Acta Universitatis Matthiae Belii ser. Mathematics, 16 (2009), 81–89.
Received: 7 July 2009, Accepted: 2 February 2010.

DECOMPOSITION AND PROJECTIVITY OF QUANTALE MODULES

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ABSTRACT. We prove that every quantale module join-generated by its subset of join-irreducible elements can be uniquely decomposed into a collection of further indecomposable submodules. This decomposition actually corresponds to the direct product when the module is "sufficiently distributive". After showing that regular projective indecomposable modules over a given quantale Q are isomorphic to Qd for an idempotent $d \in Q$, we characterize regular projective essential modules that admit this product decomposition as products of such cyclic modules.

Outside the original area of modules over unital rings, the concept of projectivity has also been investigated for sets endowed with an action of a semigroup or a monoid (so called *S-acts*, see [5]), and their partially-ordered variants [13]. Because of similarity of quantale modules to these structures, projectivity suggests to be studied in their categories as well. A recent article [3] presents use of projective objects in study of equivalences of consequence relations on powersets of propositional formulas or sequents, which form unital modules over quantales of sets of substitutions.

In this paper we follow a part of the article [13] where decomposability and projectivity were studied in the category of *S-posets*, i.e., partially ordered sets equipped with an action of a partially ordered monoid that is compatible with the order relation. In accordance with the article, we first develop a little theory of decomposability, and then we apply it to obtain the main result. For the extension to the non-unital case, we make use of the article [2]. For facts on categories of quantales and quantale modules, the reader can refer to [12] and [6].

1. Preliminaries

Our base environment will be the category of *sup-lattices*. Its objects are complete lattices but morphisms include all join-preserving maps. The greatest and

2000 *Mathematics Subject Classification.* 18G05, 06F99.
Key words and phrases. quantale module, projective module, projectivity, decomposition.

the least element of a sup-lattice are denoted by 1 and 0, respectively. A *quantale* stands for a sup-lattice endowed with associative binary multiplication '$\cdot$' distributing over arbitrary joins in both operands. Quantales possessing a multiplicative unit, denoted by e, are called *unital*. Quantale homomorphisms are then sup-lattice homomorphisms preserving multiplication as well.

Given a quantale Q, a *left Q-module M* means a sup-lattice with an associative left action of the quantale $\cdot\colon Q \times M \to M$ that distributes over joins in both components. Throughout this article, 'module' shall stand for a left module. When Q is a unital quantale and $e \cdot m = m$ for all $m \in M$, M is called *unital*, too. As we often consider all multiples of an element m by elements of a set $A \subseteq Q$, we denote by Am the set $\{a \cdot m \mid a \in A\}$, and, by analogy, $AN = \{a \cdot n \mid a \in A,\ n \in N\}$.

A *module homomorphism* f is a sup-lattice homomorphism satisfying $f(q \cdot m) = q \cdot f(m)$ for any $q \in Q$ and $m \in M$. A subset N of M is called a *submodule* if it is nonempty and closed under arbitrary joins and multiplication by elements of Q, while an *ideal* stands for a downward-closed sub-sup-lattice. A *submodule-ideal* will stand for an ideal that is a submodule as well. Submodule-ideals arise as principal downsets $\downarrow m$ given by elements m such that $1_Q \cdot m \leq m$.

When $A \subseteq M$ is closed under quantale action, the submodule of M join-generated by A shall be denoted by $\langle A \rangle$. A Q-module M satisfying $\langle QM \rangle = M$ is called *essential* [9]. The class of essential modules provides the setting for some of the results presented in this paper. In particular, unital modules belong to this class, as well as unital quantales and idempotent quantales, when one regards quantales as modules over themselves.

A nonzero element x of a lattice L is called *join-irreducible* if $x = a \vee b$ implies $x = a$ or $x = b$. To derive our results, we shall deal with modules that are join-generated by their sets of join-irreducible elements. Such lattices have been called *finitely spatial* by F. Wehrung [16]. This class includes, for instance, supercontinuous modules (see [4], Theorem I-3.16, an equivalent statement for completely distributive complete lattices and co-prime, i.e. join-prime elements).

An object P is *regular projective* when for a given regular epimorphism $g\colon A \to B$ any morphism $f\colon P \to B$ can be lifted to $h\colon P \to A$ satisfying $g \circ h = f$. In categories of quantale modules, regular epimorphisms are exactly surjective homomorphisms. As only regular projectivity is discussed in this article, it shall be called projectivity for short.

Two following propositions present well-known properties of projective modules [1, section 4.6].

Proposition 1. *For a Q-module P, the following conditions are equivalent:*

(1) *P is projective.*
(2) *Every epimorphism $f\colon R \to P$ splits, that is, a monomorphism $g\colon P \to R$ exists and satisfies $f \circ g = \mathrm{id}_P$.*

(3) *P is a retract of a free module.*

Proposition 2. *Let $P = \coprod_{i\in I} P_i$ be a coproduct of modules. Then P is projective iff each P_i is projective.*

In categories of sup-lattices and quantale modules, the notions of products and coproducts coincide, so we do not have to distinguish between them.

There exists a free Q-module over any set. Provided that Q is unital, the free module over a set X is Q^X with standard product ordering and componentwise action. For a non-unital quantale Q, the module $(\mathbf{2} \times Q)^X$ plays the role of the free object ($\mathbf{2}$ stands for the two-element quantale in which $1 \cdot 1 = 1$). Action of Q on such a module is given as follows:

$$q \cdot (b, m) = \begin{cases} (0, q \cdot m) & \text{if } b = 0, \\ (0, q \vee q \cdot m) & \text{if } b = 1. \end{cases}$$

This construction was presented in [8], an alternative formulation can be found in [6].

2. Decomposition of Modules

A module M is called *decomposable* if there exist two nontrivial submodule-ideals of M, A and B, $A \cap B = \{0\}$ that generate M as a sup-lattice by joins. Expressed using elements, there exist $a, b \in M$ satisfying $a \wedge b = 0$, $1_Q \cdot a \leq a$, $1_Q \cdot b \leq b$, and for any $m \in M$ it holds that $m = (m \wedge a) \vee (m \wedge b)$. If this does not happen, we say that M is *indecomposable*.

Lemma 3. *Let M be a Q-module and $m \in M$ be a join-irreducible element. Then $N = {\downarrow}\langle Qm \cup \{m\}\rangle = {\downarrow}((1_Q \cdot m) \vee m)$ is an indecomposable submodule-ideal.*

Proof. Obviously, N is an ideal because it is a lower set, and it is also a submodule since the quantale action is order-preserving. Suppose that N is decomposable, that is, there exist A and B, submodule-ideals of N, $A \cap B = \{0\}$ such that $m = a \vee b$ for some $a \in A$, $b \in B$. As m is join-irreducible, we have $m = a$ or $m = b$. Without loss of generality we can suppose $m = a$, thus $m \in A$, $Qm \subseteq A$, and $b = 0$. □

Lemma 4. *Let a module $M = \langle \bigcup_{i\in I} A_i\rangle = \langle \bigcup_{j\in J} B_j\rangle$ for two families $(A_i)_{i\in I}$, $(B_j)_{j\in J}$ of its submodule-ideals. Then for each $i \in I$, the submodule-ideal A_i equals $\langle \bigcup_{j\in J}(A_i \cap B_j)\rangle$.*

Proof. It is evident that $A_i \supseteq \langle \bigcup_{j\in J}(A_i \cap B_j)\rangle$. The converse inclusion also holds: if $a \in A_i$, it is a join of elements $b_k \in A_i$ where each b_k is contained in some B_j since $\bigcup_{j\in J} B_j$ generates the whole M. □

Lemma 5. *Let M_i, $i \in I$, be a family of indecomposable submodule-ideals of a finitely spatial module M satisfying $\bigcap_{i\in I} M_i \neq \{0\}$. Then the submodule $\downarrow\langle\bigcup_{i\in I} M_i\rangle$ is also an indecomposable submodule-ideal.*

Proof. Suppose $\downarrow\langle\bigcup_{i\in I} M_i\rangle$ is decomposable into A and B. Since a non-zero element, which is a supremum of join-irreducibles, belongs to $\bigcap_{i\in I} M_i$, and all M_i are lower sets, there certainly exists a join-irreducible element $m \in \bigcap_{i\in I} M_i$. If m could be written as $a \vee b$ for some $a \in A$ and $b \in B$, it would equal either a, or b. Again, without losing generality, if $m \in A$, $M_i \cap A \neq \{0\}$ for any i. By Lemma 4, $M_i = \langle(M_i \cap A) \cup (M_i \cap B)\rangle$. As all M_i are indecomposable, $M_i \cap B = \{0\}$, $M_i = \langle M_i \cap A\rangle \subseteq A$, therefore $\downarrow\langle\bigcup_{i\in I} M_i\rangle = A$. □

Theorem 6. *Every finitely spatial Q-module can be uniquely decomposed into a collection of its Q-submodule-ideals that are indecomposable and pairwise meeting in* 0 *only.*

Proof. We already know that $\downarrow\langle Qm \cup \{m\}\rangle$ is indecomposable when m is join-irreducible. Therefore, considering a join-irreducible element x, the set $D_x = \{N \mid x \in N,\ N \text{ is an indecomposable submodule-ideal}\}$ is nonempty, and $\bigcap_{N\in D_x} N \neq \{0\}$ because it contains x. From Lemma 5 it follows that the set $A_x = \downarrow\langle\bigcup_{N\in D_x} N\rangle$ is an indecomposable submodule-ideal.

Consider two join-irreducible elements $x \neq y$. Then either $A_x \cap A_y = \{0\}$, or $A_x = A_y$. This holds because if there exists $m \in A_x \cap A_y$, $m \neq 0$, also a join-irreducible element $n \leq m$ belongs to the intersection. Obviously $A_x \subseteq \downarrow\langle A_x \cup A_y\rangle$. Using Lemma 5 again, $\downarrow\langle A_x \cup A_y\rangle$ is downward-closed, indecomposable (since $A_x \cap A_y \neq \{0\}$), and containing y, so $\downarrow\langle A_x \cup A_y\rangle \subseteq A_y$ (as A_y includes all indecomposable submodule-ideals containing y). The converse inclusion can be shown in the same way.

We can therefore set an equivalence θ on join-irreducible elements of M as $x\theta y$ iff $A_x = A_y$. Since every element of M is a supremum of join-irreducibles, $M = \langle\bigcup_{x\in C} A_x\rangle$ where C is a suitable set of representatives of classes of θ.

Suppose there exist two such decompositions, so $M = \langle\bigcup_{i\in I} A_i\rangle = \langle\bigcup_{j\in J} B_j\rangle$, and consider one of the B_j. By Lemma 4, $B_j = \langle\bigcup_{i\in I}(A_i \cap B_j)\rangle$. For any $i \in I$ we then obtain a decomposition of B_j as $\langle(A_i \cap B_j) \cup \langle\bigcup_{l\neq i}(A_l \cap B_j)\rangle\rangle$. Let $b \in B_j$ be nonzero. It equals to the supremum of join-irreducibles a_k such that $a_k \leq b$ for all k. Pick any a_k of them and the submodule A_{i_k} that contains a_k. As B_j is indecomposable and $A_{i_k} \cap B_j$ is nontrivial, $\langle\bigcup_{l\neq i_k}(A_l \cap B_j)\rangle = \{0\}$, $B_j = \langle A_{i_k} \cap B_j\rangle$, and thus $B_j \subseteq A_{i_k}$. And vice versa, inclusion of A_{i_k} in some B_m (which is necessarily the considered B_j) can be shown. Identity of the collections A_i, $i \in I$, and B_j, $j \in J$, follows. □

The result can be further improved if we strengthen our assumptions on the order structure of the module. Extending the notion of 0-distributivity [15, section

3.4], we shall call a complete lattice L *infinitely 0-distributive* if L is distributive, and $a \wedge b = 0$ for all $b \in B$ implies $a \wedge \bigvee B = 0$ for any $a \in L$, $B \subseteq L$. As was pointed out by the referee, the class of infinitely 0-distributive modules includes some structures introduced by P. Resende [11]: so-called *quantal frames*, quantales in which binary meets also distribute over arbitrary joins (hence they possess the structure of a frame as well), and *stably supported quantales* where the support ςQ of such a quantale Q is a frame and becomes a left Q-module.

Theorem 7. *Every finitely spatial, infinitely 0-distributive Q-module is isomorphic to the direct product of its Q-submodule-ideals that are indecomposable and pairwise meeting in* 0 *only.*

Proof. We need to show that representation of any element x of M by a join of elements that belong to the compositing submodules is unique. Let M have a decomposition $M = \langle \bigcup_{i \in I} A_i \rangle$ as shown in the previous theorem, and let $x \in M$ satisfy $x = \bigvee \{a_i \mid a_i \in A_i\} = \bigvee \{b_i \mid b_i \in A_i\}$. Then, since the only pairwise-common element of the compositing downward-closed submodules is 0, for each $j \in I$ we have $a_j = a_j \wedge x = a_j \wedge \bigvee_{i \in I} b_i = a_j \wedge \left(\left(\bigvee_{i \neq j} b_i \right) \vee b_j \right) = \left(a_j \wedge \bigvee_{i \neq j} b_i \right) \vee (a_j \wedge b_j) = 0 \vee (a_j \wedge b_j)$, hence $a_j \leq b_j$. Using this argument we can see that the collections of a_i and b_i are equal.

The join map $f \colon \prod_{i \in I} A_i \to M$ given as $f((a_i)) = \bigvee_{i \in I} a_i$ is then a module isomorphism as it is surjective by the assumptions, injective according to the previous paragraph, and it can be verified that it is a homomorphism. □

Example. Consider the lattice $L = Idl(\mathbb{Z}_{60})$ of ideals of the ring $\mathbb{Z}_{60}$. Its subset $\mathrm{JI}(L)$ of join-irreducible elements consists of $a = ([12]_{60})$, $b = ([15]_{60})$, $c = ([20]_{60})$, and $d = ([30]_{60})$. With multiplication of ideals it becomes also a unital quantale, hence a module as well, and the cyclic submodules then look as follows: $La = \{(0), (12)\}$, $Lb = \{(0), (15), (30)\}$, $Lc = \{(0), (20)\}$, $Ld = \{(0), (30)\}$. All of them are subchains connecting 0 and the respective elements, and they are identical to their down-sets. The resulting decomposition then consists of $A_a = La$, $A_b = A_d = Lb$, and $A_c = Lc$.

A different-looking module results from the construction of the endomorphism quantale $\mathcal{Q}(L)$. Multiplication by a quantale element is then performed by endomorphism application. All the elements of L can be achieved from any join-irreducible element this way, hence all cyclic submodules $\mathcal{Q}(L)m$ generated by elements $m \in \mathrm{JI}(L)$ are equal to L, and L is therefore indecomposable as $\mathcal{Q}(L)$-module by Proposition 3.

Now look at the quantale $\mathcal{Q}(L)$. As shown more generally in [14, Proposition 1.7], it is join-generated by its join-irreducible elements of the form

$$f_{ij}(x) = \begin{cases} c_j & \text{if } x \geq c_i, \\ 0 & \text{otherwise,} \end{cases}$$

for c_i and c_j ranging over $\mathrm{JI}(L)$, and so it allows application of our results. Any endomorphism $f \in \mathcal{Q}(L)$ is uniquely determined by setting images of join-irreducible elements. This prescription can be almost arbitrary, it just has to preserve the partial order on $\mathrm{JI}(L)$. As a sup-lattice, $\mathcal{Q}(L)$ is thus isomorphic to $C = \{g \colon \mathrm{JI}(L) \to L \mid g \text{ is monotone}\}$.

The poset $\mathrm{JI}(L)$ can be viewed as a union of disjoint components (with respect to the ordering relation) $L_1 = \{a\}$, $L_2 = \{b, d\}$, $L_3 = \{c\}$. Then each $C_i = \{g \colon \mathrm{JI}(L) \to L | g \text{ is monotone, } g(x) = 0 \text{ for all } x \notin L_i\}$ is also a submodule of $\mathcal{Q}(L)$, and it can be seen that $C_1 \cup C_2 \cup C_3$ join-generates $\mathcal{Q}(L)$. Therefore, $\mathcal{Q}(Idl(\mathbb{Z}_{60}))$ can be decomposed as $Idl(\mathbb{Z}_{60}) \oplus Idl(\mathbb{Z}_{60}) \oplus Idl(\mathbb{Z}_{60})^{\mathbf{2}}$ where $S^{\mathbf{2}}$ means the poset of all monotone maps from $\mathbf{2}$ to a sup-lattice S.

3. Projective Essential Modules

Lemma 8. *Let Q be a quantale and $d \in Q$ be idempotent. Then the module Qd is projective.*

Proof. Let $f \colon Qd \to N$ be a homomorphism and $g \colon M \to N$ be a surjective homomorphism. If $f(d) = n \in N$, there exists $m \in M$ such that $g(m) = n$. Define $h \colon Qd \to M$ as $h(q \cdot d) = (q \cdot d) \cdot m$. This map is a module homomorphism because $h(r \cdot (q \cdot d)) = (r \cdot (q \cdot d)) \cdot m = r \cdot h(q \cdot d)$ and $h\left(\bigvee_{i \in I}(q_i \cdot d)\right) = \left(\bigvee_{i \in I}(q_i \cdot d)\right) \cdot m = \bigvee_{i \in I}(q_i \cdot d \cdot m) = \bigvee_{i \in I}(h(q_i \cdot d))$. The fact of d being idempotent makes the homomorphisms commute: $(g \circ h)(q \cdot d) = g(q \cdot d \cdot m) = q \cdot d \cdot g(m) = q \cdot d \cdot n = q \cdot d \cdot f(d) = f(q \cdot d \cdot d) = f(q \cdot d)$. □

Note that the above Lemma implies that unital quantales are projective when they are regarded as modules.

Proposition 9. *Let M be a Q-module and $m \in M$ belong to Qm. Then the following conditions are equivalent:*

(1) *Qm is projective.*
(2) *There exists an idempotent $d \in Q$ such that $m = d \cdot m$ and $q \cdot m \mapsto q \cdot d$ is a homomorphism.*
(3) *$Qm \cong Qd$ for some idempotent $d \in Q$.*

Proof. 1. $\Rightarrow$ 2. Let Qm be projective. Since $\psi \colon Q \to Qm$ taking q to $q \cdot m$ is onto, by Proposition 1 it is a retraction and there exists a homomorphism $g \colon Qm \to Q$ such that $\psi \circ g = \mathrm{id}_{Qm}$. Let $d \in Q$ denote the g-image of m. Then $m = \psi(g(m)) =$

$\psi(d) = d \cdot m$, and d is idempotent: $d = g(m) = g(d \cdot m) = d \cdot g(m) = d^2$. We can see that g is the desired homomorphism: $g(q \cdot m) = q \cdot g(m) = q \cdot d$.

2. $\Rightarrow$ 3. The image of g which was obtained in the previous step is Qd, and we know that g is injective.

3. $\Rightarrow$ 1. See the previous proposition. □

Proposition 10. *An indecomposable essential Q-module is projective if and only if it is isomorphic to Qd for an idempotent $d \in Q$.*

Proof. The sufficient condition for projectivity is implied by Lemma 8, so suppose that an essential Q-module P is projective and indecomposable.

Let $A = QP$ and for each $a \in A$ fix a pair q_a, r_a such that $a = q_a \cdot r_a$. As P is essential, $P = \langle A \rangle$. For each $a \in A$ we can set a map $f_a \colon Q \to P$ as $f_a(q) = q \cdot r_a$. It can be easily seen that f_a is a module homomorphism containing a in its image.

Using the homomorphisms f_a we can define a map $f\colon \prod_{a \in A} Q \to P$ as $f(x) = f((x_a)) = \bigvee_{a \in A} f_a(x_a)$. Note that $\prod_{a \in A} Q$ is also a coproduct equipped with natural injections ι_a from Q. As for every $a \in A$ $f \circ \iota_a = f_a$, universal property of the coproduct yields that f is a homomorphism. Moreover, f is a surjection — let $p \in P$ be arbitrary, then $p = \bigvee B$ for some $B \subseteq A$. Take the element $y \in \prod_{a \in A} Q$ given as

$$y_a = \begin{cases} q_a & \text{if } a \in B, \\ 0 & \text{if } a \notin B. \end{cases}$$

Then $f(y) = \bigvee_{a \in A} f_a(y_a) = \bigvee_{a \in B} (q_a \cdot r_a) = p$. By Proposition 1 there is an injective homomorphism $g\colon P \to \prod_{a \in A} Q$ satisfying $(f \circ g)(P) = P$. This gives us a submodule $g(P) \subseteq \prod_{a \in A} Q$ isomorphic to P.

Suppose there is an element $0 \neq x \in g(P)$ with x_a and x_b different from 0 for distinct $a, b \in A$. Consider the submodules $R = \{z \in g(P) \mid z_b = 0 \text{ for all } b \neq a\}$ and $S = \{z \in g(P) \mid z_a = 0\}$. By the assumption, these two submodules are nontrivial and downward-closed in $g(P)$, and they join-generate $g(P)$. However, this contradicts indecomposability of $g(P)$. Therefore all non-zero elements of $g(P)$ must be contained in a copy of Q for some $b \in A$.

Hence we obtain that $P = f(g(P)) \subseteq f\left(\prod_{a \in A} Q\right) = P$, thus $f(g(P)) = f_b(Q) = Q \cdot r_b$ for $r_b \in P$. By part 3. of Proposition 9, $P \cong Qd$ for an idempotent $d \in Q$. □

Lemma 11. *The product $M = \prod_{i \in I} M_i$ is essential iff each M_i is essential.*

Proof. If $m = \bigvee_{j \in J} (q_j \cdot n_j)$ for an index set J with $q_j \in Q$ and $n_j \in M$, then obviously every m_i, the i-th component of m, equals $\bigvee_{j \in J} (q_j \cdot (n_j)_i)$.

For the converse, let $m \in \prod_{i \in I} M_i$. For every $i \in I$ there is a set J_i such that $m_i = \bigvee_{j \in J_i} (q_j \cdot n_j)$ for $q_j \in Q$ and $n_j \in M_i$. Then $m = \bigvee_{i \in I, j \in J_i} (q_j \cdot \iota_i(n_j))$ where ι_i denotes the injection into the i-th component of the coproduct. □

Theorem 12. *An infinitely 0-distributive finitely spatial essential Q-module is projective if and only if it is isomorphic to $\prod_{i \in I} Qd_i$ where each d_i is an idempotent element of Q.*

Proof. The 'if' direction follows from the previous proposition and from the fact that products of Q-modules coincide with their coproducts.

For the converse, we have seen that every infinitely 0-distributive finitely spatial Q-module M has a unique decomposition into a set of its indecomposable submodules M_i, $i \in I$, making M isomorphic to their coproduct. As a coproduct of modules is projective iff each one is projective and the same holds true for essentiality, by Proposition 10 each M_i has to be isomorphic to Qd_i for an idempotent $d_i \in Q$. □

The example of decomposition of a quantale/module of sup-lattice endomorphisms on page 85 also illustrates the above result. Since $\mathcal{Q}(L)$ is a unital quantale, all summands can be expressed as cyclic submodules of the quantale which are generated by idempotent homomorphisms. In this case, these homomorphisms are of the form (when prescribed on join-irreducible elements) $f_i(x) = x$ if $x \in L_i$, and 0 otherwise.

Note that if the only idempotents of a unital quantale Q are 0 and the neutral element e, then every projective infinitely 0-distributive finitely spatial unital module over Q is free since its decomposition consists only of copies of Q.

Obviously, finite spatiality and infinite 0-distributivity are not necessary in the 'if' part of the main theorem. In certain cases, they may not be required in the other direction either. An instance of quantales which allow these assumptions on a projective module to be omitted is provided by supercontinuous quantales. Recall that a complete lattice is called supercontinuous if every its element x is a join of elements that lie completely below x, that is, such elements y satisfying $x \le \bigvee A \implies (\exists a \in A)(y \le a)$. G. N. Raney [10] proved that supercontinuity equals to complete distributivity, and from [4, Theorem I-3.16] it then follows that supercontinuous lattices are finitely spatial. Complete distributivity also implies infinite 0-distributivity. As products of supercontinuous complete lattices are supercontinuous as well [17] and supercontinuity is preserved by retraction of modules (shown in [7]), projective modules over supercontinuous quantales are supercontinuous, too. The above also implies that finite spatiality and infinite 0-distributivity are satisfied for all projective modules over finite quantales which are distributive as lattices.

Example. Consider the lattice $\mathcal{O}(X)$ of open sets of a topological space $X = [0,1] \cup [2,3]$ with standard topology on reals. With intersection as the binary operation, it becomes a unital idempotent commutative quantale (i.e., a frame), hence a module over itself. Although it is not finitely spatial since it lacks enough join-irreducible elements, it is a decomposable projective module because it is a unital quantale and it is join-generated by its two submodules $\mathcal{O}([0,1])$ and $\mathcal{O}([2,3])$.

Acknowledgement

The author is grateful to Jan Paseka for providing many valuable comments and directions during preparation of this paper.

References

[1] F. Borceux, *Handbook of Categorical Algebra 1: Basic Category Theory*, Cambridge University Press, Cambridge, 1994.

[2] Y. Q. Chen and K. P. Shum, *Projective and Indecomposable S-acts*, Science in China Series A: Mathematics 42 (1999), pp. 593–599.

[3] N. Galatos and C. Tsinakis, *Equivalence of consequence relations: an order-theoretic and categorical perspective*, Journal of Symbolic Logic 74(3) (2009), pp. 780–810.

[4] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, *Continuous Lattices and Domains*, Cambridge University Press, Cambridge, 2003, pp. 323–362.

[5] M. Kilp, U. Knauer, and A. V. Mikhalev, *Monoids, Acts and Categories*, Walter de Gruyter, Berlin, 2000.

[6] D. Kruml and J. Paseka, *Algebraic and categorical aspects of quantales*, in Handbook of Algebra, vol. 5, North-Holland, 2008, pp. 323–362.

[7] Y. Li, M. Zhou, and Z. Li, *Projective and injective objects in the category of quantales*, Journal of Pure and Applied Algebra 176 (2002), pp. 249–258.

[8] M. Ordelt, *Moduly v kvantových svazech* (in Czech), Master's thesis, Masaryk University, Brno, 2004.

[9] J. Paseka, *Morita equivalence in the context of Hilbert modules*, in Proceedings of the Ninth Prague Topological Symposium, Charles University and Topology Atlas, Toronto, 2002, pp. 231–258.

[10] G. N. Raney, *A Subdirect-Union Representation for Completely Distributive Complete Lattices*, Proceedings of the American Mathematical Society 4(4) (1953), pp. 518–522.

[11] P. Resende, *Étale groupoids and their quantales*, Advances in Mathematics 208 (2007), pp. 147–209.

[12] K. I. Rosenthal, *Quantales and their applications*, Pitman Research Notes in Mathematics, Longman Scientific & Technical, New York, 1990.

[13] X. Shi, Z. Liu, F. Wang, and S. Bulman-Fleming, *Indecomposable, projective and flat S-posets*, Communications in Algebra, 33 (2005), pp. 235–251.

[14] R. Šlesinger, *Projective Quantales and Quantale Modules*, Master's thesis, Masaryk University, Brno, 2008, available online at `http://is.muni.cz/th/106321/prif_m/`

[15] M. Stern, *Semimodular Lattices: Theory and Applications*, Cambridge University Press, Cambridge, 1999.

[16] F. Wehrung, *Direct decompositions of non-algebraic complete lattices*, Discrete Mathematics 263 (2003), pp. 311–321.

[17] B. Zhao and Y. Zhou, *The category of supercontinuous posets*, Journal of Mathematical Analysis and Applications 320 (2006), pp. 632–641.

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