Acta Universitatis Matthiae Belii ser. Mathematics 17 (2010), 3–19.
Received: 21 September 2007, Last revision: 23 September 2010, Accepted: 22 October 2010.
Communicated with Peter Maličký

ASYMPTOTIC MOTIONS OF THREE-PARAMETRIC ROBOT MANIPULATORS

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Abstract. The Lie algebra of screws in matrix form is used for introducing asymptotic and geodesic motions of robot manipulators. These motions are characterized as motions the covariant acceleration elements of which are tangential. In this paper the asymptotic motions of all three-parametric robot manipulators are described.

1. Introduction

We deal with robot manipulator motions which have a strong geometrical background. We use the screw method of the investigation, see [1, 7, 8, 9]. Motions of the robot effectors are determined by curves on the Lie group $E(3)$ of all Euclidean motions in the Euclidean space E_3 the Lie algebra $e(3)$ of which is the algebra of velocity twists. All motions of a n-parametric robot manipulator with n joints which are controlled by n parameters $(u) = (u_1, \dots, u_n)$ form at any position (u) the subspace $VT(u) \subset e(3)$ of the velocity twists, the subspace $AC(u) \subset e(3)$ spanned on the set of all Lie bracket values $[VT(u), VT(u)]$, the subspace $Cov(u) = span(VT(u) \cup AC(u))$ of the so-called covariant acceleration twists, the Klein subspace $K(u) \subset VT(u)$ of all twists orthogonal to $VT(u)$ according to the Klein bilinear scalar form KL on $e(3)$. We introduce the so-called asymptotic motion as a motion the covariant acceleration twists of which lie in the spaces $VT(u)$. The simple examples of the asymptotic motions are the motions when only one joint works. We describe all asymptotic motions for all three-parametric robot manipulators with revolute and prismatic joints only.

2. Basic notions of robot manipulators

The geometrical background of a kinematic and dynamic problems of the robots are both the Lie group $E(3)$ of the Euclidean motions in the Euclidean space E_3

2000 *Mathematics Subject Classification.* 53A17.
Key words and phrases. local differential geometry, robotics, Lie algebra, asymptotic motion.

and its Lie algebra $e(3)$, see for example [4, 5, 7, 8]. We use the homogeneous matrix presentation of $E(3)$, the matrix and the screw (twist) presentations of $e(3)$. We introduce a brief explanation of main ideas. Throughout our paper the notion of the robot will mean a robot manipulator with n links.

Let $\mathcal{S}_0 = (\bar{i}_0, \bar{j}_0, \bar{k}_0, O)$ or $\mathcal{S}_n = (\bar{i}_n, \bar{j}_n, \bar{k}_n, P^T)$ be the cartesian coordinate system fixed on the robot base or on the robot effector, respectively. There is a unique Euclidean transformation $\mathbf{H}$ in which the system $\mathcal{S}_0$ goes into $\mathcal{S}_n$. We get the matrix relation $\mathcal{S}_n = \mathcal{S}_0 H$ where $H = \left(\begin{smallmatrix} A & P \\ 0 & 1 \end{smallmatrix}\right)$ is the homogeneous transformation matrix; i.e., A is a $(3,3)$-matrix the columns of which are the successively coordinates of the vectors $\bar{i}_n, \bar{j}_n, \bar{k}_n$ in the base $(\bar{i}_0, \bar{j}_0, \bar{k}_0)$ and $P = (p_x, p_y, p_z)^T$ are the coordinates of the origin P of $\mathcal{S}_n$ in $\mathcal{S}_0$. Therefore $AA^T = I$ where I is the identity matrix and A^T denotes the transposed matrix to A.

Let us recall two relations according to the matrix H which we will use.

a) If $L_n = (x_n, y_n, z_n, 1)^T$ denotes the homogeneous coordinates of a point L of the effector in $\mathcal{S}_n$ then $L_0 = HL_n$ are the homogeneous coordinates of L in $\mathcal{S}_0$.

b) If $L_0 = (x_0, y_0, z_0, 1)^T$ are the coordinates of L in $\mathcal{S}_0$ and $L'_0 = (x'_0, y'_0, z'_0, 1)^T$ are the coordinates of the image of L in the Euclidean motion h in which the system $\mathcal{S}_0$ goes into $\mathcal{S}_n$ then $L'_0 = HL_0$. Evidently the Cartesian system $\mathcal{S}_0$ states the map $h \mapsto H$ which is the matrix representation of $E(3)$. We identify $E(3)$ with the group of matrices H.

A motion of the effector is expressed by the equation $\mathcal{S}_n(t) = \mathcal{S}_0 H(t)$. Then the equation $L_0(t) = H(t)L_0$ expresses the corresponding motion of an effector point L in the coordinate system $\mathcal{S}_0$. Differentiating this equation with respect to t we get $\dot{L}_0(t) = \dot{H}(t)L_0 = \dot{H}H^{-1}L_0(t)$, where $HH^{-1} = I$, $H^{-1} = \left(\begin{smallmatrix} A^T & -A^T P \\ 0 & 1 \end{smallmatrix}\right)$, $\dot{H}H^{-1} = \left(\begin{smallmatrix} \dot{A}A^T & -\dot{A}A^T P + \dot{P} \\ 0 & 0 \end{smallmatrix}\right)$ and the dot over letters denotes differentiation with respect to t. The relation $AA^T = I$ inherits the equation $\dot{A}A^T + A\dot{A}^T = 0$ and thus $\dot{A}A^T$ is skew symmetric $(\dot{A}A^T)^T = -\dot{A}A^T$. There is a unique vector $\overline{\omega} = (\omega_x, \omega_y, \omega_z)$ such that $\dot{A}A^T = \left(\begin{smallmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{smallmatrix}\right)$. A skew symmetric matrix determined in such way by the vector $\overline{\omega}$ will by denoted by C^ω. Let us recall the well known relation $C^\omega \overline{v}^T = (\overline{\omega} \times \overline{v})^T$, where $\overline{\omega} \times \overline{v}$ denotes the vector product of the vectors $\overline{\omega}$ and $\overline{v}$. If we denote $\overline{b} = B^T$, $B := -\dot{A}A^T P + \dot{P}$ we see that the matrix $\mathcal{H} := \dot{H}H^{-1}$ determines the vector couple $(\overline{\omega}, \overline{b})$ which is called briefly a twist, see [7]. The matrix $\left(\begin{smallmatrix} C^\omega & \overline{b}^T \\ 0 & 0 \end{smallmatrix}\right)$ is said to be the matrix presentation of the twist $(\overline{\omega}, \overline{b})$.

From the geometrical point of view $H(t)$ is a curve on the manifold $E(3)$ and $\dot{H}(t)$ is its tangent vector at $H(t)$. Then the right group translation by the element $H^{-1}(t) \in E(3)$ transforms $\dot{H}(t)$ into the tangent vector $\mathcal{H}(t) =$

$\dot{H}(t)H^{-1}(t)$ at the unit $I \in E(3)$. The vector space of all tangent vectors on a Lie group at its unit has a Lie algebra structure which is $e(3)$ in our case of $E(3)$. So $e(3)$ is the vector space of all twists. The Lie bracket (product) has the standard matrix representation $[\mathcal{H}_1, \mathcal{H}_2] = \mathcal{H}_1\mathcal{H}_2 - \mathcal{H}_2\mathcal{H}_1$ which corresponds with its following twist representation $[(\overline{\omega}_1, \overline{b}_1), (\overline{\omega}_2, \overline{b}_2)] = (\overline{\omega}_1 \times \overline{\omega}_2, \overline{\omega}_1 \times \overline{b}_2 - \overline{\omega}_2 \times \overline{b}_1)$.

Any twist $Y = (\overline{\omega}, \overline{b})$ determines the motion m_Y (called canonical) in the following way. If $\overline{\omega} \neq \overline{0}$ then m_Y is the uniform helical motion around the axis p of Y with the angular velocity $\overline{\omega}$ and with the translation velocity $\upsilon\overline{\omega}$ where $\upsilon = \overline{\omega} \cdot \overline{b}/\overline{\omega}^2$ is the so-called the pitch of this motion. So it is a rotation if the scalar product $\overline{\omega} \cdot \overline{b}$ is zero. Let us recall that the velocity of a point L at this motion is $\overline{v} = \overline{\omega} \times \overline{CL} + \upsilon\overline{\omega}$ and thus $\overline{v} = \overline{\omega} \times \overline{CO} + \upsilon\overline{\omega} = \overline{b}$ is the velocity of the origin O of $\mathcal{S}_0$. Let us emphasize that in the rotation case $\overline{b}$ is orthogonal to the plane (O, p). If $\overline{\omega} = \overline{0}$ then the canonical motion m_Y of $Y = (\overline{0}, \overline{b})$ means the translation with the velocity $\overline{b}$. Conversely, the uniform helical motion m around the axis $p = (C, \overline{\omega})$ where $\overline{OC}$ is perpendicular to p or $C \equiv O$, $\overline{\omega}$ is its angular velocity, $\upsilon\overline{\omega}$ is its translation velocity and $\overline{b}$ is the velocity of the origin O at this motion, determines the twist $(\overline{\omega}, \overline{b})$ the canonical motion of which is just the motion m. Evidently the translation τ with the velocity $\overline{b}$ determines the twist $(\overline{0}, \overline{b})$ the canonical motion of which is just τ. So a twist $Y = (\overline{\omega}, \overline{b})$ is called rotational or helical or translating if $\overline{\omega} \cdot \overline{b} = 0$, $\overline{\omega} \neq \overline{0}$ or $\overline{\omega} \cdot \overline{b} \neq 0$ or $\overline{\omega} = \overline{0}$ respectively.

Remark 1 (On the exponential map $\exp : e(3) \mapsto E(3)$)**.** Let $(\overline{\omega}, \overline{b})$ be a twist and $\mathcal{H} = \begin{pmatrix} C^\omega & \overline{b}^T \\ 0 & 0 \end{pmatrix}$ be its matrix presentation. Then

$$\exp t(\overline{\omega}, \overline{b}) = \exp t\mathcal{H} = I + t\mathcal{H} + t^2\mathcal{H}^2/2! + t\mathcal{H} + t^3\mathcal{H}^3/3! + \cdots \equiv \gamma(t)$$

and $\dot{\gamma}(t) = \mathcal{H} \exp t\mathcal{H}$. Let us consider the motion $L_0(t) = \gamma(t)L_0$ of a point L in the base coordinate system $\mathcal{S}_0$, $L_0 = L_0(0)$. Then for the velocity of the point $L_0(t)$ we get $\dot{L}_0(t) = \dot{\gamma}(t)L_0 = \dot{\gamma}\gamma^{-1}L_0(t) = \mathcal{H}L_0(t)$. Then in the case $\overline{\omega} \neq \overline{0}$: $\dot{L}_0(t) = \begin{pmatrix} C^\omega & \overline{b}^T \\ 0 & 0 \end{pmatrix} L_0(t) = (\overline{\omega} \times \overline{OL}_0(t) + \overline{b})^T = (\overline{\omega} \times (\overline{OC} + \overline{CL}_0(t)) + \overline{\omega} \times \overline{CO} + \upsilon\overline{\omega})^T = \overline{\omega} \times \overline{CL}_0(t) + \upsilon\overline{\omega}$. If $\overline{\omega} = \overline{0}$ then $\dot{L}_0(t) = \overline{b}^T$. We conclude: $\exp t(\overline{\omega}, \overline{b})$ is the canonical motion of the twist $(\overline{\omega}, \overline{b})$.

Remark 2. Let us recall that if $\exp t(\overline{\omega}, \overline{b})$ expresses a motion in the coordinate system $\mathcal{S}_n$, $\mathcal{S}_n = \mathcal{S}_0 H$, then $H \exp t(\overline{\omega}, \overline{b})H^{-1}$ is its expression in $\mathcal{S}_0$ and thus $Ad_H(\overline{\omega}, \overline{b}) := H\mathcal{H}H^{-1}$, $\mathcal{H} = \begin{pmatrix} C^\omega & \overline{b}^T \\ 0 & 0 \end{pmatrix}$ is the matrix presentation of $(\overline{\omega}, \overline{b})$ in $\mathcal{S}_0$.

Remark 3. The map $H \mapsto Ad_H$ is the representation of $E(3)$ in $GL(e(3))$. The axis p' of $Ad_H(\overline{\omega}, \overline{b})$ is the H-image of the axis p of $(\overline{\omega}, \overline{b})$, $p' = H(p)$.

Remark 4. The notion of the *twist* is not unified. Some authors use notions the *infinitesimal motion* or the *velocity operator*. We will refer to the *velocity twist* or the *velocity element*.

A n-parametric robot $\mathcal{R}$ consists of a sequence of n rigid links connected by joints such that every joint J_i enables the successive links to rotate around the axis o_i of J_i or to translate in the direction of o_i. Such a joint is called rotational or prismatic. So every joint determines a twist $(\overline{\omega}, \overline{b})$ in $\mathcal{S}_0$ where $\overline{\omega}^2 = 1, \overline{\omega} \cdot \overline{b} = 0$ or $\overline{\omega} = \overline{0}, \overline{b}^2 = 1$ and $\exp t(\overline{\omega}, \overline{b})$ is the unit uniform motion enabled by this joint. We will refer to the "canonical twist" of the corresponding joint. Let $X_i = (\overline{\omega}_i, \overline{b}_i)$ be the canonical twist of the ith joint J_i at the initial position $(u) = (0)$ of the robot in $\mathcal{S}_0$. Let $\exp u^i X_i$ be any motion enabled by J_i. Then the robot activity states a map $\rho : U_n \to E(3), (u) = (u_1, \dots, u_n) \mapsto \exp u^1 X_1 \dots \exp u^n X_n$, where $U_n \subset \mathbb{R}^n$ is an open neighbourhood of $(0) = (0, \dots, 0) \in \mathbb{R}^n$ of the joint parameters (of the control joint variables) the admissible values of which are determined by the robot construction. Let us consider the motion $L_0(t) = H(t)L_0$, $H(t) = \exp u^1(t)X_1 \dots \exp u^n(t)X_n$, $\dot{L}_0(t) = \dot{H}(t)H^{-1}(t)L_0(t)$. We get

$$(1) \qquad Y(t) := \dot{H}(t)H^{-1}(t) = \dot{u}^1 Y_1 + \dots + \dot{u}^n Y_n,$$

$Y_1 = X_1, Y_i = H_i X_i H_i^{-1} = Ad_{H_i} X_i, H_i = \exp u^1(t)X_1 \dots \exp u^{i-1}(t)X_{i-1}$, $\dot{L}_0(t) = Y L_0(t)$. The last equation inspire us to call Y the velocity twist or shortly a v-twist. As $Y_i = Ad_{H_i} X_i$ the axis of Y_i is the actual position of the joint axis o_i at time t. We will say that the joint J_i works or does not work at t_0 if $\dot{u}^i(t_0) \neq 0$ or $\dot{u}^i(t_0) = 0$, respectively. By the notion a "position" $(u) = (u^1, \dots, u^n)$ of the robot we mean an element $\rho(u) \in E(3)$. We say about a motion cross a position (u_0) if there is t_0 such that $(u_0) = (u(t_0))$.

Let $(u_0) = (u_0^1, \dots, u_0^n)$ be a fixed position. The curve $\gamma_i(t) = H_i(t_0) \exp(u_0^i + t)$ $X_i \exp u_0^{i+1} X_{i+1} \dots \exp u_0^n X_n$ is called an u^i-curve cross (u_0). Its tangent vector $\dot{\gamma}_i(t_0)$ will be denoted by $\partial u_i(u_0)$. By (1) it is clear that $Y_i(u_0) := Y_i(t_0) = \partial u_i(u_0) H^{-1}(u_0)$, i.e. $Y_i(u_0)$ is the image of $\partial u_i(u_0)$ by the right group translation stated by the element $H^{-1}(t_0)$. In general, a u^i-motion cross (u_0) will mean a motion $H_i(t_0) \exp u^i(t) X_i \exp u_0^{i+1} X_{i+1} \dots \exp u_0^n X_n$ when only the ith joint works.

Differentiating the equation $\dot{L}_0(t) = Y(t)L_0(t)$ we obtain $\ddot{L}_0(t) = (\dot{Y} + YY)L_0(t)$. As $Y(t) \in e(3)$ is a curve in the vector space $e(3)$ then $\dot{Y} \in e(3)$, i.e. $\dot{Y}$ is a twist. In contrary YY is not a twist. We have $\dot{Y} = \sum_{i=1}^n \ddot{u}^i Y_i + \sum_{i=1}^n \dot{u}^i \sum_{i=1}^n \frac{\partial Y_i}{\partial u^k} \dot{u}^k$. As Y_i depends on the parameters $u^1, \dots, u^{i-1}$ only then $\frac{\partial Y_i}{\partial u^k} = 0$ for $k > i$ and the direct computation leads to the expression $\frac{\partial Y_i}{\partial u^k} = [Y_k, Y_i]$, $k = 1, \dots, i-1$, see [5, 8]. We obtain

$$(2)\qquad \dot{Y} = \dot{Y}_J + \dot{Y}_C, \qquad \dot{Y}_J = \sum_{i=1}^{n} \ddot{u}^i Y_i, \qquad \dot{Y}_C = \sum_{k<i} [Y_k, Y_i]\dot{u}^k \dot{u}^i$$

$$(3)\qquad \ddot{L}_0 = (\dot{Y}_J + \dot{Y}_C + \dot{Y}_S)L_0, \qquad \dot{Y}_S = YY$$

Definition 1. An element $\dot{Y}_J$ or $\dot{Y}_C$ will be called a joint acceleration twist (shortly J-twist) or a skew acceleration twist (shortly C-twist) of the motion respectively.

3. Kinematic twist subspaces, asymptotic motions

Let us recall that the Jacobian $J(u)$ of a n-parametric robot at u is the differential $T(\rho_R^{-1}(u)\rho)$ of the composition of two maps: $\rho : U \to E(3)$ and the right side group translation $\rho_R^{-1}(u)$ by the group element $[\rho(u)]^{-1}$. Then the rank of the robot at (u) is the number $rank J(u)$.

Let us denote $VT(u) := span(Y_1(u), \dots, Y_n(u)) \subset e(3)$ the vector space of the twists at (u). Evidently $rank J(u) = \dim VT(u)$. We write shortly $VT := VT(0) = span(X_1, \dots, X_n)$.

Definition 2. A position (u) of a n-parametric robot is called regular or singular if $\dim VT(u) = n$ or $\dim VT(u) < n$ respectively. A motion is singular if its all positions are singular. A n-parametric robot is said to be the nr-robot if $\dim VT = n$.

We confine ourself on nr-robots, $\dim VT = n$. Then there is a neighourhood $U_0 \subset U$ that $\dim VT(u) = n$, $u \in U_0$ and thus the map ρ is an immersion on U_0. Then there is a neighborhood $V \subset U_0$ such that $\rho(V)$ is an immersed submanifold, see [6].

Definition 3. The subspace $AC(u) := span\{[A, B]; A, B \in VT(u)\}$ where $[A, B]$ denotes the Lie bracket of elements A, B in $e(3)$ is called the Coriolis subspace. Its elements are C-twists.

Remark 5. Let us recall that a subspace $VT(u)$ is a subalgebra of $e(3)$ iff $AC(u) \subset VT(u)$. If VT is a subalgebra then $\rho(V)$ is a submanifold of the subgroup stated by VT and $VT(u) = VT$, $u \in V$. We give the survey of all subalgebras of $e(3)$, see [4, 8]. They are:

(α) All 1-dimensional subspaces of $e(3)$.

(β) (a) All 2-dimensional subspaces of the translating twists.

(b) All 2-dimensional subspaces of the twists $(k\overline{\omega}, k\overline{b} + c\overline{\omega})$, $\overline{\omega} \neq \overline{0}$, $k, c \in \mathbb{R}$ with the same axis.

(γ) (a) All 3-dimensional subspaces of the rotational twists the axes of which have a common point or all translating twists.

(b) All 3-dimensional subspaces spanned on a rotational twist and on two translating twists where the axis of the rotational twist is orthogonal to the directions of the translating twists.

(δ) All 4-dimensional subspaces spanned on a rotational twist and on 3 independent translating twists.

Definition 4. The subspace $Cov(u) = span(VT(u) + AC(u))$ is called the covariant subspace. Its elements $\dot{Y}$ introduced by the relation (2) will be briefly called Cov-twists. A motion $u(t)$ is called geodesic if $\dot{Y}(u(t)) = 0$.

Definition 5. A motion $u(t)$ will be called asymptotic at $u_0 = u(t_0)$ or asymptotic if $\dot{Y}(t_0) \in VT(u_0)$ or $\dot{Y}(t) \in VT(u)$ for any admissible $u(t)$ respectively. A position (u) is called flat if every motion cross (u) is asymptotic at (u). The robot is flat if there is an open set $W \subset U$ such that the robot is flat at any $(u) \in W$. The subspace $AC(u) \cap VT(u)$ will be called asymptotic, its elements will be called the asymptotic twists.

The following assertions are evident.

Lemma 1.

a_1) If a robot motion is asymptotic at (u) then its C-twist $\dot{Y}_C(u)$ is asymptotic. If $\dot{Y}_C(u) = 0$ then this motion is asymptotic and thus every u^i-motion and every motions due to work of prismatic joints only are asymptotic.

a_2) Every geodesic motion is asymptotic. An asymptotic motion is geodesic if its C-twist is opposite to its J-twist, $\dot{Y}_C = -\dot{Y}_J$.

a_3) A position (u) or a robot is flat iff $VT(u)$ or VT is a subalgebra.

a_4) All motions given by work of such joints $J_{i_1}, \ldots, J_{i_k}$ that the subspaces $span(Y_{i_1}, \ldots, Y_{i_k})$ are subalgebras, are asymptotic.

Definition 6. A motion given by work of such joints $J_{i_1}, \ldots, J_{i_k}$ that the subspaces $span(Y_{i_1}, \ldots, Y_{i_k})$ are subalgebras, is called trivially asymptotic (shortly t-asymptotic). The others motions will be shortly called nt-asymptotic.

Let us recall that all u^i-motions, all motions given by work of prismatic joints only, all motions given by work of a rotational joint and a prismatic joint with parallel axes are t-asymptotic. We concentrate on nt-asymptotic motions especially on the ones with non zero C-twists.

Let us remind that in $e(3)$ the Klein bilinear form KL is defined as follows. If $X_i = (\overline{\omega}_i, \overline{b}_i) \in e(3)$, $i = 1, 2$, then $KL(X_1, X_2) := \overline{\omega}_1 \cdot \overline{b}_2 + \overline{\omega}_2 \cdot \overline{b}_1$. It means that KL is regular, symmetric and Ad-invariant.

Definition 7. Twists $Z_1, Z_2 \in e(3)$ will be called KL-orthogonal if $KL(Z_1, Z_2) = 0$. Let $V \subset e(3)$ be a vector subspace. The subspace of all twists in $e(3)$ which are KL-orthogonal to every twist $Z \in V$ will be denoted by V^{or}. The subspace $K(u) = VT(u) \cap [VT(u)]^{or}$ will be called the Klein's subspace and its elements are called the Klein's twists. We say that $VT(u)$ is isotropic if $K(u) = VT(u)$.

It is valid that any two translating twists are KL-orthogonal.

Let $Y = t_1 Y_1 + \cdots + t_n Y_n$ be a velocity twist in $VT(u)$, $Y_i = (\overline{\omega}_i, \bar{b}_i)$. Then $Y \in K(u)$ iff

$$(4) \qquad Kl(Y_i, Y) = t_1 Kl(Y_i, Y_1) + \cdots + t_n Kl(Y_i, Y_n) = 0, \qquad i = 1, \ldots, n$$

The symmetric matrix of the system (4) is the matrix of the restriction $KL|_{VT(u)}$ of the Klein form KL to the subspace $VT(u)$ and thus Y is rotational or translating iff $Kl|_{VT(u)}(Y, Y) = 0$: i.e., iff

$$(5_1) \qquad 0 = \left(\sum_{i=1}^{n} t_i \overline{\omega}_i\right) \cdot \left(\sum_{j=1}^{n} t_j \bar{b}_j\right) = \sum_{i,j=1}^{n} (\overline{\omega}_i \cdot \bar{b}_j) t_i t_j.$$

Y i.e. translating iff

$$(5_2) \qquad \bar{0} = \sum_{i=1}^{n} t_i \overline{\omega}_i.$$

Remark 6 (On screws, see [1])**.** Let $\dim VT(u)$ be $k+1$. Then all one-dimensional vector subspaces in $VT(u)$, the so-called screws, form a projective space $\mathcal{P}_k$, $\dim \mathcal{P}_k = k$. The screws in $VT(u)$ are helical, rotational, translating if their non-zero twists are helical, rotational, translating respectively. Evidently all non-zero twists which are belonging to the same screw have the same axis. All rotational and translating screws in $\mathcal{P}_k$ form a quadric subspace (q) in $\mathcal{P}_k$. The subspace (q) is given by the equation (5_1) and the projective subspace of translating screws given by (5_2) lies on the subspace (q). Their axes lie on a quadratic straight line space (Q) in E_3 extended by the infinity points. The system (4) is the system for singular points of (q). Every Klein twist is either rotational or translating and it exists only if (q) degenerates.

Let us remind the following well known properties, see [3, 8].

Lemma 2.

(a) Two rotational twists are KL-orthogonal iff their axes are intersecting or parallel. A rotational or helical twist is KL-orthogonal to a translating twist iff its axis is orthogonal to the direction of the translating twist.

(b) Let X, Y be two not translating twists with the axes x, y. Then the axis of $[X, Y]$ orthogonally intersects the axes x, y. If Y is translating with the direction $\overline{y}$ then $[X, Y]$ is translating and its direction is orthogonal to x and $\overline{y}$. If X, Y are translating then $[X, Y] = 0$.

Corollary 1.

(a) The twist $[X, Y]$ is KL-orthogonal to X and to Y.

(b) If a Klein twist $\hat{Y}$ is rotational then its axis intersects or is parallel with the axis of any rotational twist in $VT(u)$ and it is orthogonal to all directions of all translating twists in $VT(u)$. If $\hat{Y}$ is translating then its direction is orthogonal to the axes of all rotational twists in $VT(u)$.

Remark 7 (The Background of the notions "asymptotic", "geodesic")**.** In our consideration an essential role has the right group translations on $E(3)$. The parallel transport stated by these translations introduces an affine connection ${}^R\Gamma$ on $E(3)$, see [6, 8]. In general, let Γ be an affine connection on an arbitrary manifold M and let ${}^\Gamma\nabla$ be the covariant derivative stated by the parallel transport of Γ. Let $N \subset M$ be a submanifold. A curve γ on N is Γ-asymptotic or geodesic if ${}^\Gamma\nabla_{\dot{\gamma}}\dot{\gamma}$ is tangent on N or ${}^\Gamma\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ respectively. Our concept of asymptotic and geodesic motions coincides with the ones above with respect to the connection ${}^R\Gamma$ on $E(3)$.

4. Three-parametric robot manipulators

Definition 8. A $(3, k)$-robot means a three-parametric robot the maximal number of the independent directions of their rotational joint axes is just k. By $\tau(u)$ or $\tau_D(u)$ we will denote the subspace of all translating twists in $VT(u)$ or the subspace of their directions in E_3 respectively.

4.1. **(3,0)-robots.** All joints of a $(3, 0)$-robot are translating and thus $AC(u) = 0$ and $VT(u)$ is a subalgebra, $\dot{Y}_C(u) = 0$. A motion of $(3, 0)$-robot is geodesic iff $\ddot{u}^1(t) = 0$, $\ddot{u}^2(t) = 0$, $\ddot{u}^3(t) = 0$; i.e., iff this motion is uniform. As $rank(KL|_{VT}) = 0$ therefore for the Klein space it holds $K(u) = VT(u)$ and $VT(u)$ is isotropic.

4.2. **(3,1)-robots.** These robots are described in detail in the paper [2]. Therefore we give here a short survey of properties of these robots only.

Let us remind that in the case of the $(3, 1)$-robots $\rho(V)$ lies on a 4-dimensional subgroup generated by a 4-dimensional subalgebra spanned on VT and all translating twists in $e(3)$. This fact has no influence on our considerations.

All rotational joint axes of the $(3, 1)$-robot are parallel at any position (u). Therefore every velocity twist Y_i, $i = 1, 2, 3$, is of the form $Y_i = (\delta_i\overline{\omega}, \overline{m}_i)$, $\delta_i \in \mathbb{R}$. Neglecting ordering there are three cases: $\{Y_1 = (\overline{\omega}, \overline{m}_1), Y_2 = (\overline{0}, \overline{m}_2), Y_3 = (\overline{0}, \overline{m}_3)\}$, $\{Y_1 = (\overline{\omega}, \overline{m}_1), Y_2 = (\overline{\omega}, \overline{m}_2), Y_3 = (\overline{0}, \overline{m}_3)\}$, $\{Y_1 = (\overline{\omega}, \overline{m}_1), Y_2 =$

$(\overline{\omega}, \overline{m}_2), Y_3 = (\overline{\omega}, \overline{m}_3)\}, \overline{\omega}\cdot\overline{m}_i = 0$. It means that in $VT(u)$ there are always velocity elements $B_1 = (\overline{s}, \overline{b}_1), B_2 = (\overline{0}, \overline{b}_2), B_3 = (\overline{0}, \overline{b}_3)$, $\overline{s}\cdot\overline{b}_i = 0$, $i = 1, 2, 3$, such that $VT(u) = span(B_1, B_2, B_3)$. Now $AC(u) = span([B_1, B_2] = (\overline{0}, \overline{s}\times\overline{b}_2), [B_1, B_3] = (\overline{0}, \overline{s}\times\overline{b}_3), [B_2, B_3] = (\overline{0}, \overline{0}))$, $\tau = span(B_2, B_3)$, $\tau_D = span(\overline{b}_2, \overline{b}_3)$ and the equations (5_1) and (5_2) are of the forms $\left[(\overline{s}\cdot\overline{b}_2)t_2 + (\overline{s}\cdot\overline{b}_3)t_3\right] t_1 = 0$ and $t_1 = 0$. From this we infer the following properties.

Proposition 1. Let ρ be a $(3,1)$-robot. Then

(1) A position (u) is singular iff $\dim \tau_D = 1$. At a singular position $VT(u)$ can not be a subalgebra.
(2) At a regular position $VT(u)$ is a subalgebra iff the direction $\overline{s}$ of the rotational joint axes is orthogonal to τ_D.
(3) The rank of $KL|_{VT(u)}$ is 0 or 2. The relation $rank\left(KL|_{VT(u)}\right) = 0$, i.e. $K(u) = VT(u)$, is true iff $\overline{s}$ is orthogonal to τ_D. There is not a helical twist in $VT(u)$ in this case. If $\overline{s}$ is not orthogonal to τ_D then rank of $KL|_{VT(u)}$ is 2, $\dim K(u) = 1$, $K(u) = span(\hat{Y})$, $\hat{Y} = (\overline{0}, \overline{m}) \in \tau$ where $\overline{m}$ is orthogonal to $\overline{s}$. In this case the axes of all rotational twists in $VT(u)$ form a bundle of the parallel straight lines the plane of which is orthogonal to the direction $\overline{m}$ of the Klein twist $\hat{Y}$.
(4) $\dim AC(u) = 1$ if and only if the direction of the rotational joint axes is parallel with τ_D or (u) is singular. In this case the asymptotic space $AC(u) \cap VT(u)$ is zero.
(5) If the direction of the rotational joint axes is not parallel with τ_D and (u) is regular then $\dim AC(u) = 2$ and $AC(u)$ is a subspace of all translating twists the directions of which are orthogonal to the rotational joint axes. If $AC(u) = \tau$ then $VT(u)$ is a subalgebra. If $AC(u) \neq \tau$ then $AC(u) \cap VT(u) = K(u) = span(\hat{Y})$ and thus a motion is asymptotic iff $\dot{Y}_C = \lambda\hat{Y}$.

We will specify these assertions for concrete cases of robots in the corollary form.

Corollary 2. For $(3,1)$-robots with one rotational joint, (RTT, TRT, TTR), we get:

(a) A singular position there is only in the case TRT when $\angle(o_1, o_2) = \angle(o_3, o_2)$.
(b) The subspace $VT(u)$ is a subalgebra iff (u) is regular and the rotational joint axis is orthogonal to the prismatic joint axes.
(c) Nontrivial asymptotic motions there are in two cases:
    c_1) In the cases RTT, TTR when all joint axes are complanar, the rotational joint works and the ratio of the prismatic joint velocities is equal to the ratio of the corresponding coordinates of the unit vector of the angular velocity (of the rotational joint) in the base formed by the unit direction vectors of the prismatic joint axes.

c$_2$) If the joint axes are not complanar and $VT(u)$ is not a subalgebra then $AC(u) \cap VT(u) = K(u) = span(\hat{Y})$ and just the motions satisfying the equation $\dot{Y}_C = \lambda\hat{Y}$, $\lambda \neq 0$ are *nt*-asymptotic.

Corollary 3. In the case of (3,1)-robots with two rotational joints, RRT, RTR, TRR, we have:

(a) The subspace τ_D is spanned on a direction vector of the prismatic joint axis and a normal vector of the plane ξ of the rotational joint axes. Thus a position (u) is singular iff the prismatic joint axis is orthogonal to the plane ξ.

(b) $VT(u)$ is a subalgebra iff the prismatic joint axis is orthogonal to the rotational joint axes but it is not orthogonal to the plane ξ.

(c) The rotational joint axes are parallel with τ_D at any position (u) iff all joint axes are parallel. There are only *t*-asymptotic motions in this case.

(d) A *nt*-asymptotic motion there is only in the case when the rotational joint axes are not parallel with τ_D and $VT(u)$ is not a subalgebra. It is determined by the equation $\dot{Y}_C = \lambda\hat{Y}$, $\lambda \neq 0$, $AC(u) \cap VT(u) = K(u) = span(\hat{Y})$.

Corollary 4. At a (3,1)-robot with three rotational joints, RRR, we have. The subspace τ_D is spanned on the normal vectors of the planes $\xi_1 = (o_1, o_2)$, $\xi_2 = (o_1, o_3)$ and thus $VT(u)$ is always subalgebra excepting singular positions when $\xi_1 = \xi_2$. There are not *nt*-asymptotic motions.

4.3. **(3,2)-robots.** Excluding helical joints there are the cases: RRT, RTR, TRR, RRR. A robot RRR is a $(3,2)$-robot at any position only in two cases: $o_1 \parallel o_2$ or $o_3 \parallel o_2$. At any position (u) there are at least two rotational joints with non parallel axes and thus in every space $VT(u)$ there are such velocity twists $B_1 = (\bar{s}_1, \bar{b}_1)$, $\bar{s}_1 \cdot \bar{b}_1 = 0$, $B_2 = (\bar{s}_2, \bar{b}_2)$, $\bar{s}_2 \cdot \bar{b}_2 = 0$, $\bar{s}_1 \times \bar{s}_2 \neq \bar{0}$, $B_3 = (\bar{0}, \bar{b}_3 \neq \bar{0})$ that $VT(u) = span(B_1, B_2, B_3)$, $\tau = span(B_3)$, $\tau_D = span(\bar{b}_3)$, $AC(u) = span([B_1, B_2] = (\bar{s}_1 \times \bar{s}_2, \bar{s}_1 \times \bar{b}_2 + \bar{s}_2 \times \bar{b}_1, [B_1, B_3] = (\bar{0}, \bar{s}_1 \times \bar{b}_3), [B_2, B_3] = (\bar{0}, \bar{s}_2 \times \bar{b}_3))$.

Proposition 2. In the case of a $(3,2)$-robot there is not a singular position and $VT(u)$ can not be a subalgebra.

Proof. As $\bar{s}_1 \times \bar{s}_2 \neq \bar{0}$ then $\bar{s}_1 \times \bar{s}_2 \notin span(\bar{s}_1, \bar{s}_2)$ and thus $VT(u)$ can not be a subalgebra. A position (u) is singular iff $\bar{b}_3 = \bar{0}$, i.e. $\tau_D = \bar{0}$. It is impossible at RRT, RTR, TRR and also at RRR as $o_1 \neq o_2$, $o_2 \neq o_3$. □

Lemma 3. At any three-parametric robot if $\dim AC(u) = 3$ and $AC(u) \cap VT(u) = \bar{0}$ then a motion is asymptotic iff it is an u^i-motion, $i = 1, 2, 3$.

Proof. If $AC(u) \cap VT(u) = \bar{0}$ then a motion is asymptotic iff $\bar{0} = \dot{Y}_C(u) = \sum_{i<j}^{3} \dot{u}^i \dot{u}^j [Y_i, Y_j]$. As $\dim AC(u) = 3$ then $\dot{Y}_C = \bar{0}$ iff $\dot{u}^1\dot{u}^2 = 0$, $\dot{u}^1\dot{u}^3 = 0$, $\dot{u}^2\dot{u}^3 = 0$. It completes our proof. □

Let $\Omega(u)$ denote the vector space spanned on the direction vectors of all rotational joint axes at a position (u).

Lemma 4. At any $(3,2)$-robot we have: $\dim AC(u) < 3$ iff $\tau_D(u) \subset \Omega(u)$.

Proof. $\dim AC(u) < 3$ iff $\overline{0} = (\overline{s}_1 \times \overline{b}_3) \times (\overline{s}_2 \times \overline{b}_3)$; i.e., iff $(\overline{s}_1 \times \overline{s}_2) \cdot \overline{b}_3 = 0$. The proof is finished. □

Proposition 3. At any $(3,2)$-robot the equation $AC(u) \cap VT(u) = \overline{0}$ is valid.

Proof. he relation $k_1[B_1,B_2] + k_2[B_1,B_3] + k_3[B_2,B_3] = c_1B_1 + c_2B_2 + C_3B_3$ is true iff $k_1\overline{s}_1 \times \overline{s}_2 = c_1\overline{s}_1 + c_2\overline{s}_2$, $k_1(\overline{s}_1 \times \overline{b}_2 - \overline{s}_2 \times \overline{b}_1) + k_2\overline{s}_1 \times \overline{b}_3 + k_3\overline{s}_2 \times \overline{b}_3 = c_1\overline{b}_1 + c_2\overline{b}_2 + c_3\overline{b}_3$; i.e., iff $k_1 = 0$, $c_1 = 0$, $c_2 = 0$ and $(k_2\overline{s}_1 + k_3\overline{s}_2) \times \overline{b}_3 = c_3\overline{b}_3$. Evidently $\dim AC(u)$ is 2 or 3. If $\dim AC(u) = 2$ then we can suppose that $AC(u) = span([B_1,B_2],[B_1,B_3])$; i.e.; $k_3 = 0$. Then $k_2\overline{s}_1 \times \overline{b}_3 = c_3\overline{b}_3$ is true iff $k_2 = 0$, $c_3 = 0$. In the case $\dim AC(u) = 3$ the equation $(k_2\overline{s}_1 + k_3\overline{s}_2) \times \overline{b}_3 = c_3\overline{b}_3$ is equivalent to $k_2 = k_3 = c_3 = 0$. It completes our proof. □

Corollary 5. At any $(3,2)$-robot we have if $\dim AC(u) = 3$ then by *Proposition 3* and *Lemma 4* only u^i-motions are asymptotic.

In the case $\tau_D(u) \subset \Omega(u)$, $\overline{b}_3 = c_1\overline{s}_1 + c_2\overline{s}_2$, we compute successively:

$$(6_1) \qquad \begin{gathered} RRT: Y_1 = B_1, Y_2 = B_2, Y_3 = B_3, \\ \dot{Y}_C = \dot{u}^1\dot{u}^2[B_1,B_2] + \dot{u}^3(c_2\dot{u}^1 - c_1\dot{u}^2)(\overline{0}, \overline{s}_1 \times \overline{s}_2), \end{gathered}$$

$$(6_2) \qquad \begin{gathered} RTR: Y_1 = B_1, Y_2 = B_3, Y_3 = B_2, \\ \dot{Y}_C = \dot{u}^1\dot{u}^3[B_1,B_2] + \dot{u}^2(c_2\dot{u}^1 + c_1\dot{u}^3)(\overline{0}, \overline{s}_1 \times \overline{s}_2), \end{gathered}$$

$$(6_3) \qquad \begin{gathered} TRR: Y_1 = B_3, Y_2 = B_1, Y_3 = B_2, \\ \dot{Y}_C = \dot{u}^2\dot{u}^3[B_1,B_2] + \dot{u}^1(-c_2\dot{u}^2 + c_1\dot{u}^3)(\overline{0}, \overline{s}_1 \times \overline{s}_2), \end{gathered}$$

$$(6_4) \qquad \begin{gathered} RRR, o_1 \parallel o_2: Y_1 = B_1, Y_2 = B_3 + B_1, Y_3 = B_2, \\ \dot{Y}_C = \dot{u}^3(\dot{u}^1 + \dot{u}^2)[B_1,B_2] + \dot{u}^2(c_2\dot{u}^1 + c_1\dot{u}^3)(\overline{0}, \overline{s}_1 \times \overline{s}_2), \end{gathered}$$

$$(6_5) \qquad \begin{gathered} RRR, o_2 \parallel o_3: Y_1 = B_1, Y_2 = B_2, Y_3 = B_3 + B_2, \\ \dot{Y}_C = \dot{u}^1(\dot{u}^2 + \dot{u}^3)[B_1,B_2] + \dot{u}^3(c_2\dot{u}^1 - c_1\dot{u}^2)(\overline{0}, \overline{s}_1 \times \overline{s}_2). \end{gathered}$$

In the case of RRR, $\tau_D = span(\overline{b}_3)$ is orthogonal to the plane ξ of the parallel joint axes. Evidently the relation $\tau \subset \Omega(u)$ at RRT, RTR, TRR means complanarity of all joint axes. This property is preserved at all positions in a RTR-robot but in the case of robots RRT, TRR only iff the prismatic joint axis is parallel with the axis of the neighboring rotational joint. In the case of robots RRT, TRR there always is a position $u_2 = \tilde{u}_2$ so that $\tau_D(\tilde{u}) \subset \Omega(\tilde{u})$ at $(\tilde{u}) = (u_1, \tilde{u}_2, u_3)$. At RRR-robots $\overline{b}_3 = \overline{n}_\xi$ is a normal vector of the plane ξ. Therefore at RRR-robots the relation $\tau_D \subset \Omega(u)$ is true at a position $u_2 \overset{\circ}{=\rightarrow} u_2$ when the joint axis which is not parallel with o_2 is complanar with the plane ξ. This parameter $u_2 \stackrel{\circ}{=\!\!\to} u_2$ always exists but this property is not preserved. By *Proposition 3* a motion is asymptotic iff $\dot{Y}_C = 0$. We conclude from the relations (6).

Proposition 4. The motions, when $\tau_D \subset \Omega(u)$ for all positions $(u(t))$, can not be nt-asymptotic. They can be only t-asymptotic: u^i-motions or motions when only the prismatic and the rotational joint with parallel axes work.

Corollary 6. At any (3,2)-robots there are not nontrivial asymptotic motions.

Let us turn to rotational velocity twists in $VT(u)$ and to the Klein form $KL|_{VT(u)}$.

The equation (5_1) for $Y = t_1B_1 + t_2B_2 + t_3B_3$ to be rotational or translating have the form

$$(5_1') \qquad \begin{array}{c} t_1t_2(\overline{s}_1 \cdot \overline{b}_2 + \overline{s}_2 \cdot \overline{b}_1) + t_1t_3(\overline{s}_1 \cdot \overline{b}_3) + t_2t_3(\overline{s}_2 \cdot \overline{b}_3) = 0, \\ \det KL|_{VT(u)} = 2(\overline{s}_1 \cdot \overline{b}_2 + \overline{s}_2 \cdot \overline{b}_1)(\overline{s}_1 \cdot \overline{b}_3)(\overline{s}_2 \cdot \overline{b}_3). \end{array}$$

A twist Y is translating iff $Y \in \tau$; i.e., iff $t_1 = 0, t_2 = 0$. Y is rotational iff $t_1^2 + t_2^2 \neq 0$ and $(5_1')$ is satisfied. By *Lemma 2* the form $KL|_{VT(u)}$ is singular; i.e., $\det KL|_{VT(u)} = 0$, in two cases: a) two rotational joint axes are intersecting, b) at least one rotational joint axis is perpendicular to τ_D. Then $KL|_{VT(u)}$ is always singular at a (3,2)-robot RRR. Evidently the point of intersection of the neighboring rotational joint axes is preserved. At RTR the point of intersection of the axes o_1, o_3 is preserved iff the axes o_1, o_2, o_3 are complanar.

Let us recall that the equation $(5_1')$ determines in the projective plane $\mathcal{P}_2$ of all screws in $VT(u)$ a quadratic curve (q) on which the translating screw τ lies. Therefore the infinity axis of τ belongs to the set Q of all axes of the velocity screws from (q) and thus if $KL|_{VT(u)}$ is regular then Q is a system of straight lines in a hyperbolic paraboloid. If (q) is singular then the Klein space $K(u)$ is not zero. We describe the set Q using *Corollary 1.* Evidently the rank of $KL|_{VT(u)}$ is 0 or 2. The rank of $KL|_{VT(u)} = 0$ iff two rotational joint axes are intersecting and they are orthogonal to τ_D. In this case $K(u) = VT(u)$. There are no helical twists in $VT(u)$ and Q is the space of all straight lines in the plane orthogonal to τ_D. If the rank of $KL|_{VT(u)}$ is 2 then $\dim K(u) = 1$, $K(u) = span(\hat{Y})$. There are two cases:

(a) $\hat{Y}$ is not translating. Then its axis $\hat{o}$ intersects or it is parallel with the rotational joint axes and it is orthogonal to τ_D. The set Q is decomposed on the bundle Q_1 and Q_2. The bundle Q_1 forms intersecting lines involving $\hat{o}$ and the rotational joint axis which is not orthogonal to τ_D. The bundle Q_2 forms parallel lines (including $\hat{o}$ and the rotational joint axis which is orthogonal to τ) the plane of which is orthogonal to τ_D.

(b) $\hat{Y}$ is translating, $\hat{Y} = (\overline{0}, \overline{m})$, $\tau_D = span(\overline{m})$. Then the axis of any rotational twist in $VT(u)$ is orthogonal to $\overline{m}$. The set Q is decomposed on two bundles Q_1, Q_2 of parallel straight lines the planes of which are orthogonal to τ_D.

4.4. **(3,3)-robots.** Let us analyze a singular position (u) of (3,3)-robots. The velocity joint twists Y_1, Y_2, Y_3 at a position (u) are of the forms: $Y_1 = (\overline{\omega}_1, \overline{0})$, $Y_2 = (\overline{\omega}_2, \overline{m}_2)$, $Y_3 = (\overline{\omega}_3, \overline{m}_3)$, $\overline{\omega}_2 \cdot \overline{m}_2 = 0$, $\overline{\omega}_3 \cdot \overline{m}_3 = 0$, $\overline{\omega}_1 \times \overline{\omega}_2 \neq \overline{0}$. Then $\dim VT(u) < 3$ iff $k_1\overline{\omega}_1 + k_2\overline{\omega}_2 + k_3\overline{\omega}_3 = \overline{0}$, $k_2\overline{m}_2 + k_3\overline{m}_3 = \overline{0}$, $k_1, k_2, k_3 \in \mathbb{R}$, $k_1^2+k_2^2+k_3^2 \neq 0$. As $\overline{\omega}_1 \times \overline{\omega}_2 \neq \overline{0}$ therefore $k_3 \neq 0$. The following cases are possible: a) $k_2 = 0$. Then $\overline{m}_3 = \overline{0}$, $\overline{\omega}_3 = k\overline{\omega}_1$ and thus $o_1 \equiv o_3$; i.e., at the rotation around o_2 the joint axis o_3 is identified with o_1. It is possible iff the axes o_1, o_3 lie on the same rotational cone or belong to the same system of straight lines of a one sheet rotational hyperboloid with axis o_2. b) $k_2 \neq 0$. Then $\overline{m}_3 = k\overline{m}_2$ and thus the plane (O, o_2) is identified with the plane (O, o_3). As $\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3$ are complanar then $o_1 \in (O, o_2)$ and o_1 intersect o_2. If we choose the origin $O = o_1 \cap o_2$ then $\overline{m}_2 = \overline{0} = \overline{m}_3$. It means that o_1, o_2, o_3 are intersecting at a common point. We conclude: if a (3,3)-robot all joint axes of which are intersecting at a common point or the axes o_1, o_3 belong to the same system of straight lines of a one sheet rotational hyperboloid the axis of which is o_2 then the (3,3)-robot has singular positions for some value $u^2 = \overline{u}^2$.

The equation (5_1) for a velocity twist $Y = t_1Y_1 + t_2Y_2 + t_3Y_3$ to be rotational or translating have the form

$$(5_1") \qquad \begin{gathered} t_1t_2(\overline{\omega}_1 \cdot \overline{m}_2) + t_1t_3(\overline{\omega}_1 \cdot \overline{m}_3) + t_2t_3(\overline{\omega}_2 \cdot \overline{m}_3 + \overline{\omega}_3 \cdot \overline{m}_2) = 0, \\ \det KL|_{VT(u)} = 2(\overline{\omega}_1 \cdot \overline{m}_2)(\overline{\omega}_1 \cdot \overline{m}_3)(\overline{\omega}_2 \cdot \overline{m}_3 + \overline{\omega}_3 \cdot \overline{m}_2). \end{gathered}$$

As the axes o_1, o_2 and o_2, o_3 can not be parallel then due to the equation (5_2) at a position (u) there is a translating twist in $VT(u)$ iff $\overline{\omega}_3 \in span(\overline{\omega}_1, \overline{\omega}_2)$; i.e., iff the axes o_1, o_2, o_3 are complanar. Then if $KL|_{VT(u)}$ is regular and $\overline{\omega}_3 \notin span(\overline{\omega}_1, \overline{\omega}_2)$ or $\overline{\omega}_3 \in span(\overline{\omega}_1, \overline{\omega}_2)$ then the set Q of all axes of all rotational twists in $VT(u)$ is a system of the straight lines on an one sheet hyperboloid or in a hyperbolic paraboloid respectively.

The form $KL|_{VT(u)}$ is singular; i.e., $\det KL|_{VT(u)} = 0$, iff at least two joint axes are intersecting. Its rank can be 0 or 2. It is zero in two cases: a) All joint axes have a common point (a spherical robot) b) The axis o_2 intersects o_1 and o_3 in two different points and the axis o_3 rotating around o_2 gets at a value $u^2 = \overset{\circ}{\rightarrow} u^2$ into plane (o_1, o_2) and intersects o_1. The rank of $KL|_{VT(u)}$ is two iff joint axes are intersecting excluding two of them. In this case the Klein space $K(u)$ is spanned on a twist $\hat{Y}$ and the axes of all rotational twists in $VT(u)$ form two bundles of the straight lines with the common axis $\hat{o}$ of $\hat{Y}$. If the joint axes are complanar at (u) then there is a twist $(\overline{0}, \overline{m}) \in VT(u)$ and thus one bundle is a bundle of the parallel straight lines of the plane which is orthogonal to $\overline{m}$.

Let us turn to the investigation of asymptotic motions. In the case when all joint axes have a common point (a spherical robot) all motions at the regular position are asymptotic. A singular motions can be asymptotic only if it is u^i-motion.

Let use analyze cases when o_1, o_2 or o_2, o_3 are intersecting and all axes have no a common point. We have the following cases:

a) $o_1 \cap o_2 = M$. We choose $M = O$ as the origin of $\mathcal{S}_0$. Then $Y_1 = (\overline{\omega}_1, \overline{0})$, $Y_2 = (\overline{\omega}_2, \overline{0})$, $Y_3 = (\overline{\omega}_3, \overline{m}_3)$, $\overline{m}_3 \neq \overline{0}$, $\overline{\omega}_3 \cdot \overline{m}_3 = 0$, $[Y_1, Y_2] = (\overline{\omega}_1 \times \overline{\omega}_2, \overline{0})$, $[Y_1, Y_3] = (\overline{\omega}_1 \times \overline{\omega}_3, \overline{\omega}_1 \times \overline{m}_3)$, $[Y_2, Y_3] = (\overline{\omega}_2 \times \overline{\omega}_3, \overline{\omega}_2 \times \overline{m}_3)$. Then $Y = t_1[Y_1, Y_2] + t_2[Y_1, Y_3] + t_3[Y_2, Y_3]$ lies in $VT(u)$ iff $Y = k_1 Y_1 + k_2 Y_2 + k_3 Y_3$; i.e., iff the following equalities are valid:

$$(7) \qquad k_1\overline{\omega}_1 + k_2\overline{\omega}_2 + k_3\overline{\omega}_3 = t_1\overline{\omega}_1 \times \overline{\omega}_2 + t_2\overline{\omega}_1 \times \overline{\omega}_3 + t_3\overline{\omega}_2 \times \overline{\omega}_3$$

$$(8) \qquad k_3\overline{m}_3 = t_2\overline{\omega}_1 \times \overline{m}_3 + t_3\overline{\omega}_2 \times \overline{m}_3 = (t_2\overline{\omega}_1 + t_3\overline{\omega}_2) \times \overline{m}_3.$$

We are interested in the case $Y \neq 0$ (we find asymptotic motions that are not u^i-motions). The vector scalar multiplication of (8) by $\overline{m}_3$ gives $k_3 = 0$. Then (8) is satisfied iff $\overline{m}_3 = k(t_2\overline{\omega}_1 + t_3\overline{\omega}_2)$, $0 \neq k \in \mathbb{R}$; i.e., iff the plane $\beta = (O, o_3)$ is orthogonal to the plane $\alpha = (o_1, o_2)$; i.e., iff the plane β includes the straight line p orthogonal to α and going cross O. Then o_3 intersects p or it is parallel with p. This property is preserved at a rotation around o_2 iff β is orthogonal to o_2. When β is not orthogonal to o_2 then the straight line o_3 creates at a rotation around o_2 a surface $\mathcal{F}$. If p intersects or does not intersect $\mathcal{F}$ then there is a value $u^2 = \tilde{u}^2$ such that o_3 intersects p or o_3 is parallel with p and thus β is orthogonal to α at $\tilde{u} = (u^1, \tilde{u}^2, u^3)$. At a motion $u^2 = \tilde{u}^2$, $\dot{u}^2 = 0$, it holds $\dot{Y}_C = \dot{u}^1\dot{u}^3[Y_1, Y_3]$. For $Y = [Y_1, Y_3]$; i.e., for $t_1 = 0$, $t_2 = 1$, $t_3 = 0$, the equations (7), (8) are satisfied; i.e., $[Y_1, Y_3] \in VT(\tilde{u})$, iff $k_1\overline{\omega}_1 + k_2\overline{\omega}_2 = \overline{\omega}_1 \times \overline{\omega}_3$, $\overline{\omega}_1 \times \overline{m}_3 = \overline{0}$. In this case $\overline{m}_3 = c\overline{\omega}_1$, (i.e., $\overline{\omega}_1 \cdot \overline{\omega}_3 = 0$), $k_1\overline{\omega}_2 \cdot \overline{\omega}_3 = 0$, $\overline{\omega}_2 \cdot \overline{\omega}_3 = 0$ and so o_1 is orthogonal to β, o_3 is orthogonal to α, $\overline{\omega}_1 \times \overline{\omega}_3$ is a direction vector of the axis $\hat{o} = \alpha \cap \beta$ of the Klein twist $\hat{Y}$. Let us analyze the case when β is orthogonal to o_2. Then β is always orthogonal to α, $\overline{m}_3 = k\overline{\omega}_2$ and the relation (8) is satisfied iff $k_3 = 0$, $t_2 = 0$. As $\hat{o} = \alpha \cap \beta$ then $\hat{Y} = (\hat{\overline{\omega}}, \overline{0})$, $\hat{\overline{\omega}} \cdot \overline{\omega}_2 = 0$, $\hat{\overline{\omega}} \in span(\overline{\omega}_1, \overline{\omega}_2)$, $\hat{\overline{\omega}} \in span((\overline{\omega}_1 \times \overline{\omega}_2), (\overline{\omega}_2 \times \overline{\omega}_3))$ as $\hat{o} \in \beta$. Then (7) is satisfied iff $k_1\overline{\omega}_1 + k_2\overline{\omega}_2 = \hat{\overline{\omega}} = t_1\overline{\omega}_1 \times \overline{\omega}_2 + t_3\overline{\omega}_2 \times \overline{\omega}_3$. It means that $AC(u) \cap VT(u) = span(\hat{Y}(u))$ for any (u). Then a motion is asymptotic iff $\dot{Y}_C = \lambda\hat{Y}$; i.e., iff $\dot{u}^1\dot{u}^2[Y_1, Y_2] + \dot{u}^1\dot{u}^3[Y_1, Y_3] + \dot{u}^2\dot{u}^3[Y_2, Y_3] = \lambda(c_1[Y_1, Y_2] + c_2[Y_2, Y_3])$, $c_1^2 + c_2^2 \neq 0$, i.e. iff $\dot{u}^1\dot{u}^2 = \lambda c_1$, $\dot{u}^1\dot{u}^3 = 0$, $\dot{u}^2\dot{u}^3 = \lambda c_2$. If $\lambda = 0$ then only u^i-motions can be asymptotic. When $\lambda \neq 0$ then we have the following cases: i) If $\dot{u}^1 = 0$ then $c_1 = 0$. Then $\hat{Y} = c_2[Y_2, Y_3]$; i.e., $\hat{\overline{\omega}} = c_2\overline{\omega}_2 \times \overline{\omega}_3$. Then o_3 is orthogonal to α. This position is possible only for a special value $u^2 = \tilde{u}^2$. Then $\dot{u}^2 = 0$ and $c_2 = 0$. Then $\hat{Y} = \overline{0}$. It is impossible. ii) If $\dot{u}^3 = 0$ then $c_2 = 0$. We get $\hat{Y} = c_1[Y_1, Y_2]$, i.e. $\hat{\overline{\omega}} = c_1\overline{\omega}_1 \times \overline{\omega}_2$. It is impossible. We summarize the previous results.

Proposition 5. Let o_1, o_2 be intersecting in $O = o_1 \cap o_2$ and $O \notin o_3$. Let o_3 does not lie in the plane orthogonal to o_2 cross O. Let $\hat{o}$ be the straight line in the plane $\alpha = (o_1, o_2)$ orthogonal to o_1 cross O. Then only if o_3 lies in the plane κ orthogonal to o_2 and its distance from o_2 is equal to $|MN|$, $M = \kappa \cap o_2$, $N = \kappa \cap \hat{o}$, there are asymptotic motions with non zero C-twists. They are given by the equation $u^2 = \tilde{u}^2$, $\dot{u}^2 = 0$ where $\tilde{u} = (u^1, \tilde{u}^2, u^3)$ is a position when $o_3(\tilde{u})$ is orthogonal to α. Their C-twists are of the form $\dot{Y}_C(\tilde{u}) = \dot{u}^1 \dot{u}^3 k \tilde{Y}(\tilde{u})$ where $\tilde{Y}(\tilde{u})$ is the Klein twist in $VT(\tilde{u})$. If o_3 lies in the plane orthogonal to o_2 cross O then $VT(u) \cap AC(u) = K(u) = span(\hat{Y})$ at any (u) but there is not an asymptotic motion the C-twist of which is $\lambda \hat{Y}$, $\lambda \neq 0$.

b) In the case when o_2, o_3 are intersecting and o_2 does not intersect o_1 we have:

Proposition 6. Let o_2, o_3 be intersecting in $M = o_2 \cap o_3$ and o_1 does not intersect o_2. Let p be the straight line orthogonal to the plane $\alpha = (o_1, M)$ going cross M. Let $o_2 \neq p$. Then only in the case when o_1 is orthogonal to the plane $\beta = (o_2, o_3)$ and $\angle(o_2, o_3) = \angle(o_2, p)$ there are asymptotic motions with non zero C-twists. They are given by the equations $u^2 = \tilde{u}^2$, $\dot{u}^2 = 0$, $\dot{Y}_C = \dot{u}^1 \dot{u}^3 k \hat{Y}$, $k \neq 0$ where $\tilde{u} = (u^1, \tilde{u}^2, u^3)$ is a position at which $o_3 = p$ and $\hat{Y}$ is the Klein twist. If $p = o_2$ then $AC(u) \cap VT(u) = span(\hat{Y}(u))$ at any (u) but an asymptotic motion the C-twist of which is $\lambda \hat{Y}$, $\lambda \neq 0$, there is iff o_1 is orthogonal to the plane (o_2, o_3). Its equation is $\dot{u}^3 = 0$.

Proof. Evidently $\alpha \cap \beta = \hat{o}$ is the axis of the Klein twist $\hat{Y} \neq \overline{0}$ and intersects o_1 at $N = o_1 \cap \hat{o}$. Now $\beta = (o_2, \hat{o})$. When the only second joint works the plane α is stable and the point N is moving along o_1. Evidently $VT(u) = span(Y_1, \hat{Y}, Y_2)$. We choose N as the origin of $\mathcal{S}_0$ at $u(t)$. Then $Y_1 = (\overline{\omega}_1, \overline{0})$, $\hat{Y} = (\hat{\overline{\omega}}, \overline{0})$, $Y_2 = (\overline{\omega}_2, \overline{m}_2)$, $\overline{\omega}_2 \cdot \overline{m}_2 = 0$, $\overline{m}_2 \neq \overline{0}$, $\hat{\overline{\omega}} \cdot \overline{m}_2 = 0$, $VT(u) = span(Y_1, \hat{Y}, Y_2)$, $[Y_1, \hat{Y}] = (\overline{\omega}_1 \times \hat{\overline{\omega}}, \overline{0})$, $[Y_1, Y_2] = (\overline{\omega}_1 \times \overline{\omega}_2, \overline{\omega}_1 \times \overline{m}_2)$, $[\hat{Y}, Y_2] = (\hat{\overline{\omega}} \times \overline{\omega}_2, \hat{\overline{\omega}} \times \overline{m}_2)$, $Y_3 = (\overline{\omega}_3, \overline{m}_3) = z_1 \hat{Y} + z_2 Y_2$, $\overline{m}_3 = z_2 \overline{m}_2$, $z_1 \neq 0$. Solving the question of existence of non zero asymptotic twists $Y \in AC(u) \cap VT(u)$ we can repeat our considerations on the relation (7) and (8) where instead Y_2, Y_3 we put $\hat{Y}, Y_2$ respectively. If there are non zero asymptotic twists then the plane β must be orthogonal to α; i.e., the plane β includes the line p. There are the following cases: a) If $o_2 \neq p$ then there are only special values $u_2 = \tilde{u}_2$ when $\beta \equiv \xi = (o_2, p)$. At the position $\tilde{u} = (u^1, \tilde{u}^2, u^3)$ β is orthogonal to α and $\overline{m}_2 = y_1 \overline{\omega}_1 + y_2 \hat{\overline{\omega}}$. For the motions $u^2 = \tilde{u}^2$, $\dot{u}^2 = 0$ we have $\dot{Y}_C = \dot{u}^1 \dot{u}^3 [Y_1, Y_3]$. Then $\dot{Y}_C$ is asymptotic iff $[Y_1, Y_3] = k_1 Y_1 + k_2 \hat{Y} + k_3 Y_2$; i.e., iff $z_1(\overline{\omega}_1 \times \hat{\overline{\omega}}) + z_2(\overline{\omega}_1 \times \overline{\omega}_2) = k_1 \overline{\omega}_1 + k_2 \hat{\overline{\omega}} + k_3 \overline{\omega}_2$, $z_2 \overline{\omega}_1 \times \overline{m}_2 = k_3 \overline{m}_2$. The last relation is true iff $k_3 = 0$, $z_2 = 0$ or $k_3 = 0$, $\overline{\omega}_1 = k \overline{m}_2$, $0 \neq k \in \mathbb{R}$. In the former, $(o_3 \equiv \hat{o})$, the first relation $z_1(\overline{\omega}_1 \times \hat{\overline{\omega}}) = k_1 \overline{\omega}_1 + k_2 \hat{\overline{\omega}}$ can not be satisfied. In the further case, scalar multiplication of the firs relation by $\overline{\omega}_1 = k \overline{m}_2$ gives $k_1 = 0$. Then $\overline{\omega}_1 \times (z_1 \hat{\overline{\omega}} + z_2 \overline{\omega}_2) = \overline{\omega}_1 \times \overline{\omega}_3 = k_2 \hat{\overline{\omega}}$ is satisfied iff $\overline{\omega}_3$ is orthogonal to α. Now $\dot{Y}_C = \dot{u}^1 \dot{u}^3 k_2 \hat{Y} \in VT(\tilde{u})$. It means that if $p \neq o_2$ then only in the case when $\overline{\omega}_1$ is orthogonal to β and $\angle(o_2, o_3) = \angle(o_2, p)$ there are positions $\tilde{u} = (u^1, \tilde{u}^2, u^3)$ at which $\overline{0} \neq \dot{Y}_C(\tilde{u}) \in VT(\tilde{u})$. Let us analyse the case b) $o_2 = p$. Now β is always orthogonal to α; i.e., $\overline{m}_2 \in span(\overline{\omega}_1, \hat{\overline{\omega}})$. The relations (7), (8) in the base $Y_1, \hat{Y}, Y_2$ are of the form:

$$(7') \qquad k_1\overline{\omega}_1 + k_2\hat{\overline{\omega}} + k_3\overline{\omega}_2 = t_1\overline{\omega}_1 \times \hat{\overline{\omega}} + t_2\overline{\omega}_1 \times \overline{\omega}_2 + t_3\hat{\overline{\omega}} \times \overline{\omega}_2$$

$$(8') \qquad k_3\overline{m}_2 = t_2\overline{\omega}_1 \times \overline{m}_2 + t_3\hat{\overline{\omega}} \times \overline{m}_2 = (t_2\overline{\omega}_1 + t_3\hat{\overline{\omega}}) \times \overline{m}_2$$

The relation $(8')$ is true iff $k_3 = 0$, $t_2\overline{\omega}_1 + t_3\hat{\overline{\omega}} = k\overline{m}_2$. Multiplying the equation $(7')$ by $\overline{\omega}_2$ we get $t_1 = 0$. Then $(7')$ is of the form $k_1\overline{\omega}_1 + k_2\hat{\overline{\omega}} = k\overline{m}_2 \times \overline{\omega}_2$. Let us multiply the last equation by $\overline{m}_2$. As $\overline{\omega}_1 \cdot \overline{m}_2 \neq 0$ we have $k_1 = 0$ and then $k_2\hat{\overline{\omega}} = k\overline{m}_2 \times \overline{\omega}_2$. It means that $VT(u) \cap AC(u) = span(\hat{Y}(u))$ at any (u). A motion is asymptotic iff $\dot{Y}_C = \lambda\hat{Y}$; i.e., iff $(\dot{u}^1\dot{u}^2 + z_2\dot{u}^1\dot{u}^3)[Y_1, Y_2] + \dot{u}^1\dot{u}^3z_1[Y_1, \hat{Y}] - \dot{u}^2\dot{u}^3z_1[\hat{Y}, Y_2] = \lambda(t_2[Y_1, Y_2] + t_3[\hat{Y}, Y_2]$; i.e., iff $\dot{u}^1\dot{u}^2 + z_2\dot{u}^1\dot{u}^3 = \lambda t_2$, $\dot{u}^1\dot{u}^3z_1 = 0$, $-\dot{u}^2\dot{u}^3z_1 = \lambda t_3$. If $\lambda = 0$ then only u^i-motions can be asymptotic. For $\lambda \neq 0$ we have following cases: i) $\dot{u}^1 = 0$, $t_2 = 0$. Then $t_3\hat{\overline{\omega}} = k\overline{m}_2$. It is impossible. ii) $\dot{u}^3 = 0$, $t_3 = 0$. Then $t_2\overline{\omega}_1 = k\overline{m}_2$; i.e., o_1 is orthogonal to β. It completes our proof. □

Finally we will deal with the case when any joint axes are not intersecting at a position (u). If (u) is a regular position then in $VT(u)$ there are velocity twists $B_1 = (\overline{s}_1, \overline{0}), B_2 = (\overline{s}_2, \overline{b}_2), B_3 = (\overline{s}_3, \overline{b}_3)$ such that $(\overline{s}_1, \overline{s}_2, \overline{s}_3)$ is an orthonormal base and $\overline{s}_1 \times \overline{s}_2 = \overline{s}_3$, $\overline{s}_1 \times \overline{s}_3 = -\overline{s}_2$, $\overline{s}_2 \times \overline{s}_3 = \overline{s}_1$. Let us remark that B_2, B_3 can be helical. Let $\overline{b}_i = b_i^1\overline{s}_1 + b_i^2\overline{s}_2 + b_i^3\overline{s}_3$, $i = 1, 2, 3$, $b_1^k = 0$, $k = 1, 2, 3$. We get $[B_1, B_2] = (\overline{s}_1 \times \overline{s}_2 = \overline{s}_3, \overline{s}_1 \times \overline{b}_2 = b_2^2\overline{s}_3 + b_2^3\overline{s}_2)$, $[B_1, B_3] = (\overline{s}_1 \times \overline{s}_3 = -\overline{s}_2, \overline{s}_1 \times \overline{b}_3 = b_3^2\overline{s}_3 - b_3^3\overline{s}_2)$, $[B_2, B_3] = (\overline{s}_2 \times \overline{s}_3 = \overline{s}_1, \overline{s}_2 \times \overline{b}_3 - \overline{s}_3 \times \overline{b}_2 = \overline{s}_1(b_3^3 + b_2^2) - b_2^1\overline{s}_2 - b_3^1\overline{s}_3)$. A velocity twist $Y = k_1B_1 + k_2B_2 + k_3B_3$ lies in the asymptotic space $A_3(u) \cap AC(u)$ iff there are real numbers t_1, t_2, t_3 such that $t_1[B_1, B_2] + t_2[B_1, B_3] + t_3[B_2, B_3] = k_1B_1 + k_2B_2 + k_3B_3$; i.e., iff $t_1 = k_3, t_2 = -k_2, t_3 = k_1$ and

$$(9) \qquad \begin{aligned} (b_2^2 + b_3^3)k_1 - b_2^1k_2 - b_3^1k_3 = 0 \\ -b_2^1k_1 + (b_2^2 - b_3^3)k_2 + (-b_2^3 - b_3^2)k_3 = 0 \\ -b_3^1k_1 + (-b_2^3 - b_3^2)k_2 + (b_2^2 - b_3^3)k_3 = 0. \end{aligned}$$

Let D be the determinant of the system (9). If $D = 0$ then there is a non zero C-twist $\dot{Y}_C \in AC(u) \cap VT(u)$ in $VT(u)$. If $D \neq 0$ then $AC(u) \cap VT(u) = \overline{0}$. As $\dim AC(u) = 3$ then by *Lemma 3* only u^i-motions are asymptotic. The technical and geometrical interpretation of the condition $D = 0$ or $D \neq 0$ rest open.

Remark 8. As $\det KL|_{VT(u)} \neq 0$ then $K(u) = 0$; i.e., there is not a non zero Klein twist in $VT(u)$. If the axes o_1, o_2, o_3 are not complanar then there is not a translating twist in $VT(u)$ and the axes of all rotational velocity twists in $VT(u)$ form a system of the straight lines in a one sheet hyperboloid.

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