Thue [13] there exists a non-repetitive edge 3-colouring of the path $P = v_1, v_2, \ldots, v_r$, see Figure 1, that uses the colours 1, 2 and 3. Let us colour the edges of the path $Q = u_2, u_3, \ldots, u_r, u_1$ with the colours 1, 2 and 3 in such a way that an edge $u_{i+1}u_{i+2}$ has the same colour as an edge v_iv_{i+1} for $i = 1, 2, \ldots, r-1$. Colour the edges v_rv_1 and u_1u_2 with the colour 4.

Then we have to distinguish four situations to show that our colouring fulfills the required conditions:

Without loss of generality we can assume, that edge $v_{r-1}v_r$ received colour 1 and a $v_{r-2}v_{r-1}$ colour 2.

Case 1: If colour v_1v_2 is 1 and v_2v_3 is 2, then we shall colour the edges u_rv_r, u_1v_1 and u_2v_2 with the colour 3 and the remaining edges with the colour 4.

Case 2: If colour v_1v_2 is 1 and v_2v_3 is 3, then we shall colour the edges u_rv_r, u_1v_1 with the colour 3; the edge u_2v_2 with the colour 2 and the remaining edges with the colour 4.

Case 3: If colour v_1v_2 is 2, then we shall colour the edges u_rv_r and u_1v_1 with the colour 3.
If the colour of the edge v_2v_3 is 1, then we shall colour the edge u_2v_2 with the

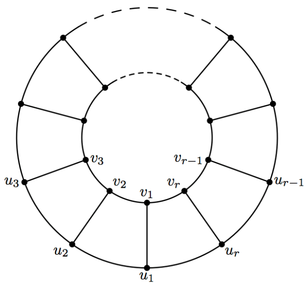

FIGURE 1. The prism

colour 3 too, otherwise we shall colour it 1. For colouring of the remaining edges we can use colour 4.

Case 4: If colour v_1v_2 is 3, then we shall colour the edge u_rv_r with the colour 3 and the edge u_1v_1 with the colour 2.
If the colour of the edge v_2v_3 is 2, then we shall colour the edge u_2v_2 with the colour 1; otherwise we shall colour it with the colour 2. For colouring of the remaining edges we shall use colour 4.

It is easy to see that in each case the obtained colouring is a facial non-repetitive 4-edge-colouring.
Now we are going to show that the lower bound of the facial Thue chromatic index of D_r is 4; $r \geq 4$: Suppose, that there exist facial non-repetitive 3-edge-colouring of D_r, $r \geq 4$. In this case on the r-gonal face of D_r there exist a sequence of edges v_iv_{i+1}, $v_{i+1}v_{i+2}$, $v_{i+2}v_{i+3}$, $v_{i+3}v_{i+4}$ coloured with colours a, b, a, c. Thus both of the edges $v_{i+1}u_{i+1}$ and $v_{i+2}u_{i+2}$ have to be coloured with the colour c and the edge $v_{i+3}u_{i+3}$ has to be coloured with b. Hence the edge $u_{i+2}u_{i+3}$ have to be coloured with the colour a. But then the colour a, as well as b and c, could not be used for colouring the edge $u_{i+1}u_{i+2}$ – a contradiction. □

4. Platonic polyhedra

The set of Platonic solids consists of five polyhedra:

(i) the tetrahedron or the $(3,3,3)-$ *polyhedron*,
(ii) the cube or the $(4,4,4)-$ *polyhedron*,
(iii) the octahedron or the $(3,3,3,3)-$ *polyhedron*,
(iv) the dodecahedron or the $(5,5,5)-$ *polyhedron* and
(v) the icosahedron or the $(3,3,3,3,3)-$ *polyhedron*.

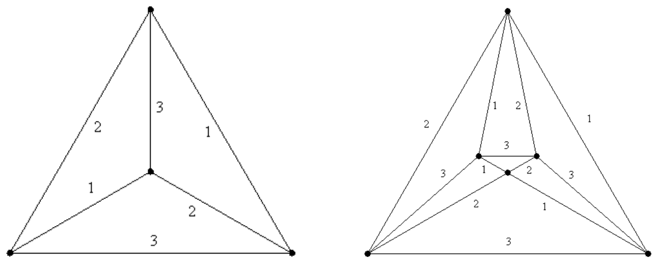

FIGURE 2. The tetrahedron and the octahedron

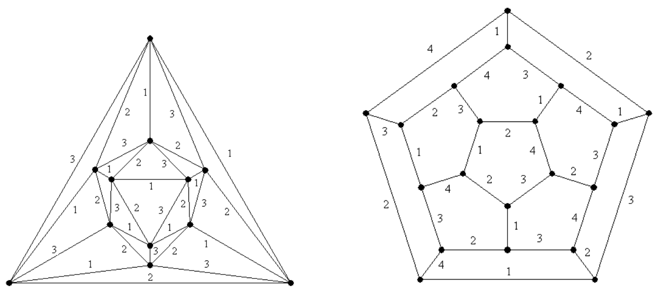

FIGURE 3. The isosahedron and the dodecahedron

Theorem 5. *If G is the tetrahedron, the octahedron or the icosahedron, then $\pi'_f(G) = 3$. If G is the dodecahedron or the cube, then $\pi'_f(G) = 4$.*

Proof. Theorem 2 implies that $\pi'_f(G) \geq 3$ for G being the tetrahedron, the octahedron, the cube or the icosahedron and $\pi'_f(G) \geq 4$ for G being the dodecahedron. From Figures 2 and 3 we can observe that except of the cube these bounds are achieved. The cube Q is in a family of prisms hence according to the Theorem 4 we have $\pi'_f(Q) = 4$. □

5. Archimedean polyhedra

The set of Archimedean solids consists of thirteen polyhedra:

(i) the cub-octahedron or the $(3,4,3,4) - polyhedron$,
(ii) the rhomb-cub-octahedron or the $(3,4,4,4) - polyhedron$,
(iii) the snub cube or the $(3,3,3,3,4) - polyhedron$,
(iv) the truncated dodecahedron or the $(3,10,10) - polyhedron$,
(v) the truncated icosi-dodecahedron or the $(4,6,10) - polyhedron$ or the great rhomb-icosi-dodecahedron,

(vi) the truncated icosahedron or the $(5,6,6) - \mathit{polyhedron}$,
(vii) the icosi-dodecahedron or the $(3,5,3,5) - \mathit{polyhedron}$,
(viii) the rhomb-icosi-dodecahedron or the $(3,4,5,4) - \mathit{polyhedron}$,
(ix) the snub dodecahedron or the $(3,3,3,3,5) - \mathit{polyhedron}$,
(x) the truncated tetrahedron or the $(3,6,6) - \mathit{polyhedron}$,
(xi) the truncated octahedron or the $(4,6,6) - \mathit{polyhedron}$,
(xii) the truncated cube or the $(3,8,8) - \mathit{polyhedron}$ and
(xiii) the truncated cub-octahedron or the $(4,6,8) - \mathit{polyhedron}$, or the great rhomb-cub-octahedron.

Theorem 6. *If G is a plane graph of the $(3,4,3,4)$-polyhedron, the $(3,4,4,4)$-polyhedron or the $(3,3,3,3,4)$-polyhedron, then $\pi'_f(G) = 3$.*
If G is a plane graph of the $(3,10,10)$-polyhedron, the $(4,6,10)$-polyhedron, the $(5,6,6)$-polyhedron, the $(3,5,3,5)$-polyhedron, the $(3,4,5,4)$-polyhedron or the $(3,3,3,3,5)$-polyhedron, then $\pi'_f(G) = 4$.
If G is a plane graph of the $(3,6,6)$-polyhedron, the $(4,6,6)$-polyhedron, the $(3,8,8)$-polyhedron or the $(4,6,8)$-polyhedron, then $\pi'_f(G) = 4$.

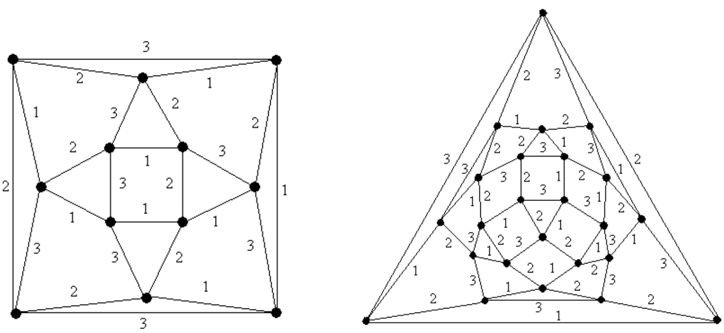

FIGURE 4. The $(3,4,3,4)$-polyhedron and the $(3,4,4,4)$-polyhedron

Proof. Theorem 2 gives the lower bound for the facial Thue chromatic index of Archimedean solids. From Figures $4-9$ we can observe that except of the $(4,6,6)$-polyhedron, the $(3,6,6)$-polyhedron, the $(3,8,8)$-polyhedron and the $(4,6,8)$-polyhedron these bounds are achieved.

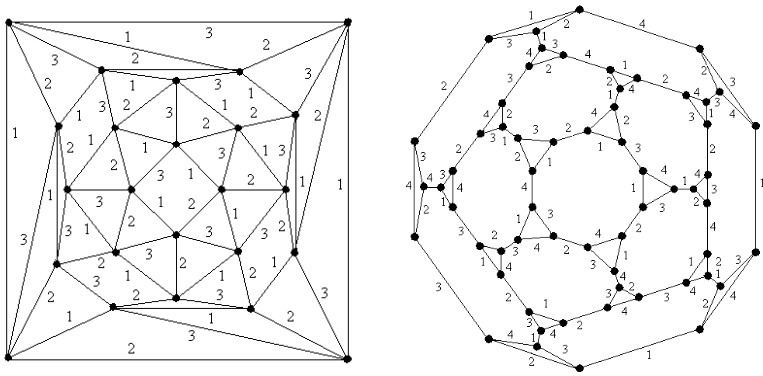

FIGURE 5. The $(3,3,3,3,4)$-polyhedron and the $(3,10,10)$-polyhedron

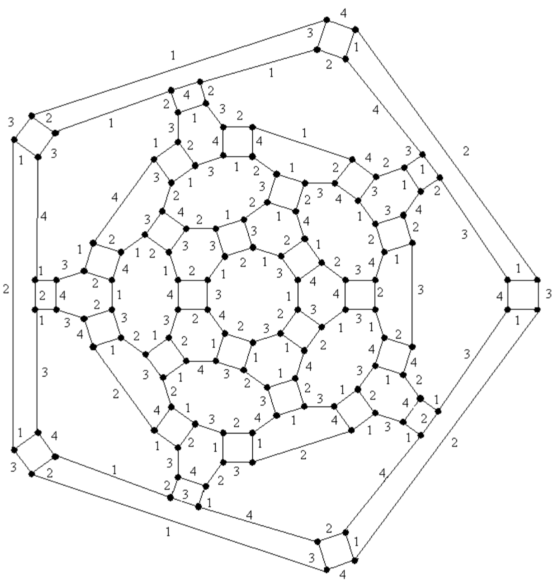

FIGURE 6. The $(4,6,10)$-polyhedron

For these four exceptions Theorem 2 gives $\pi'_f(G) \geq 3$. In what follows we show that 3 colours are not enough to colour their edges facially non-repetitively.

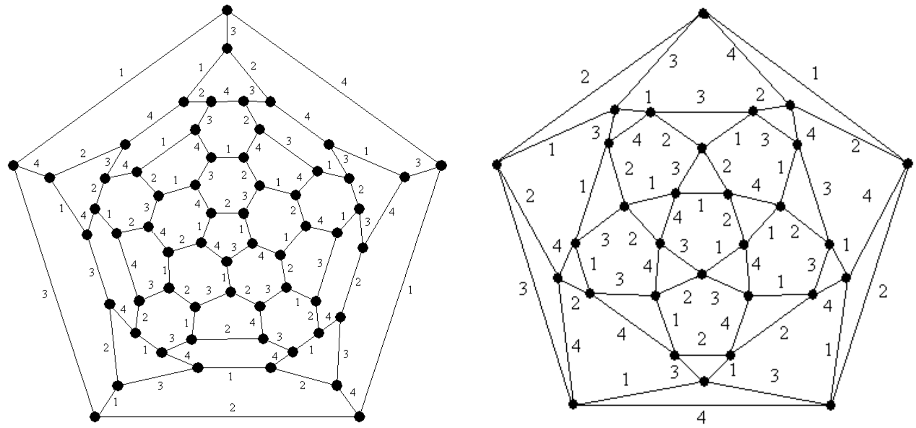

FIGURE 7. The $(5,6,6)$-polyhedron and the $(3,5,3,5)$-polyhedron

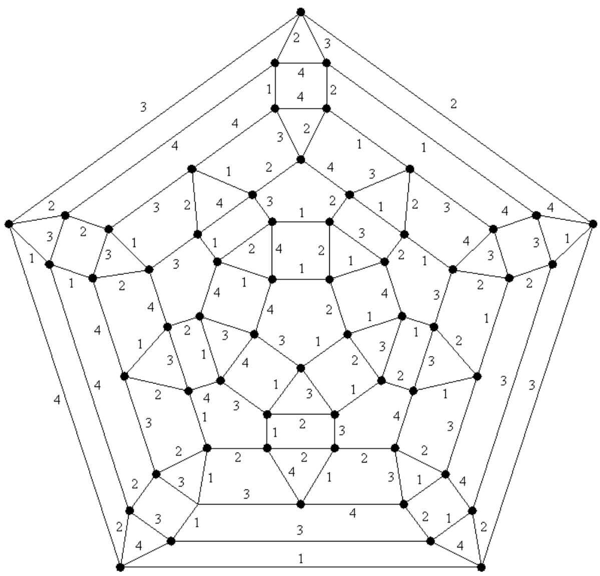

FIGURE 8. The $(3,4,5,4)$-polyhedron

Case 1: The $(4,6,6)$-polyhedron
Consider a graph of the $(4,6,6)$-polyhedron depicted on the Figure 10 (do not

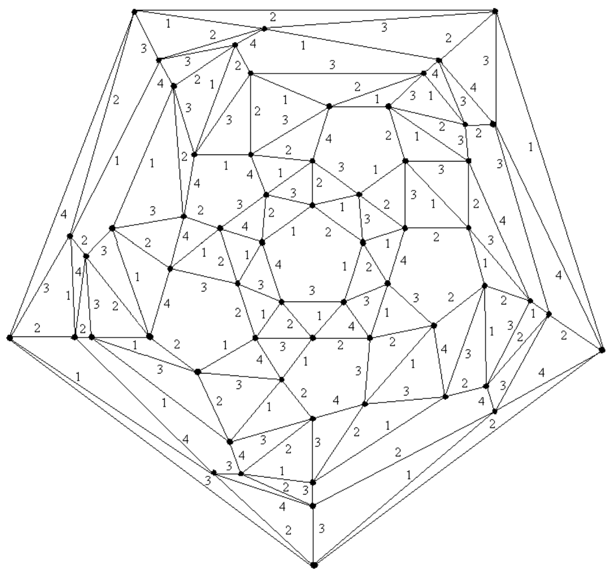

FIGURE 9. The $(3,3,3,3,5)$-polyhedron

consider the edge labelling there).
By a way of contradiction let us suppose that there exists a facial non-repetitive edge 3-colouring of the $(4,6,6)$-polyhedron. In such a case there exist a 4-gonal face that edges v_1v_2, v_2v_3, v_3v_4, v_4v_1 are coloured w.l.o.g. with colours 1, 2, 1, 3. Then the edges v_2v_6 and v_3v_7 have to have the colour 3 and the edges v_4v_8 and v_1v_5 have to have the colour 2. The edge v_5v_{10} has to have the colour 1, because in other case there would be either a repetition 2, 2 or a repetition 2, 3, 2, 3. Hence the edge v_5v_{11} has to have the colour 3. By the similar reasons the edge v_6v_{13} has to have the colour 1. Thus the edge v_6v_{12} has to have the colour 2. But in that case the edge $v_{11}v_{12}$ has to have the colour 1 and there is a repetitive sequence of colours 2, 3, 1, 2, 3, 1 on edges of one face of $(4,6,6)$-polyhedron – a contradiction.

For a facial non-repetitive 4-edge-colouring of $(4,6,6)$-polyhedron see Figure 10 where the numbers on edges are colours of these edges.

Case 2: The $(3,6,6)$-polyhedron
Consider a graph of the $(3,6,6)$-polyhedron depicted on the Figure 11 (do not consider the edge labelling).
By a way of contradiction let us suppose that there exists a facial non-repetitive

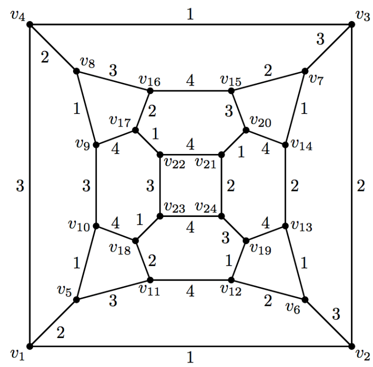

FIGURE 10. Case 1 - the $(4, 6, 6)$-polyhedron

edge 3-colouring of the $(3, 6, 6)$-polyhedron. Then there exist a 3-gonal face with edges v_1v_2, v_2v_3, v_3v_1 coloured w.l.o.g. 1, 2 and 3. Hence the edge v_2v_5 have to have the colour 3 and the edge v_3v_6 have to have the colour 1. Then one of the edges v_5v_{10}, v_5v_{11} is coloured with the colour 1 and the other one with the colour 2. Thus the edge $v_{10}v_{11}$ is coloured with the colour 3. Similarly one of the edges v_6v_8, v_6v_9 is coloured with the colour 2 and the other one with the colour 3. Hence the edge v_8v_9 is coloured with the colour 1. But then the edge v_9v_{10} has to have the colour 2 and the edge v_5v_{10} has to have the colour 1. But then the edge v_6v_9 has to have the colour 3 and thus there is a repetitive sequence of colours 1, 2, 3, 1, 2, 3 on edges of one face of the $(3, 6, 6)$-polyhedron – a contradiction. For a facial non-repetitive edge 4-colouring of $(3, 6, 6)$-polyhedron see Figure 11.

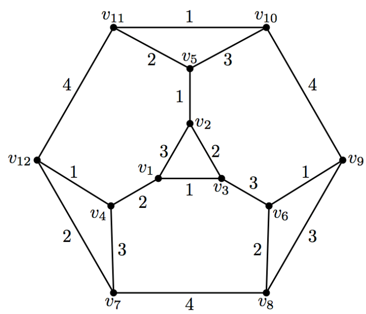

FIGURE 11. Case 2 - the $(3, 6, 6)$-polyhedron

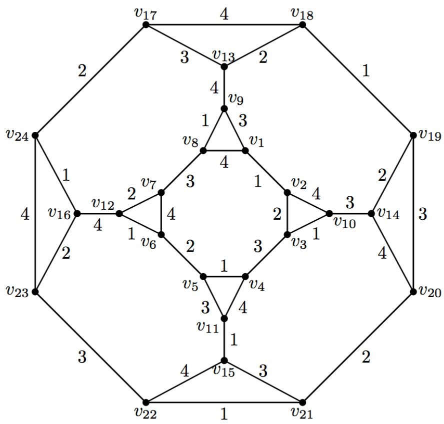

FIGURE 12. Case 3 - the $(3,8,8)$-polyhedron

Case 3: the $(3,8,8)$-polyhedron

Consider a graph of the $(3,8,8)$-polyhedron depicted on the Figure 12 (do not consider the edge labelling). By a way of contradiction let us suppose that there exists a facial non-repetitive edge 3-colouring of the $(3,8,8)$-polyhedron. Notice that there exist unique facial non-repetitive edge colouring of the cycle C_8 with three symbols. Thus there exists an 8-gonal face of $(3,8,8)$-polyhedron that edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_6v_7, v_7v_8, v_8v_1 are coloured either with the sequence of colours $S_1 = 1,2,1,3,2,1,2,3$, or with the sequence of colours $S_2 = 3,1,2,1,3,2,1,2$.

If the 8-gonal face is coloured with the sequence of colours S_1, the edges v_2v_{10} and v_3v_{10} have to have the colour 3 – a contradiction.

Now suppose that the 8-gonal face mentioned above is coloured with the sequence of colours S_2. In such a case the edges v_1v_9 and v_6v_{12} have to have the colour 1, the edges v_4v_{11}, v_7v_{12} and v_8v_9 have to have the colour 3 and the edge v_5v_{11} has to have the colour 2. Hence the edges v_9v_{13} and $v_{12}v_{16}$ have to have the colour 2 too and the edge $v_{11}v_{15}$ has to have the colour 1. Then one of the edges $v_{15}v_{21}$, $v_{15}v_{22}$ is coloured with the colour 2 and the other one with the colour 3 thus the edge $v_{21}v_{22}$ has to have the colour 1. Similarly one of the two edges $v_{16}v_{23}$, $v_{16}v_{24}$, likewise one of the two edges $v_{13}v_{17}$, $v_{13}v_{18}$, has to have the colour 1 and the other one the colour 3. Thus the edges $v_{23}v_{24}$, $v_{17}v_{18}$ have to have the colour 2. Hence the edge $v_{22}v_{23}$ has to have the colour 3; the edge $v_{15}v_{22}$ has to have the colour 2 and the edge $v_{16}v_{23}$ the colour 1. But then the edge $v_{17}v_{24}$ has to have the colour 1 and both of the edges $v_{13}v_{17}$, $v_{16}v_{24}$ have to have the colour

3. Hence there is a repetition 1, 3, 2, 3, 1, 3, 2, 3, of colours on edges of one face of the $(3, 8, 8)$-polyhedron – a contradiction.

For a facial non-repetitive 4-edge-colouring of $(3, 8, 8)$-polyhedron see Figure 12.

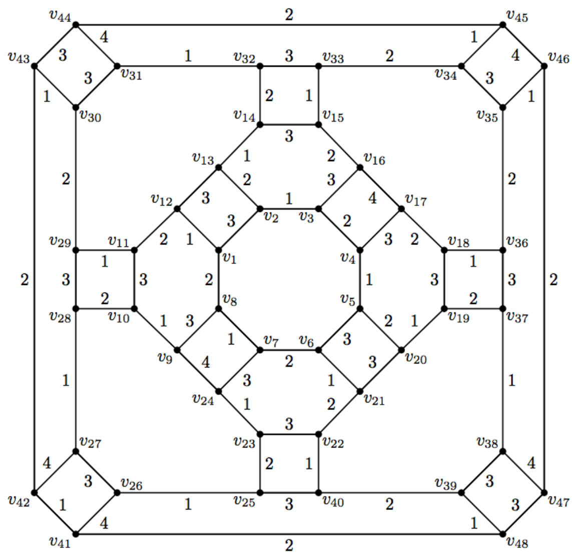

FIGURE 13. Case 4 - the $(4, 6, 8)$-polyhedron

Case 4: the $(4, 6, 8)$-polyhedron
Consider a graph of the $(4, 6, 8)$-polyhedron depicted on the Figure 13 without labellings of edges.
By a way of contradiction let us suppose that there exists a facial non-repetitive 3-edge-colouring of the $(4, 6, 8)$-polyhedron.
The unique non-repetitive colouring of C_8 gives two possibilities how the edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_6v_7, v_7v_8 and v_8v_1 of an 8-gonal face of $(4, 6, 8)$-polyhedron are coloured.

First let us suppose that the edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_6v_7, v_7v_8 and v_8v_1 are coloured with the sequence of colours 1, 2, 1, 3, 2, 1, 2, and 3, respectively. In such a case the edges v_2v_{13} and v_3v_{16} have to have the colour 3 and the edge v_4v_{17} has to have the colour 2. Thus the edge $v_{16}v_{17}$ has to have the colour 1. But then the edge $v_{15}v_{16}$ has to have the colour 2 and there is a repetition 2, 3, 2, 3 of colours on edges of one face of the $(4, 6, 8)$-polyhedron – a contradiction.

Now suppose that the edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_6v_7, v_7v_8 and v_8v_1 of 8-gonal face are coloured consequently with colours 3, 1, 2, 1, 3, 2, 1, 2. Then the edges v_3v_{16} and v_4v_{17} have to have the colour 3, the edge v_1v_{12} has to have the colour 1 and the edge v_2v_{13} has to have the colour 2. Hence the edge $v_{16}v_{17}$ has to have the colour 1 and the edges $v_{15}v_{16}$, $v_{17}v_{18}$ have to have the colour 2. Then the edges $v_{12}v_{13}$ and $v_{14}v_{15}$ have to have the colour 3 and the edge $v_{13}v_{14}$ has to have the colour 1. But in such a case the edge $v_{15}v_{33}$ has to have the colour 1 and there is a repetition 1, 2, 1, 2 of colours on edges of one face of the $(4, 6, 8)$-polyhedron – a contradiction.

For a facial non-repetitive edge 4-colouring the $(4, 6, 8)$-polyhedron see Figure 13. □

6. Pseudo-Archimedean polyhedron

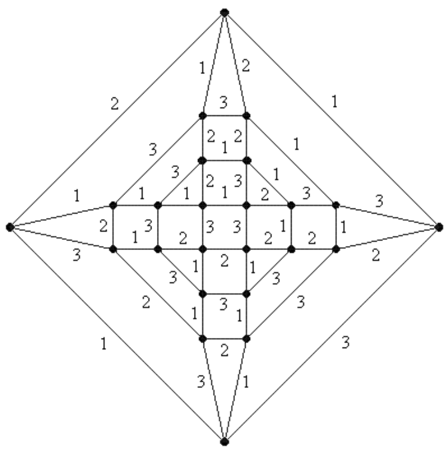

Figure 14. the Miller polyhedron

Theorem 7. *Let G be a graph of the Miller polyhedron. Then $\pi'_f(G) = 3$.*

Proof. Theorem 2 gives $\pi'_f(G) \geq 3$ for G being a graph of Miller polyhedron. A facial non-repetitive edge 3-colouring of G is at Figure 14, thus $\pi'_f(G) = 3$. □

7. Discussion

In [6] there was conjectured that for every 3-connected plane graph G the facial Thue chromatic index $\pi'_f(G) \leq 6$. In the present paper we have found the exact values of the facial Thue chromatic index for semiregular polyhedra. We showed that $\pi'_f(G)$ is either 3 or 4 for graphs of semiregular polyhedra, which is the first step towards the conjecture mentioned.

By Theorem 1 for every cycle C_n, where $n \in \{2, 5, 7, 9, 10, 14, 17\}$ holds $\pi'_f(C_n) = 4$. We showed that even if the 3-connected plane graph does not contain any face of degree $n \in \{2, 5, 7, 9, 10, 14, 17\}$ its facial Thue chromatic index could be 4 (see Figure 10 – 13) or greater in general case.

The existence of a plane graph G for which $\pi'_f(G) \geq 5$ is still an open question.

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Institute of Mathematics, Faculty of Science, P. J. Šafárik University, Jesenná 5, 040 01 Košice, Slovakia
E-mail address, S. Jendroľ: stanislav.jendrol@upjs.sk

E-mail address, E. Škrabuľáková: erika.skrabulakova@upjs.sk