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STABILITY OF HOMOMORPHISMS BETWEEN COMPACT ALGEBRAS

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Dedicated to the 70th birthday of Alfonz Haviar

Abstract. We generalize a result on stability of continuous homomorphisms between compact groups to continuous homomorphisms between compact topological algebras. In general, a continuous function between such algebras which is almost a homomorphism need not be uniformly close to a homomorphism. Positive results can be obtained introducing some control over the continuity of the functions resembling homomorphisms by means of a "continuity scale."

The question we are dealing with in this paper can be roughly speaking formulated as follows: if a continuous function $g\colon A \to B$ between two topological universal algebras A and B of the same similarity type behaves almost like a homomorphism, is it then necessarily uniformly close to a genuine continuous homomorphism $h\colon A \to B$? Question of this type, made precise for various types of algebras and functions, use to be called *stability problems*.

The problem of ε-stability of additive functions $\mathbb{R} \to \mathbb{R}$, as well as its generalization to mappings between arbitrary metrizable groups, was raised by S. M. Ulam — cf. [9], [10]. Since then the topic was thoroughly examined and generalized in various respects — see [2], [4], [5] and [7] for surveys of further development.

There are many known examples showing that "arbitrarily good almost homomorphisms" need not be close to homomorphisms. In the paper [8] by J. Špakula and the present author a stability result for continuous homomorphisms between compact topological groups was established. This was enabled by controlling the continuity of the almost homomorphisms by means of a "continuity scale." We also remarked there that this result could readily be generalized to homomorphisms between any compact universal algebras of a *finite* type. In this note we

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prove such a generalization for compact universal algebras of an *arbitrary* similarity type (signature). Finally, we state separately the particular version of this result for metrizable algebras and quote a family of counterexamples from [8] showing that one cannot resign on the continuity control.

1. The stability theorem

Our standard references for universal algebra and general topology are the books [3] by G. Grätzer and [1] by R. Engelking, respectively.

Under the term *universal algebra*, or just *algebra* we always mean an algebra of the same fixed but otherwise arbitrary similarity type, with the set of (finitary) operation symbols denoted by F. However, unlike in [3], we denote the algebra $\left(A, f^A\right)_{f\in F}$ by the same character as its underlying set A. A *topological (universal) algebra* A is usually defined as an algebra endowed with a topology making all the operations $f^A\colon A^n \to A$, $f \in F$, continuous. We additionally include the Hausdorff or T_2 separation property into this definition. A topological algebra A is called *compact* or *completely regular* if the topological space A is compact or completely regular, respectively.

If A is a completely regular topological algebra then the topology of A can be induced by a uniformity $\mathcal{U}$ on A. However, the operations $f^A\colon A^n \to A$ are neither explicitly required nor need to be (unless A is compact) uniformly continuous with respect to $\mathcal{U}$.

Definition 1. Let A, B be topological algebras of the same type, such that the topology of B is induced by a uniformity $\mathcal{U}$ on B. Given an entourage $U \in \mathcal{U}$, a function $g\colon A \to B$ is called a *U-homomorphism* if for each n-ary operation symbol $f \in F$ and all $a_1, \dots, a_n \in A$ we have

$$\left(gf^A(a_1, \dots, a_n),\ f^B(ga_1, \dots, ga_n)\right) \in U.$$

Two functions $g, h\colon A \to B$ are U-close if $\left(g(a), h(a)\right) \in U$ for each $a \in A$. The pair (A, B) is said *to have stable homomorphisms with respect to $\mathcal{U}$* if for each $V \in \mathcal{U}$ there exists a $U \in \mathcal{U}$ such that for every continuous U-homomorphisms $g\colon A \to B$ one can find a continuous homomorphism $h\colon A \to B$ such that g is V-close to h.

The answer to the stability problem is negative in general. A counterexample, comprising an infinite family of pairs of compact metrizable abelian groups (A, B) none of which has stable homomorphisms, will be presented in the final part of the next section, devoted to the metrizable case.

Thus in order to get some positive results, some additional assumptions are unavoidable. One possibility consists in introducing a kind of control over the continuity of functions $A \to B$.

Definition 2. Let X and Y be two topological spaces such that the topology of Y is induced by a uniformity $\mathcal{U}$ on Y with a basis $\mathcal{U}_0 \subseteq \mathcal{U}$. An $(X,Y,\mathcal{U})$-*continuity scale* is any mapping Γ assigning to each pair $(x,U) \in X \times \mathcal{U}_0$ a neighborhood $\Gamma(x,U)$ of the point $x \in X$. Then a function $g\colon X \to Y$ is said to be Γ*-continuous* if for all $U \in \mathcal{U}_0$ and $x,y \in X$ the condition $y \in \Gamma(x,U)$ implies $\bigl(g(x), g(y)\bigr) \in U$.

Obviously, a Γ-continuous function $g\colon X \to Y$ is continuous. The point is that any family of Γ-continuous function $g\colon X \to Y$ already is *equicontinuous.* The other way round, using the axiom choice one can easily show that any equicontinuous family of functions $X \to Y$ is Γ-continuous with respect to some $(X,Y,\mathcal{U})$-continuity scale Γ.

Definition 3. Let A, B be topological algebras such that the topology of B is induced by a uniformity $\mathcal{U}$ and Γ be an $(A,B,\mathcal{U})$-continuity scale. The pair (A,B) is said *to have stable homomorphisms with respect to the continuity scale* Γ if for each $V \in \mathcal{U}$ there exists a $U \in \mathcal{U}$ such that for every Γ-continuous U-homomorphism $g\colon A \to B$ one can find a continuous homomorphism $h\colon A \to B$ such that g is V-close to h.

Now, everything is ready to state and prove the announced stability theorem.

Theorem 1. *Let A, B be compact topological algebras of the same similarity type. Then the pair (A,B) has stable homomorphisms with respect to every $(A,B,\mathcal{U})$-continuity scale Γ, where $\mathcal{U}$ is the (unique) uniformity inducing the topology of B.*

Proof. Assume the contrary and fix some compact topological algebras A, B, a uniformity $\mathcal{U}$ on B with a basis $\mathcal{U}_0$, an $(A,B,\mathcal{U})$-continuity scale Γ with domain $A \times \mathcal{U}_0$ and an entourage $V \in \mathcal{U}$ witnessing it.

Let D denote the set of all functions $g\colon A \to B$ such that g is not V-close to any continuous homomorphism $h\colon A \to B$. For $U \in \mathcal{U}$ denote by E_U the set of all Γ-continuous U-homomorphisms $g\colon A \to B$. Obviously, $E_U \subseteq E_{U'}$ for any $U \subseteq U'$ in $\mathcal{U}$. By the assumption, $D \cap E_U \neq \emptyset$ for each $U \in \mathcal{U}$. Then $D \cap E_U$ is an equicontinuous family of functions $A \to B$, and — as B is compact — the set

$$\{g(x) \mid g \in D \cap E_U\} \subseteq B$$

is relatively compact in B for each $x \in A$. Hence, by the Arzelà-Ascoli theorem, the closure

$$H_U = \mathrm{cl}(D \cap E_U)$$

is a compact subset of the space $\mathcal{C}(A,B)$ of all continuous functions $A \to B$ with the compact-open topology (which, by the compactness of A, coincides with the topology of uniform convergence on A). As $\emptyset \neq H_U \subseteq H_{U'}$ for $U \subseteq U'$ in $\mathcal{U}$, the intersection

$$H = \bigcap_{U \in \mathcal{U}} H_U$$

is nonempty, as well. Take any $h \in H$; it obviously is a continuous function $A \to B$. (Moreover, if all the entourages in the basis $\mathcal{U}_0$ are closed as subsets of the topological space $B \times B$, which can be assumed without loss of generality, then h is even Γ-continuous.)

We show that h is a homomorphisms. Take any n-ary operation symbol $f \in F$, $a_1, \dots, a_n \in A$, and a symmetric entourage $U \in \mathcal{U}$. By the continuity of f^B one can find a symmetric $U' \in \mathcal{U}$ such that $U' \subseteq U$ and

$$\big(f^B(ha_1, \dots, ha_n),\ f^B(b_1, \dots, b_n)\big) \in U$$

for all $b_1, \dots, b_n \in B$ satisfying $\big(h(a_i), b_i\big) \in U'$. As $h \in H_U$, there is a $g \in D \cap E_U$ such that $\big(h(x), g(x)\big) \in U'$ for each $x \in A$. Hence, in particular,

$$\big(hf^A(a_1, \dots, a_n),\ gf^A(a_1, \dots, a_n)\big) \in U'.$$

Since g is an U-homomorphism,

$$\big(gf^A(a_1, \dots, a_n),\ f^B(ga_1, \dots, ga_n)\big) \in U.$$

Finally, as $\big(h(a_i), g(a_i)\big) \in U'$ for $i \leq n$, we have

$$\big(f^B(ga_1, \dots, ga_n),\ f^B(ha_1, \dots, ha_n)\big) \in U.$$

Consequently,

$$\big(hf^A(a_1, \dots, a_n),\ f^B(ha_1, \dots, ha_n)\big) \in U' \circ U \circ U \subseteq U^3.$$

As the entourages of the form U^3, with $U \in \mathcal{U}$ symmetric, form a basis of the Hausdorff uniformity $\mathcal{U}$, we get the homomorphy condition

$$f^B(ha_1, \dots, ha_n) = hf^A(a_1, \dots, a_n).$$

Choose a symmetric $W \in \mathcal{U}$ such that $W^4 \subseteq V$. We will show that for any continuous homomorphism $\varphi\colon A \to B$ there is an $x \in A$ such that $\big(h(x), \varphi(x)\big) \notin W$. Assume this is not the case, i.e., there exists a continuous homomorphism $\varphi\colon A \to B$ which is W-close to h. Then for any $U \in \mathcal{U}$, $U \subseteq V$, there is a $g_U \in D \cap E_U$ such that $\big(g_U(x), h(x)\big) \in U$ for all $x \in A$, and an $x_U \in A$ such that $\big(g_U(x_U), \varphi(x_U)\big) \notin V$. As A is compact, there is a basis $\mathcal{U}_1 \subseteq \mathcal{U}$ such that $U \subseteq V$ for each $U \in \mathcal{U}_1$ and the net $(x_U)_{U \in \mathcal{U}_1}$ converges to a point $x \in A$. By the continuity of the functions h and φ there is a neighborhood $N \subseteq A$ of x such that for any $y \in N$ we have $\big(h(x), h(y)\big) \in W$, as well as $\big(\varphi(x), \varphi(y)\big) \in W$. Then there is a $U_1 \in \mathcal{U}_1$ such that for each $U \in \mathcal{U}_1$ the condition $U \subseteq U_1$ implies $x_U \in N$. Choose any $U \in \mathcal{U}_1$ such that $U \subseteq U_1 \cap W$. Then

$$\begin{aligned} \big(g_U(x_U), h(x_U)\big) &\in U, \\ \big(h(x_U), h(x)\big) &\in W, \\ \big(h(x), \varphi(x)\big) &\in W, \\ \big(\varphi(x), \varphi(x_U)\big) &\in W, \end{aligned}$$

hence

$$\big(g_U(x_U), \varphi(x_U)\big) \in U \circ W \circ W \circ W \subseteq W^4 \subseteq V,$$

and we have a contradiction.

As h itself is a continuous homomorphisms $A \to B$, it follows that in particular h is not W-close to h. This contradiction concludes the proof of the theorem. □

2. The metrizable case

Assume that the topologies of A and B stem from metrics ρ and σ, respectively. Then for any strictly decreasing sequence $(\eta_k)_{k\in\mathbb{N}}$ of positive reals converging to 0, the relations $\big\{(u,v) \in B \times B \mid \sigma(u,v) \le \eta_k\big\}$ form a basis of the uniformity $\mathcal{U}_\sigma$ on B. Due to compactness of A the notions of continuity and uniform continuity coincide for functions $A \to B$. Hence a closed and symmetric $(A, B, \mathcal{U}_\sigma)$-continuity scale can be represented as a sequence $\Gamma = \big((\gamma_k, \eta_k)\big)_{k\in\mathbb{N}}$, where $(\gamma_k)_{k\in\mathbb{N}}$ is a decreasing sequence of positive reals. It is more natural to call the sequence Γ a (ρ, σ)-*continuity scale* in this case. Then a function $g\colon A \to B$ is called Γ-*continuous* if for each k and all $x, y \in A$ the condition $\rho(x,y) \le \gamma_k$ implies $\sigma(g(x), g(y)) \le \eta_k$.

For completeness' sake we still quote the obvious translations of the intuitive concepts of closeness and almost homomorphy into the metric terms. Let $\varepsilon > 0$. Two functions $g, h\colon A \to B$ are said to be ε-*close* if $\sigma(g(a), h(a)) \le \varepsilon$ for each $a \in A$; a function $g\colon A \to B$ is called an ε-*homomorphism* if for each n-ary operation symbol $f \in F$ and all $a_1, \dots, a_n \in A$ we have

$$\sigma\big(gf^A(a_1, \dots, a_n),\ f^B(ga_1, \dots, ga_n)\big) \le \varepsilon.$$

For metrizable algebras the stability theorem 1 can be stated in the following more usual form.

Theorem 2. *Let A, B be compact topological algebras of the same similarity type, ρ and σ be two metrics inducing the topology of A and B, respectively, and $\Gamma = \big((\gamma_k, \eta_k)\big)_{k\in\mathbb{N}}$ be a (ρ, σ)-continuity scale. Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that every Γ-continuous δ-homomorphism $g\colon A \to B$ is ε-close to a continuous (even Γ-continuous) homomorphism $h\colon A \to B$.*

Let us close with the announced counterexample, showing that one cannot even prove the stability of continuous homomorphisms between compact metrizable abelian groups, unless some additional assumptions are fulfilled. In particular, one cannot get rid of mentioning the continuity scale Γ in theorems 1 and 2. The construction is taken from [8]; it is based on an example from [6], forming its initial part.

Example. Let p be an arbitrary prime and $\mathbb{Z}_p$ denote the compact metric abelian group of p-adic integers, i.e., the completion of the ring $\mathbb{Z}$ with respect to the norm

$$|a|_p = p^{-o_p(a)},$$

where $p^{o_p(a)}$ is the highest power of p dividing the integer $a \neq 0$, and $|0|_p = 0$. Mapping the remainder $x \in \{0, 1, \ldots, p^n - 1\}$ mod p^n onto the corresponding integer $g_n(x) = x \in \mathbb{Z} \subseteq \mathbb{Z}_p$ defines a p^{-n}-homomorphisms of the finite cyclic group $\mathbb{Z}/(p^n)$ into $\mathbb{Z}_p$ for every $n \in \mathbb{N}$. Indeed, the difference $g_n(x) + g_n(y) - g_n(x+y)$ is either 0 or p^n for any $x, y \in \mathbb{Z}/(p^n)$, hence

$$\left|g_n(x) + g_n(y) - g_n(x+y)\right|_p \leq |p^n|_p = p^{-n}.$$

However, as $\mathbb{Z}_p$ is torsionfree, there is no homomorphism $\mathbb{Z}/(p^n) \to \mathbb{Z}_p$ except for the trivial one.

The direct product $A_p = \prod_{n\in\mathbb{N}} \mathbb{Z}/(p^n)$ with the product topology is a compact metrizable abelian group; denote by $\pi_n\colon A_p \to \mathbb{Z}/(p^n)$ the projection onto the nth factor and $\iota_n\colon \mathbb{Z}/(p^n) \to A_p$ its embedding into the product. Then $g_n \circ \pi_n\colon A_p \to \mathbb{Z}_p$ is a continuous p^{-n}-homomorphisms, however, for each homomorphism $h\colon A_p \to \mathbb{Z}_p$ we have

$$\begin{aligned}\sup_{u\in A_p} \left|(g_n \circ \pi_n)(u) - h(u)\right|_p &\geq \max\nolimits_{x\in\mathbb{Z}/(p^n)} \left|g_n(x) - (h \circ \iota_n)(x)\right|_p \\ &= \max\nolimits_{0\leq x<p^n} |x|_p = 1,\end{aligned}$$

since the homomorphism $h \circ \iota_n\colon \mathbb{Z}/(p^n) \to \mathbb{Z}_p$ necessarily is constantly 0. We can conclude that, for any prime p, the pair $(A_p, \mathbb{Z}_p)$ of compact metrizable abelian groups does not have stable homomorphisms.

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