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HOMOMORPHIC EXTENSIONS OF PSEUDOCOMPLEMENTED SEMILATTICES

TIBOR KATRIŇÁK AND JAROSLAV GURIČAN

Dedicated to the 70th birthday of Alfonz Haviar

Abstract. Our aim is to study and characterize extensions to a homomorphism in the class of pseudocomplemented semilattices. We present here such a description.

1. Introduction

We shall deal with the question in which circumstances a mapping f from a generating set X of a pseudocomplemented semilattice S into a pseudocomplemented semilattice M can be extended to a homomorphism $g : S \to M$. Such an extension, if it exists, is uniquely determined.

It is a well-known fact (see [5]) that the class of all pseudocomplemented semilattices is equational with only one non-trivial subvariety, namely, the class of Boolean algebras. The preceding question found an answer for Boolean algebras (see [9] and especially Sikorski's extension criterion). We shall use these results as a motivation for our task.

2. Preliminaries

A pseudocomplemented semilattice (= PCS) is an algebra $(S; \wedge, ^*, 0, 1)$ of type (2,1,0,0), where $(S; \wedge, 0, 1)$ is a bounded meet-semilattice and, for every $a \in S$, the element a^* is a *pseudocomplement* of a, i.e. $x \leq a^*$ if and only if $x \wedge a = 0$. A PCS S is said to be *non-trivial*, whenever $\mid S \mid \geq 2$. An element $a \in S$ is called *closed*, if $a = a^{**}$. Let $B(S)$ denote the set of all closed elements of S. It is known that

$$(B(S); +, \wedge, ^*, 0, 1)$$

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forms a Boolean algebra with

$$a + b = (a^* \wedge b^*)^*$$

(see [1] and [3]). (Clearly, a PCS S is a Boolean algebra if and only if S satisfies the identity $x = x^{**}$.)

Here are some rules of computation with * and $\wedge$ (see [1] or [3]):

(1) $x \wedge x^* = 0$.

(2) $x \leq y$ implies that $x^* \geq y^*$.

(3) $x \leq x^{**}$.

(4) $x^* = x^{***}$.

(5) $(x \wedge y)^{**} = x^{**} \wedge y^{**}$.

(6) $0^* = 1$ and $1^* = 0$.

The following result can be easily verified (see [7]).

Lemma 2.1. *Let S be a PCS and let $X \subseteq S$. Then S is generated by X, i.e. $S = [X]$ if and only if $[X^{**}]_{Bool} = B(S)$ and $S = [X \cup B(S)]_{sem}$, that means, $B(S)$ is generated by $X^{**} = \{x^{**} : x \in X\}$ as a Boolean algebra and S is generated by $X \cup B(S)$ as a semilattice.*

Let S and T be PCSs. A function $f : S \to T$ is called a *homomorphism* (of PCSs) if $f(x \wedge y) = f(x) \wedge f(y)$, $f(x)^* = f(x^*)$ for $x, y \in S$. We observe that $f(0) = 0$, and $f(1) = 1$.

The definitions of the concepts discussed in this paper may be found in [1] and [3].

3. Extensions

Let S and K be PCSs and let K be a subalgebra of S, that means, S is an *extension* of K. (Notation: $K \leq S$.) In addition, we set $K[X] = [K \cup X]$, whenever $X \subseteq S$. We say that S is a *finite* (*simple*) extension of its subalgebra K, if $S = K[X]$ for some finite (one-element) set $X \subseteq S$.

Proposition 3.1. *Let K and S be PCSs. Then S is a simple extension of K, that means, $S = K[x]$ for some $x \in S$, if and only if*

(i) $B(S) = [B(K) \cup \{x^{**}\}]_{Bool}$,

(ii) $S_1 = [B(S) \cup K]_{sem}$ *is a subalgebra of S and*

(iii) $S = [S_1 \cup \{x\}]_{sem}$.

Proof. Assume first $S = K[x]$. Then (i) is straightforward (see Lemma 2.1). (ii) We have only to show that $u \in S_1$ implies $u^* \in S_1$. But this follows from the fact that $u^* \in B(S) \subseteq S_1$. Thus S_1 is a subalgebra of S. (iii) Set $M = [S_1 \cup \{x\}]_{sem}$. We claim that M is a subalgebra of S. Similarly as above, we have only to show that $u \in M$ implies $u^* \in M$. Since $B(S) \subseteq S_1 \subseteq M$ and $u^* \in B(S)$, we see that $u^* \in M$. Finally, since $K \cup \{x\} \subseteq M$, we obtain $M = S$.

To prove the converse, assume that the conditions (i)-(iii) are satisfied. It is easy to see that $K \leq S$. Therefore, $K[x] = [K \cup \{x\}] \subseteq S$. On the other hand, $B(S) \subseteq K[x]$ by (i). Consequently, $S \subseteq K[x]$ by (ii) and (iii), and the proof is complete. $\square$

Proposition 3.1 generalizes immediately to arbitrary set X (instead of one-element set $\{x\}$).

Theorem 3.2. *Let K and S be PCSs. Then $S = K[X]$ for some $X \subseteq S$ if and only if*

(i) $B(S) = [B(K) \cup X^{**}]_{bool}$,
(ii) $S_1 = [B(S) \cup K]_{sem}$ *is a subalgebra of* S *and*
(iii) $S = [S_1 \cup X]_{sem}$.

Corollary 3.3. *Let $S = K[X]$ and let $u \in S$. Then there exist $s \in K$ and a finite $U \subseteq X$ such that*

$$u = u^{**} \wedge s \wedge \bigwedge(x : x \in U).$$

For our next result we need the following concept:

Definition 3.4. *Let K and S be bounded meet-semilattices* (PCSs) *such that $K \leq S$. Then K is said to be* relatively complete *in S, if for each $b \in S$ there exists a smallest $a \in K$ such that $b \leq a$. In notation:*

$$a = \Pr(b) = \Pr_K^S(b) = \min\{x \in K \mid b \leq x\}.$$

Write $K \leq_{rc} S$ if K is relatively complete in S. See also [6] or [9] for relatively complete lattices or Boolean algebras.

Using the notation from the preceding theorem, we can formulate the following result:

Corollary 3.5. *Let $K \leq S$ for PCSs. Then $K \leq_{rc} S$ if and only if*

$$K \leq_{rc} S_1 \leq_{rc} S,$$

where $S_1 = [B(S) \cup K]_{sem}$.

Proof. Let $K \leq_{rc} S$. (Clearly, $S = K[X]$ for some $X \subseteq S$.) It follows that $B(K) \leq_{rc} B(S)$ and $K \leq_{rc} S_1$. It remains to prove $S_1 \leq_{rc} S$. Let $u \in S$ and $u \leq v$ for some $v \in S_1$. It is easy to see that $v = a \wedge t$ for some $a \in B(S)$ and $t \in K$. Now, $u \leq v$ if and only if $u \leq a$ and $u \leq t$ in S. But $u \leq a$ if and only if $u^{**} \leq a$. The second relation $u \leq t$ is equivalent to $u \leq \Pr_K^S(u) \leq t$. Therefore,

$$u \leq u^{**} \wedge \Pr_K^S(u) \leq a \wedge t = v.$$

Since $u^{**} \wedge \Pr_K^S(u) \in S_1$, we have $S_1 \leq_{rc} S$. The converse implication is straightforward. $\square$

4. Extension to a homomorphism

In this section we shall examine the following situation: Let K, M and $S = K[X]$ be PCSs. Let $f_0 : K \to M$ be a homomorphism and $f : X \to M$ be a mapping. The question concerning f is whether or not there exists a homomorphism $g : S \to M$ such that $g \restriction_{K \cup X} = f_0 \cup f$ (= the restriction of g to $K \cup X$). It is easy to see that g, whenever it exists, is uniquely determined. In this case we say that g is an *extension of* $f_0 \cup f$ *to a homomorphism.*

Notice that a specialization of our question for Boolean algebras has been considered by R. Sikorski. He found a useful characterization of those mappings f, for which there exists an extension to a Boolean homomorphism (see Sikorski's extension criterion in [9]).

The next theorem is concerned with a more general situation and will frequently be useful:

Theorem 4.1. *Let K, M and S be PCSs and let S be an extension of K, that means, $S = K[X]$ for some $X \subseteq S$. Assume that $f_0 : K \to M$ is a homomorphism and let $f : X \to M$ be a mapping. Then there exists a homomorphism $g : S \to M$ extending $f_0 \cup f$ if and only if the following conditions are fulfilled:*

(i) *there is a Boolean homomorphism $h : B(S) \to B(M)$, which is an extension of $(f_0)_B : B(K) \to B(M)$ (i.e. $(f_0)_B$ is a restriction of f_0 to $B(K)$) such that*

$$h(x^{**}) = f(x)^{**}$$

for every $x \in X$;

(ii) *if $S_1 = [B(S) \cup K]_{sem}$, then there exists a meet-semilattice homomorphism $f_1 : S_1 \to M$ such that f_1 is an extension of $f_0 \cup h$;*

(iii) *there exists a meet-semilattice homomorphism $g : S \to M$ which is an extension of $f_1 \cup f$.*

In addition, the homomorphism $g : S \to M$, if it exists, is uniquely determined. If $u \in S$, then

$$g(u) = h(u^{**}) \wedge f_0(s) \wedge \bigwedge(f(x) : x \in U) = f_1(u^{**} \wedge s) \wedge \bigwedge(f(x) : x \in U)$$

for some $s \in K$ and a finite $U \subseteq X$ (see Corollary 3.3).

Proof. The necessity of (i)-(iii) is straightforward (see Lemma 2.1 and Theorem 3.2). Conversely, assume conditions (i) - (iii). First we show that $f_1 : S_1 \to M$ is a PCS-homomorphism. Really, suppose $u \in S_1$. By Theorem 3.2, $u = a \wedge s$ for some $a \in B(S)$ and $s \in K$. Therefore,

$$f_1(u) = f_1(a \wedge s) = h(a) \wedge f_0(s),$$

by (ii). Now,

$$\begin{aligned} f_1(u)^{**} &= (h(a) \wedge f_0(s))^{**} = h(a)^{**} \wedge f_0(s)^{**} = h(a) \wedge f_0(s^{**}) \\ &= h(a) \wedge h(s^{**}) = h(a \wedge s^{**}) = h(u^{**}) = f_1(u^{**}), \end{aligned}$$

by (i) and (ii). Hence,

$$f_1(u)^* = f_1(u)^{***} = h(u^{**})^* = h(u^*) = f_1(u^*),$$

as h is a Boolean homomorphism. Clearly, f_1 is a PCSs homomorphism and an extension of $f_0 \cup h$.

Now, we can show that meet-semilattice homomorphism $g : S \to M$ satisfies $g(u)^* = g(u^*)$ for any $u \in S$ as well. Really, take $u \in S$. By Theorem 3.2, either $u \in S_1$ or $u = s \wedge (\bigwedge X_1)$ for some $s \in S_1$ and a finite non-empty $X_1 \subseteq X$. The first case is straightforward: $g(u) = f_1(u)$. Let us consider the second event. By hypothesis,

$$g(u) = g(s \wedge (\bigwedge X_1)) = g(s) \wedge \bigwedge(g(y) : y \in X_1) = f_1(s) \wedge \bigwedge(g(y) : y \in X_1).$$

Since $g(y)^{**} = f(y)^{**} = h(y^{**})$, for $y \in X_1$, we get

$$g(u)^{**} = f_1(s)^{**} \wedge \bigwedge(g(y)^{**} : y \in X_1) = h(s^{**}) \wedge h(\bigwedge X_1^{**}) = h(u^{**}).$$

It follows that

$$g(u)^* = g(u)^{***} = (g(u)^{**})^* = h(u^{**})^* = h(u^*) = f_1(u^*) = g(u^*),$$

by (i) - (iii). Now, it is easy to see that g is the required homomorphism extending $f_0 \cup f$. The last statement follows from Theorem 3.2 and Corollary 3.3. The proof is complete. □

Corollary 4.2. *Under the assumptions of Theorem 4.1 and the additional hypothesis that $B(K) = B(S)$, the following statements are equivalent:*

- (i) *There exists a PCS-homomorphism $g : S \to M$, which is an extension of $f_0 \cup f$.*
- (ii) *There exists a meet-semilattice homomorphism $g : S \to M$, which is an extension of $f_0 \cup f$.*

Proof. Clearly, $B(K) = B(S)$ yields that $h \subseteq f_0$. Hence $f_1 = f_0$ and the rest follows from Theorem 4.1. □

Theorem 4.1 shows that an extension of a PCS-homomorphism can be reduced to three special parts: one extension of a Boolean homomorphism and two extensions of bounded meet-semilattice homomorphisms. More precisely, let K, M and S be PCSs and let $S = K[X]$. Assume that there exist a PCS-homomorphism $f_0 : K \to M$ and a mapping $f : X \to M$. Then there exists

(I) a Boolean homomorphism $(f_0)_B : B(K) \to B(M)$, where $(f_0)_B$ is the restriction of f_0 to $B(K)$ (Lemma 2.1). In addition, there is a mapping $f^+ : X^{**} \to B(M)$ defined by the rule

$$f^+(x^{**}) = f(x)^{**}.$$

The first question concerning $(f_0)_B$ is whether or not there is an extension of $(f_0)_B \cup f^+$ to a Boolean homomorphism $h : B(S) \to B(M)$. (Notice that $[B(K) \cup X^{**}]_{Bool} = B(S)$ by Lemma 2.1.) The answer to this question comes from the following lemma, due to R. Sikorski (see [9], Theorem 5.5). First we need a new notation: Let B be a Boolean algebra. For $x \in B$ and $\varepsilon \in \{+1, -1\}$, define the element x^ε of B by

$$x^{+1} = x, \quad x^{-1} = x^*.$$

Lemma 4.3. *A Boolean homomorphism $h : B(S) \to B(M)$ is an extension of $(f_0)_B \cup f^+$ if and only if*

$$a^{\varepsilon_0} \wedge (x_1^{**})^{\varepsilon_1} \wedge \cdots \wedge (x_k^{**})^{\varepsilon_k} = 0$$

in $B(S)$ for $a \in B(K)$, $x_1^{**}, \cdots, x_k^{**} \in X^{**}$ and $\varepsilon_i \in \{+1, -1\}$ implies

$$f_0(a)^{\varepsilon_0} \wedge f(x_1^{**})^{\varepsilon_1} \wedge \cdots \wedge f(x_k^{**})^{\varepsilon_k} = 0$$

in $B(M)$.

(II) Suppose now that a Boolean homomorphism $h : B(S) \to B(M)$ exists and h is an extension of $(f_0)_B \cup f^+$. In addition, there exists $S_1 \leq S$ and we can ask again whether or not there exists a meet-semilattice homomorphism $f_1 : S_1 \to M$, which is an extension of $f_0 \cup h$. The answer can be formulated as follows:

Lemma 4.4. *Let $h : B(S) \to B(M)$ be a Boolean homomorphism and an extension of $(f_0)_B \cup f^+$. Then there exists a meet-semilattice homomorphism $f_1 : S_1 \to M$, which is an extension of $f_0 \cup h$ if and only if*

$$a \wedge s = b \wedge t$$

implies

$$h(a) \wedge f_0(s) = h(b) \wedge f_0(t)$$

for any $a, b \in B(S)$ and $s, t \in K$.

The result requires only routine verification, and the proof can be omitted.

(III) It remains to establish the third part. We thus have a semilattice homomorphism $f_1 : S_1 \to M$, which is an extension of $f_0 \cup h$. Since $S = [S_1 \cup X]_{sem}$ (Theorem 3.2), it is reasonable to ask again whether or not there exists a meet-semilattice homomorphism $g : S \to M$, which is an extension of $f_1 \cup f$. The following lemma yields a solution:

Lemma 4.5. *Let $f_1 : S_1 \to M$ be a semilattice homomorphism extending $f_0 \cup h$. Then there exists a semilattice homomorphism $g : S \to M$, which is an extension of $f_1 \cup f$ if and only if*

$$s \wedge \bigwedge(y : y \in Y) = t \wedge \bigwedge(z : z \in Z)$$

implies

$$f_1(s) \wedge \bigwedge(f(y) : y \in Y) = f_1(t) \wedge \bigwedge(f(z) : z \in Z)$$

for any $s, t \in S_1$ and arbitrary finite $Y, Z \subseteq X$.

The proof is routine.

We conclude this section by observing that Lemmas 4.3-4.5 complete Theorem 4.1. The interested reader should have no serious difficulty in reconstructing the corresponding theorem.

5. Simple extensions

In the last section (Theorem 4.1) we saw how a PCS-homomorphism $f_0 : K \to M$ can be extended to a PCS-homomorphism $g : S \to M$, where $K \leq S$. Unfortunately, our characterization is of general nature, that means, the result is not useful enough. The purpose of this section is to find sufficient conditions under which we can easily say that an extension exists or not. For this reason we perform some specializations (simple extensions, retractions) and a generalization (meet-semilattices). (See the discussion in the preceding section.)

Proposition 5.1. *Let $f : T \to M$ be a homomorphism of non-trivial bounded meet-semilattices. Assume that the bounded meet-semilattice $S = T[x]$ is a simple extension of T and u is an element of M. Moreover, assume that the element $\mathrm{Pr}_T^S(x)$ exists and, that we have a retraction $\alpha : T[x] \to T$, that means, $\alpha(a) = a$ for any $a \in T$, such that $\alpha(x) = \mathrm{Pr}_T^S(x)$. Then there exists a meet-semilattice homomorphism*

$$g : S = T[x] \to M$$

extending f and mapping x to $u \in M$ if and only if

$$a \leq x \text{ in } S \text{ and } a \in T \text{ imply } f(a) \leq u \leq f(\mathrm{Pr}_T^S(x)) \text{ in } M.$$

Proof. Necessity of the condition is obvious. As to sufficiency, it is known that an arbitrary element $v \in S$ can be written in the form

$$v = a \wedge x^r,$$

where $r \in \{0, 1\}$ and $a \in T$. (Note that $x^0 = 1$ and $x^1 = x$.) Now, we can define

$$g : S \to M$$

by

$$g(v) = f(a) \wedge u^r.$$

First we have to show that g is well-defined, that is,

$$c \wedge x^r = d \wedge x^s \quad \text{implies} \quad f(c) \wedge u^r = f(d) \wedge u^s,$$

for $c, d \in T$. We have to verify two cases only:

$$c \wedge x = d \wedge x \quad \text{and} \quad c = d \wedge x.$$

Writing $\mathrm{Pr}(x)$ for $\mathrm{Pr}_T^S(x)$ we get in the first event

$$\alpha(c \wedge x) = c \wedge \mathrm{Pr}(x) = d \wedge \mathrm{Pr}(x) = \alpha(d \wedge x),$$

by the hypothesis on α. Therefore,

$$f(c) \wedge f(\mathrm{Pr}(x)) = f(c \wedge \mathrm{Pr}(x)) = f(d \wedge \mathrm{Pr}(x)) = f(d) \wedge f(\mathrm{Pr}(x)),$$

as f is a homomorphism. Since $u \leq f(\mathrm{Pr}(x))$, we obtain

$$f(c) \wedge u = f(d) \wedge u.$$

Considering the second case $c = d \wedge x$, we see that $c \leq x$. Hence $f(c) \leq u$, by the hypothesis on f. Using the same reasoning as above, we obtain

$$f(c) = f(d) \wedge u,$$

and g is well-defined. The element 0 in S can be expressed as $0 = 0 \wedge x$. Therefore,

$$g(0) = f(0) \wedge u = 0$$

in M. Similarly, $g(1) = 1$. Now, it can be readily shown that g is a meet-semilattice homomorphism extending f with the required properties. □

Lemma 5.2. *Let $S = K[x]$ be a simple extension of PCSs. Assume that there exists $\mathrm{Pr}_K^S(x)$. Then there exists $\mathrm{Pr}_{S_1}^S(x)$ (for S_1 see Section 3) and*

$$\mathrm{Pr}_{S_1}^S(x) = x^{**} \wedge \mathrm{Pr}_K^S(x).$$

Proof. Clearly, $x \leq x^{**} \wedge \mathrm{Pr}_K^S(x) \in S_1$. On the other hand, let $x \leq v$ for some $v \in S_1$. By Theorem 3.2, $v = a \wedge t$ for some $a \in B(S)$ and $t \in K$. Now, $x \leq a \wedge t$ implies $x^{**} \leq a$ in $B(S)$ and $x \leq t$ in K. Hence

$$x^{**} \wedge \mathrm{Pr}_K^S(x) \leq a \wedge t = v.$$

□

As a consequence of these results we have

Theorem 5.3. *Let K, M and S be PCSs, let $S = K[x]$ be a simple extension of K for some $x \in S$ and let $u \in M$. Let $f_0 : K \to M$ be a PCS-homomorphism. Assume that the element $\mathrm{Pr}_K^S(x)$ exists and that we have (in the notation of Section 3) a retraction $\alpha : S_1[x] \to S_1$ such that $\alpha(x) = x^{**} \wedge \mathrm{Pr}_K^S(x)$. Then there exists a PCS-homomorphism*

$$g : S \to M$$

extending f_0 and mapping x to $u \in M$ if and only if

(i) *there exists a meet-semilattice homomorphism $f_1 : S_1 \to M$ which is an extension of $f_0 \cup h$ (see Theorem 4.1) and, we have*

$$f_1(x^{**}) = h(x^{**}) = u^{**},$$

(ii) *$t \leq x$ in S and $t \in S_1$ imply $f_1(t) \leq u \leq f_1(\mathrm{Pr}^S_{S_1}(x))$ in M.*

Proof. Suppose that $g : S \to M$ is an extension of f_0 such that $g(x) = u$. Since g is a PCS-homomorphism, condition (ii) follows easily. Condition (i) is a consequence of Theorem 4.1.

To prove the remaining half, let us suppose (i) and (ii). We shall proceed by Theorem 4.1. We start by establishing a Boolean homomorphism $h : B(S) \to B(M)$ which is an extension of $(f_0)_B$ (see Theorem 4.1) such that $h(x^{**}) = u^{**}$. It is easy to check that $[B(K) \cup \{x^{**}\}]_{Bool} = B(S)$. Moreover, from (ii) and the hypothesis that $\mathrm{Pr}^S_K(x)$ exists, it follows that

$$a^{**} \leq x^{**} \leq b^{**} \text{ in } S \text{ implies } f_0(a^{**}) \leq u^{**} \leq f_0(\mathrm{Pr}^S_K(x)^{**}) \leq f_0(b^{**}) \text{ in } M$$

for any $a, b \in K$. Now we can apply ([9], Corollary 5.8) of Sikorski's extension criterion for Boolean algebras. It does ensure that there is a Boolean homomorphism $h : B(S) \to B(M)$ extending $(f_0)_B : B(K) \to B(M)$ such that $h(x^{**}) = u^{**}$.

By (i) we see that $f_1 : S_1 \to M$ is a meet-semilattice homomorphism extending $f_0 \cup h$. It remains to show that there exists a meet-semilattice homomorphism $g : S \to M$ extending $f_1 \cup \{(x, u)\}$. Evidently, $S = S_1[x]$ is a simple meet-semilattice extension. Now, we can apply Proposition 5.1. By Lemma 5.2 and the hypothesis that f_1 ia a meet-semilattice homomorphism, we get

$$u^{**} \wedge f_0(\mathrm{Pr}^S_K(x)) = h(x^{**}) \wedge f_1(\mathrm{Pr}^S_K(x)) = f_1(\mathrm{Pr}^S_{S_1}(x)).$$

Now, setting T for S_1 in (ii), we get the main condition of Proposition 5.1. It follows that there exists a meet-semilattice homomorphism $g : S \to M$ extending $f_1 \cup \{(x, u)\}$. Ultimately Theorem 4.1 implies that g is a PCS-homomorphism, and the proof of the theorem is complete. □

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Katedra AGDM FMFI, Comenius University, Mlynská dolina, 842 15 Bratislava, Slovakia

E-mail address, T. Katriňák: katrinak@fmph.uniba.sk

E-mail address, J. Guričan: gurican@fmph.uniba.sk