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GENERATED FUZZY IMPLICATIONS AND KNOWN CLASSES OF IMPLICATIONS

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Abstract. In MV-logic we use a mapping $I : [0,1]^2 \to [0,1]$, called a fuzzy implication, which is a monotonous extension of classical implication on the unit interval. In this paper we deal with one of possible extensions of classical implication. Our implications are generated. Some properties of these implications have already been given in [7]. Well-known classes of implications are (S,N)-implications and R-implications. Some connections between class of our generated implications on one side, and (S,N) and R-implications on the other side will be given. The aim of this paper is to study their properties and to investigate connections between mentioned classes.

1. Preliminaries

We briefly recall definitions and properties of the most important connectives in MV-logic.

Definition 1.1. *A unary operator $n : [0,1] \to [0,1]$ is called a fuzzy negation if, for any $x, y \in [0,1]$,*

- $x < y \Rightarrow n(y) \le n(x)$,
- $n(0) = 1, n(1) = 0$.

The negation n is called a *strict negation* if and only if the mapping n is continuous and strictly decreasing. A strict negation is strong if it is an involution.

Example 1.2. *The following are some examples of fuzzy negations:*

- $N_s(x) = 1-x$ — *strong negation, standard negation,*
- $n(x) = 1 - x^2$ — *strict, not strong negation,*
- $n(x) = \sqrt{1 - x^2}$ — *strong negation,*
- $N_{G_1}(1) = 0, N_{G_1}(x) = 1$ *if* $x < 1$ — *non-continuous, greatest, Gödel negation,*
- $N_{G_2}(0) = 1, N_{G_2}(x) = 0$ *if* $x > 0$ — *non-continuous, smallest, dual Gödel negation.*

Note that the dual negation based on a negation n is given by $n^d(x) = 1-n(1-x)$.

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Definition 1.3. *A non-decreasing mapping* $C : [0,1]^2 \to [0,1]$ *is called a conjunctor if, for any* $x, y \in [0,1]$, *it holds*

- $C(x,y) = 0$ *whenever* $x = 0$ *or* $y = 0$,
- $C(1,1) = 1$.

Commonly used conjunctors in MV-logic are the triangular norms.

Definition 1.4. *A triangular norm (t-norm for short) is a binary operation on the unit interval* $[0,1]$, *i.e., a function* $T : [0,1]^2 \to [0,1]$ *such that for all* $x, y, z \in [0,1]$, *the following four axioms are satisfied:*
(T1) Commutativity $T(x,y) = T(y,x)$,
(T2) Associativity $T(x, T(y,z)) = T(T(x,y), z)$,
(T3) Monotonicity $T(x,y) \leq T(x,z)$ *whenever* $y \leq z$,
(T4) Boundary Condition $T(x,1) = x$.

Remark 1.5. *Note that the dual operator to the conjunctor* C, *defined by* $D(x,y) = 1 - C(1-x, 1-y)$ *is called the disjunctor. Equivalently, a disjunctor can be defined as a non-decreasing mapping* $D : [0,1]^2 \to [0,1]$ *such that* $D(x,y) = 1$ *whenever* $x = 1$ *or* $y = 1$ *and* $D(0,0) = 0$. *Commonly used disjunctors in MV-logic are the triangular conorms. A triangular conorm (also called a* $t-$*conorm) is a binary operation* S *on the unit interval* $[0,1]$ *which, for all* $x, y, z \in [0,1]$, *satisfies* $(T1) - (T3)$ *and* $(S4)$ $S(x,0) = x$. *The original definition of* $t-$*conorms given in* [8] *is completly equivalent to the previous axiomatic definition, where the* $t-$*conorm is based on a given* $t-$*norm* T *by formula*

$$S(x,y) = 1 - T(1-x, 1-y).$$

For more information, see [4].

In the literature, we can find several different definitions of fuzzy implications. In this paper we will use the following one, which is equivalent to the definition introduced by Fodor and Roubens in [3]. The readers can obtain more information in [2] and [5].

Definition 1.6. *A function* $I : [0,1]^2 \to [0,1]$ *is called a fuzzy implication if it satisfies the following conditons:*

(I1) I *is decreasing in its first variable,*
(I2) I *is increasing in its second variable,*
(I3) $I(1,0) = 0$, $I(0,0) = I(1,1) = 1$.

Now, we recall definitions of some important properties of implications, which we will investigate.

Definition 1.7. *A fuzzy implication* $I : [0,1]^2 \to [0,1]$ *satisfies:*

(NP) *the left neutrality property, or is called left neutral, if*

$$I(1, y) = y;\ \ y \in [0, 1],$$

(EP) *the exchange principle if*

$$I(x, I(y, z)) = I(y, I(x, z)) \text{ for all } x, y, z \in [0, 1],$$

(IP) *the identity principle if*

$$I(x, x) = 1;\ \ x \in [0, 1],$$

(OP) *the ordering property if*

$$x \leq y \iff I(x, y) = 1;\ \ x, y \in [0, 1],$$

(CP) *the contrapositive symmetry with respect to a given negation n if*

$$I(x, y) = I(n(y), n(x));\ \ x, y \in [0, 1].$$

Definition 1.8. *Let $I : [0, 1]^2 \to [0, 1]$ be a fuzzy implication. The function N_I defined by $N_I(x) = I(x, 0)$ for all $x \in [0, 1]$, is called the natural negation of I.*

One of well-known classes of implications is represented by (S, N)-implications, which are based on given t-conorm and negation N.

Definition 1.9. *A function $I : [0, 1]^2 \to [0, 1]$ is called an $(S, N)-$implication if there exist a t-conorm S and fuzzy negation N such that*

$$I(x, y) = S(N(x), y),\ \ x, y \in [0, 1].$$

If N is a strong negation, then I is called a strong implication.

The following characterization of $(S, N)-$implications is from [1].

Theorem 1.10. *(Baczyňski and Jayaram* [1], *Theorem 5.1) For a function I : $[0, 1]^2 \to [0, 1]$, the following statements are equivalent:*

- *I is an (S, N)-implication generated from some t-conorm and some continuous (strict, strong) fuzzy negation N.*
- *I satisfies (I2), (EP) and N_I is a continuous (strict, strong) fuzzy negation.*

Another way of extending the classical binary implication operator to the unit interval $[0, 1]$ uses the *residuation* I with respect to a left-continuous triangular norm T

$$I(x, y) = \max\{z \in [0, 1]; T(x, z) \leq y\}.$$

The following characterization of $R-$implications is from [3].

Theorem 1.11. *(Fodor and Roubens* [3], *Theorem 1.14) For a function I : $[0, 1]^2 \to [0, 1]$, the following statements are equivalent:*

- *I is an R-implication based on some left-continuous t-norm T.*

- *I satisfies (I2), (OP), (EP), and $I(x,.)$ is a right-continuous for any $x \in [0,1]$.*

Our constructions of implications will make use extensions of the classical inverse of function. One way of extending is described in next definitions.

Definition 1.12. *Let $\varphi : [0,1] \to [0,\infty]$ be a non-decreasing function. The function $\varphi^{(-1)}$ which is defined by*

$$\varphi^{(-1)}(x) = \sup\{z \in [0,1]; \varphi(z) < x\},$$

is called the pseudo-inverse of the function φ, with the convention $\sup \emptyset = 0$.

Definition 1.13. *Let $f : [0,1] \to [0,\infty]$ be a non-increasing function. The function $f^{(-1)}$ which is defined by*

$$f^{(-1)}(x) = \sup\{z \in [0,1]; f(z) > x\},$$

is called the pseudo-inverse of the function f, with the convention $\sup \emptyset = 0$.

One of main contributions of our paper are, in fact, corollaries of the following technical result.

Proposition 1.14. *Let c be a positive real number. Then the pseudo-inverse of a positive multiple of any monotone function $f : [0,1] \to [0,\infty]$ satisfies*

$$(c \cdot f)^{(-1)}(x) = f^{(-1)}\left(\frac{x}{c}\right).$$

Proof. Let f be a non-decreasing function. Then

$$f^{(-1)}(x) = \sup\{z \in [0,1]; f(z) < x\}$$

and

$$(c \cdot f)^{(-1)}(x) = \sup\{z \in [0,1]; c \cdot f(z) < x\} =$$
$$= \sup\left\{z \in [0,1]; f(z) < \frac{x}{c}\right\} = f^{(-1)}\left(\frac{x}{c}\right).$$

Now, the proof for the case of non-increasing function is analogous. □

2. New generated implications

It is well-known that it is possible to generate t-norms from one variable functions. It means it is enough to consider one variable function instead of two variable function. Moreover, we can generate implications in a similar way as t-norms. One of these possibilities is described in the next theorem and example.

Theorem 2.1. *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function such that $f(1) = 0$. Then the function $I_f(x,y) : [0,1]^2 \to [0,1]$ which is given by*

$$I_f(x,y) = \begin{cases} 1 & \text{if } x \le y, \\ f^{(-1)}(f(y^+) - f(x)) & \text{otherwise,} \end{cases}$$

where $f(y^+) = \lim\limits_{y \to y^+} f(y)$ and $f(1^+) = f(1)$ is a fuzzy implication.

Proof. We proceed by the points of the Definition 1.6.

(I1)
- Let $x_1, x_2, y \in [0,1]$ and $x_1 \le x_2$ and $x_1 \ge y$. Function f is decreasing and therefore $f(x_1) \ge f(x_2)$ and $f(y^+) - f(x_1) \le f(y^+) - f(x_2)$. Pseudoinverse $f^{(-1)}$ of function f is decreasing, too, and $f^{(-1)}(f(y^+) - f(x_1)) \ge f^{(-1)}(f(y^+) - f(x_2))$. Therefore $I_f(x_1, y) \ge I_f(x_2, y)$ and it means that the function I_f is decreasing in its first variable.
- If $x_1 \le y \le x_2$, then $I_f(x_1, y) = 1$ and $I_f(x_2, y) \le 1$.
- If $x_1 \le x_2 \le y$, then $I_f(x_1, y) = I_f(x_2, y) = 1$.

(I2)
- Let $x, y_1, y_2 \in [0,1]$ and $y_1 \le y_2$ and $x \ge y_2$. Function f is decreasing and therefore $f(y_1^+) \ge f(y_2^+)$ and $f(y_1^+) - f(x) \ge f(y_2^+) - f(x)$. Pseudoinverse $f^{(-1)}$ of function f is decreasing too and $f^{(-1)}\big(f(y_1^+) - f(x)\big) \le f^{(-1)}\big(f(y_2^+) - f(x)\big)$. Therefore $I_f(x, y_1) \le I_f(x, y_2)$ and this means that the function I_f is increasing in its second variable.
- If $y_1 \le x \le y_2$, then $I_f(x, y_2) = 1$ and $I_f(x, y_1) \le 1$.
- If $x \le y_1 \le y_2$, then $I_f(x, y_1) = I_f(x, y_2) = 1$.

(I3) From the formula for function I_f we get $I_f(0,0) = I_f(1,1) = 1$ and for $I_f(1,0)$ we have

$$I_f(1,0) = f^{(-1)}\big(f(0^+) - f(1)\big) = f^{(-1)}(f(0^+)) = \sup\{z \in [0,1] f(z) > f(0)\} = 0.$$

□

For illustration we introduce some examples of generated implications.

Example 2.2. *Let $f_1, f_2, f_3 : [0,1] \to [0,\infty]$ be functions defined as follows:*

- $f_1(x) = \begin{cases} 1 - x & \text{if } x \le 0.5, \\ 0.5 - 0.5x & \text{otherwise,} \end{cases}$
- $f_2(x) = \frac{1}{x} - 1,$
- $f_3(x) = -\ln(x).$

Note, that all three functions are decreasing. For $f_1^{(-1)}, f_2^{(-1)}, f_3^{(-1)}$, we get:

- $f_1^{(-1)}(x) = \begin{cases} 1 - 2x & \text{if } x \le 0.25, \\ 0.5 & \text{if } 0.25 < x \le 0.5, \\ 1 - x & \text{otherwise,} \end{cases}$

- $f_2^{(-1)}(x) = \min\left\{\frac{1}{1+x}, 1\right\}$,
- $f_3^{(-1)}(x) = \min\{e^{-x}, 1\}$.

For our functions f_1, f_2, f_3 we get

- $I_{f_1}(x,y) = \begin{cases} 1 & \text{if } x \le y, \\ 1 - 2x + 2y & \text{if } x \le 0.5, y < 0.5, x - y \le 0.25, x > y, \\ 0.5 & \text{if } x \le 0.5, y < 0.5, x - y > 0.25, \\ 0.5 & \text{if } x > 0.5, y < 0.5, x \le 2y, \\ 0.5 + y - 0.5x & \text{if } x > 0.5, y < 0.5, x > 2y, \\ 1 - x + y & \text{if } x > 0.5, y \ge 0.5, \end{cases}$
- $I_{f_2}(x,y) = \begin{cases} 1 & \text{if } x \le y, \\ \frac{1}{\frac{1}{y} - \frac{1}{x} + 1} & \text{otherwise}, \end{cases}$
- $I_{f_3}(x,y) = \begin{cases} 1 & \text{if } x \le y, \\ \frac{y}{x} & \text{otherwise}. \end{cases}$

Remark 2.3. *All three implications satisfy IP, NP and OP. Note that I_{f_3} is the well-known Goguen implication.*

3. Properties of I_f implications

In this section we investigate the properties of I_f implications. We turn our attention to relations with (S, N) and R implications and our implications. Directly from Definition 1.7 and the following equivalence for strictly decreasing function f

$$f^{(-1)}(x_0) = 1 \iff x_0 \le \lim_{x \to 1^-} f(x) = f(1^-),$$

we get the condition for NP. The part concerning OP is explained in subsequent example.

Proposition 3.1. *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function such that $f(1) = 0$. Then I_f satisfies IP and NP. Moreover, f is continuous in $x = 1$ if and only if I_f satisfies OP.*

The meaning of continuity of the function f in $x = 1$ is introduced in the next example.

Example 3.2. *Let us have function $f : [0,1] \to [0,1]$ given by*

$$f(x) = \begin{cases} 1 - \frac{x}{2} & x \in [0, 1[, \\ 0 & x = 1. \end{cases}$$

Pseudoinverse $f^{(-1)} : [0,1] \to [0,1]$ *will be given by*

$$f^{(-1)}(x) = \begin{cases} 1 & x \leq 0.5, \\ 2 - 2x & x \in]0.5, 1]. \end{cases}$$

Implication $I_f : [0,1]^2 \to [0,1]$ *will be given by*

$$I_f(x,y) = \begin{cases} y & x = 1, \\ 1 & \text{otherwise.} \end{cases}$$

For this implication it holds $I_f(0.5, 0.4) = 1$*. Therefore* I_f *doesn't have OP. It is due to the fact that* $f^{(-1)}(x) = 1$ *for some* $x > 0$*, which is a consequence of violation of continuity of* f *at* $x = 1$*. From continuity in* $x = 1$ *we have* $f^{(-1)}(x) = 1$ *only for* $x = 0$ *and from strictly decreasing function* f *we have* $f(y^+) - f(x) = 0$ *only for* $x = y$*, where* $x, y \in [0,1]$*. It means that continuity in* $x = 1$ *is equivalent with OP for implication* I_f*.*

Continuity of strictly decreasing function f implies that $f \circ f^{(-1)}(x) = x$. Therefore we get for EP and CP the propositions.

Proposition 3.3. *Let* $f : [0,1] \to [0,\infty]$ *be a continuous strictly decreasing function such that* $f(1) = 0$*. Then the implication* I_f *satisfies EP.*

Proof. Since f is continuous function on $[0,1]$ and by definition $f(1^+) = f(1)$, we have $f(z^+) = f(z)$ $\forall z \in [0,1]$. Also $\forall z \in [0,1] : f^{(-1)} \circ f(z) = z$ and $\forall z \in [0, f(0)] : f \circ f^{(-1)}(z) = z$.

- Let $x > z$ and $y > z$. Then

$$f(I_f(y,z)) = f(f^{(-1)}(f(z) - f(y))) = f(z) - f(y),$$

  and

$$I_f(x, I_f(y,z)) = \begin{cases} 1 & x \leq I_f(y,z), \\ f^{(-1)}(f(z) - f(y) - f(x)) & \text{otherwise.} \end{cases}$$

  Analogously

$$I_f(y, I_f(x,z)) = \begin{cases} 1 & y \leq I_f(x,z), \\ f^{(-1)}(f(z) - f(y) - f(x)) & \text{otherwise.} \end{cases}$$

  If $x \leq I_f(y,z)$, then $f(x) \geq f(f^{(-1)}(f(z) - f(y)))$ and then $f(x) \geq f(z) - f(y)$ or equivalently $f(y) \geq f(z) - f(x)$. The last inequality implies $y \leq I_f(x,z)$ and it means $I_f(x, I_f(y,z)) = I_f(y, I_f(x,z))$
- Let $x > z$ and $y \leq z$. It is clear that $I_f(x, I_f(y,z)) = I_f(x,1) = 1$. Since f is decreasing function f, we have $f(y) \geq f(z) - f(x)$ and this implies $y \leq f^{(-1)}(f(z) - f(x))$. From previous inequality and the formula for I_f we get $I_f(y, I_f(x,z)) = I_f(y, f^{(-1)}(f(z) - f(x)) = 1$.

- The proof is very similar to the previous point for $x \leq z$ and $y > z$.
- Let $x \leq z$ and $y \leq z$. Then obviously $I_f(x, I_f(y,z)) = I_f(y, I_f(x,z)) = 1$.

□

Remark 3.4. *We study the properties of implications I_f under which they are $(S,N)-$ or $R-$ implications. Because there are relations between $(S,N)-$ implications and EP and continuity of N_{I_f}, the previous proposition leads us to dealing with continuous function f. Continuity of function f implies continuity of natural negation based on I_f. Moreover for continuous and bounded strictly decreasing function f such that $f(1) = 0$ and $f(0) = c$ the natural negation N_{I_f} is strong.*

Proposition 3.5. *Let $f : [0,1] \to [0,c]$ be a continuous bounded decreasing function such that $f(1) = 0$. The I_f possess CP only with respect to its natural negation $N_{I_f}(x) = f^{-1}(f(0) - f(x))$.*

Proof. Let $f : [0,1] \to [0,c]$ be a continuous bounded decreasing function, such that $f(1) = 0$ and $N_{I_f}(x) = f^{-1}(f(0) - f(x))$. Since we deal with classical inverse function, we have

$$\forall x \in [0,1]; f(N_{I_f}(x)) = f(0) - f(x),$$

and therefore

$$\forall x, y \in [0,1]^2; f(N_{I_f}(x)) - f(N_{I_f}(y)) = f(y) - f(x).$$

Since f is continuous, we get $f(y^+) = f(y)$. Since N_{I_f} is strictly decreasing, the conditions $x < y$ and $N_{I_f}(x) > N_{I_f}(y)$ are equivalent and

$$I_f(x,y) = \begin{cases} f^{-1}(f(y) - f(x)) & x > y, \\ 1 & \text{othervise.} \end{cases}$$

Therefore I_f possess CP. Let I_f possess CP w.r. to $n(x)$. We have:

$$I_f(x,0) = \begin{cases} 1 & x = 0, \\ f^{-1}(f(0) - f(x)) & \text{otherwise,} \end{cases}$$

and

$$I_f(1, n(x)) = \begin{cases} 1 & n(x) = 1, \\ f^{-1}(f(n(x))) & \text{otherwise.} \end{cases}$$

Since $I_f(1, n(x)) = I_f(x, 0)$, we have that $n(x) = f^{-1}(f(n(x))) = f^{-1}(f(0) - f(x)) = N_{I_f}(x)$ for all $x > 0$ and $n(0) = 1$. □

Remark 3.6. *This proposition is a corollary of Proposition 2.5.28 of* [2], *since continuity of function f implies that we have $R-$implication (Theorem 1.16 from* [3]*).*

It is well known that generators of continuous Archimedean t-norms are unique up to a positive multiplicative constant, and this is also true for the f generators of I_f implications. The next theorem is a corollary of Proposition 1.14.

Theorem 3.7. *Let c be a positive constant and $f : [0,1] \to [0,\infty]$ be a strictly decreasing function. Then the implications I_f and $I_{c\cdot f}$ which are based on functions f and $c \cdot f$ are identical.*

Proof.
- Let $x, y \in [0,1], x \leq y$ and c be a positive real number. From Theorem 2.1 we get $I_{c\cdot f}(x,y) = I_f(x,y) = 1$.
- Let $x, y \in [0,1], x > y$ and c be a positive real number. Then from Theorem 2.1 and Proposition 1.14 we get

$$I_{c\cdot f}(x,y) = (c \cdot f)^{(-1)} \big((c \cdot f)(y^+) - (c \cdot f)(x)\big) =$$

$$= f^{(-1)} \left(\frac{(c \cdot f)(y^+) - (c \cdot f)(x)}{c} \right) = f^{(-1)} \big(f(y^+) - f(x)\big) = I_f(x,y).$$

□

The last theorem of this sections is corollary of previous propositions and Theorems 1.10 and 1.11.

Theorem 3.8. *Let $f : [0,1] \to [0,\infty]$ be a continuous strictly decreasing function such that $f(1) = 0$. Then I_f is an R-implication given by some left-continuous t-norm, and more if $f(0) < \infty$ then I_f is an (S,N)-implication, too.*

The full characterization of $f-$ generated fuzzy implications is yet unknown, and is significant enough to merit attention. Our future endeavors will be along these lines. Note that similar problems relating $QL-$implications and $R-$ and $(S,N)-$ implications were recently studied in [6].

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