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REGULAR ORIENTED HYPERMAPS UP TO FIVE HYPERFACES

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Abstract. The chiral hypermaps with at most four hyperfaces were classified in [A. Breda d'Azevedo and R. Nedela, *Chiral hypermaps with few hyperfaces*, Math. Slovaca, 53 (2003), n.2, 107–128]. It arises from this classification that all chiral hypermaps are "canonical metacyclic", that is, the one-step rotation about a hypervertex, or about a hyperedge, or about a hyperface, generates a normal subgroup in the orientation-preserving automorphism group. In this paper we complete the above classification by classifying the reflexible regular oriented hypermaps with three and four hyperfaces, and extend the classification to five hyperfaces. The chiral hypermaps arising in this work will be either canonical metacyclic or coverings of canonical metacyclic hypermaps. All have metacyclic monodromy groups and cyclic chirality groups.

1. Introduction

Regular oriented hypermaps algebraically correspond to two-generated groups G with a prescribed couples of generators a and b. Geometrically they determine cellular embeddings of hypergraphs (bipartite graphs) in orientable compact and connected surfaces. Endowing the compact surface with an orbifold-induced metric, the edges of the bipartite map can be seen as geodesics. The genus of the compact surface is the *genus* of the hypermap. If $\mathcal{H} = (G; a, b)$ is a hypermap, the Euler characteristic of $\mathcal{H}$ (that is, the Euler characteristic of its underlying surface) is calculated according to the formula

$$\chi(\mathcal{H}) = V + E + F - |G|,$$

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where V is the number of orbits of $\langle b \rangle$ (the *hypervertices* of $\mathcal{H}$), E is the number of orbits of $\langle ab \rangle$ (the *hyperedges* of $\mathcal{H}$) and F is the number of orbits of $\langle a \rangle$ (the *hyperfaces* of $\mathcal{H}$), under the right action of $G = \langle a, b \rangle$ on itself. If ab is an involution, $\mathcal{H}$ is a *map*. A common central problem in the theory of maps/hypermaps has been the classification of regular oriented hypermaps either by size (order of G) [22, 25], by number of hyperfaces [26, 6], by underlying graph [18, 24, 12], by automorphism group [3], or by genus [14, 15, 16, 21]. As a consequence of the well known Hurwitz bound, the number of regular oriented hypermaps of genus $g > 1$ is finite and bounded by $84(g-1)$. Regular oriented hypermaps on the sphere are easily deduced from the Euler formula, viz. the five Platonic solids, two infinite families of types $(1, n, n)$ and $(2, 2, n)$, plus their duals. A classification of the regular oriented maps on the torus can be seen in Coxeter and Moser [11]. The generalisation to hypermaps was done by Corn and Singerman [10]. The classification problem for double torus was settled in [4] and for higher genera only partial results are known. Conder and Dobcsanyi [9], with computational support, classified all regular oriented maps[1] from genus 8 to 15, raising previous classifications of Sherk [21], Grek [14, 15, 17] and Garb [13]. On the other hand, Breda and Nedela [7] classified the chiral hypermaps of genus up to 4. According to this classification any chiral hypermap must have at least 3 hyperfaces. Chiral hypermaps are regular oriented hypermaps that are not isomorphic with their mirror images (see for instance [1, 6, 5, 11, 19, 23]). We say that a regular oriented hypermap $\mathcal{H} = (G; a, b)$ is *canonical metacyclic* if the rotation one-step about a hyperface, or about a hypervertex or about a hyperedge, generates a normal cyclic subgroup of G (the automorphism group of $\mathcal{Q}$). This is equivalent to say that a rotation one-step about a hyperface (or a hyperedge or a hypervertex) fixes all the hyperfaces (resp. all the hyperedges or all the hypervertices). The monodromy (or automorphism) group of a canonical metacyclic hypermap is a metacylcic group, but the converse is not true. One feature standing out from the classification [7] is that the chiral hypermaps with 3 and 4 hyperfaces are all canonical metacyclic.

The classification by number of faces appears most frequently inside other classifications. In [8] it was classified the *reflexible* hypermaps (the regular oriented hypermaps that are isomorphic to their mirror images) with one and two hyperfaces. In [6] we find a classification of chiral hypermaps up to 4 hyperfaces and in [26] a classification of the non-orientable reflexible maps with a prime number of faces and the non-orientable reflexible hypermaps with 1, 2, 3 and 5 hyperfaces. With 4 hyperfaces only a partial result has been established. Non-orientable reflexible hypermaps $\mathcal{H}$ are regular hypermaps on non-orientable compact surfaces - these correspond to groups G with prescribed involutory triples of generators

[1]By the time we have finished writing this paper Conder released in his web-homepage a computer-classification of all regular and chiral hypermaps up to genus 101.

r_0, r_1, r_2 such that the even words r_0r_1 and r_1r_2 still generate G. In this settlement, hypervertices, hyperedges and hyperfaces correspond to orbits of $\langle r_1, r_2\rangle$, $\langle r_2, r_0\rangle$ and $\langle r_0, r_1\rangle$. If r_0r_1 and r_1r_2 generate instead a proper subgroup G^+ (necessarily a normal subgroup of index 2 in G), the reflexible hypermap is orientable and therefore accomplished by the reflexible regular oriented hypermap $\mathcal{H}^+ = (G^+; r_0r_1, r_1r_2)$; in other words, both $\mathcal{H}$ and $\mathcal{H}^+$ determine the same hypergraph cellular embedding on the same compact orientable surface.

Contrary to hypermap theory, in map theory classifying by size is the same as classifying by the number of edges since the size of a regular oriented map with E number of edges is $2E$. This has been done extensively by Wilson [22] with a quasi complete classification of regular oriented maps up to 100 edges, now collected and better completed in the census [25]. More recently Orbanic′ [20] gave a classification of reflexible maps up to 100 edges which are not parallel-product decomposable. On the other hand, we can find a classification of reflexible hypermaps of size $2p$ (p prime) in [2] and a classification of non-orientable reflexible hypermaps of size a power of 2 in [26].

In this paper we complete the classification [6] by computing the reflexible hypermaps with 3 and 4 hyperfaces, and extend this classification to 5 hyperfaces. A complete list of the regular oriented hypermaps (up to duality) with at most 5 hyperfaces can be seen in the Table 1.1.

Most of the definitions and notations are borrowed from [6], where we can also find a more deep introduction to maps, hypermaps and chirality. For short, by a *reflexible hypermap* we mean a reflexible regular oriented hypermap (usually refered as *regular hypermap*).

# faces	extra relations ($\langle a, b \mid a^n = 1, \text{xtra relations}\rangle$)	κ	X
1	$b = a^s$ (cyclic group C_n)	1	1
2	$[a, b] = 1$, $b^2 = a^u$	1	1
2	$b^2 = (ab)^2 = 1$ (dihedral group D_n)	1	1
2	$(ab)^2 = 1$, $b^2 = a^{\frac{n}{2}}$ n even	1	1
3	$[a, b] = 1$, $b^3 = a^u$	1	1
3	$b^3 = a^u$, $bab^{-1} = a^t$ $n \geq 7$, $(t-1)u = 0 \bmod n$, $t^3 = 1 \bmod n$, $t \neq 1 \bmod n$	$\frac{n}{(n,t^2-1)}$	$\langle a^{t^2-1}\rangle$
3	$[a^2, b] = 1$, $b^3 = a^u$, $(ab)^2 = a^v$ n, u, v even and $3v - 2u = 6 \bmod n$	1	1
4	$b^4 = a^u$, $[a, b] = 1$	1	1
4	$b^4 = 1$, $bab^{-1} = a^{-1}$	1	1

4	$b^4 = a^{\frac{n}{2}}$, $bab^{-1} = a^{-1}$ n even	1	1
4	$b^4 = a^u$, $bab^{-1} = a^t$ $n \geq 5$, $t^4 = 1 \bmod n$, $t^2 \neq 1 \bmod n$, $u(t-1) = 0 \bmod n$	$\frac{n}{(n,t^2-1)}$	$\langle a^{t^2-1} \rangle$
4	$[a^2, b^2] = 1, b^4 = a^u, (ab)^2 = a^v, b^{-2}ab^2 = a^t$ n, u, v even, t odd, $2(t-1) = 0 \bmod n$ and $(2v-u-t-3)\frac{u}{2} = (2v-u-t-3)\frac{v}{2} = 0 \bmod n$ $(2v-u-t-3)\frac{t-1}{2} = 0 \bmod n$	1	1
4	$[a^3, b] = 1, b^3 = a^u, (ab)^2 = a^v$ n, u, $v = 0 \bmod 3$ and $-4u + 6v = 12 \bmod n$	1	1
4	$[a^3, b] = 1, b^3 = a^u, (ab)^3 = a^{3u+3v-3}, (ab^{-1})^2 = a^v$ n, u, $v = 0 \bmod 3$ and $4u + 6v = 12 \bmod n$	1	1
5	$[a, b] = 1$, $b^5 = a^u$	1	1
5	$b^5 = a^u$, $bab^{-1} = a^t$ $n \geq 5$, $t^5 = 1 \bmod n$, $t \neq 1 \bmod n$, $u(t-1) = 0 \bmod n$	$\frac{n}{(n,t^2-1)}$	$\langle a^{t^2-1} \rangle$
5	$b^5 = a^u, (ab)^2 = a^v, b^{-1}ab^{-1} = a^{3-v}$ n, u, v even and $-2u + 5v = 10 \bmod n$	1	1
5	$[a^4, b] = 1, b^4 = a^{4t}, (ab)^2 = a^{2t+2}, b^2ab^{-1} = a^{t+1}$ $n = 0 \bmod 4$ and $t = 1 \bmod 4$	5	$\langle [a, b] \rangle$
5	$[a^4, b] = 1, b^5 = a^{5(t-1)}, b^2ab^{-1} = a^t$ $n = 0 \bmod 4$ and $t = 1 \bmod 4$	5	$\langle [a, b] \rangle$

Table 1.1: The regular oriented hypermaps (up to duality and a chiral pair) with 1, 2, 3, 4 and 5 hyperfaces ("faces" in the table) and their chirality indices κ and chirality groups X. Chirality index 1 means reflexible. All the information were collected from [6, 8] and this paper.

Apart from the brief introduction to the classification made in section 2 where the necessary tools and notation are given, the rest of the paper is organised in two sections, one dealing with reflexible hypermaps with 3 and 4 hyperfaces and the other with reflexible and chiral hypermaps with 5 hyperfaces. The sensation of repetition cannot be avoided at all since the relations that appear are different in each case.

2. Preamble to the classification

In what follows, let $\mathcal{Q} = (D; a, b)$ be a regular oriented hypermap of type (l, m, n) with n hyperfaces. We use the bipartite map representation of a hypermap: black and white vertices are the hypervertices and the hyperedges respectively, while faces are the hyperfaces. The set D is the set of *darts* (of the bipartite map), that is, a pair of hyperedge-hypervertex incident flags (see Fig. 1), where a *flag* is a (local) mutually incident triple hypervertex-hyperedge-hyperface (usually represented by a little triangle). The permutation a permutes the darts around

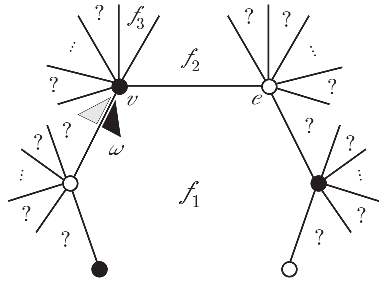

FIGURE 1

hyperfaces as local one-step counter-clockwise face-rotations, while permutation b permutes the darts around hypervertices as local one-step counter-clockwise vertex-rotations. The group Q generated by a and b is called the *monodromy* group of $\mathcal{Q}$ and often denoted by $Mon(\mathcal{Q})$. We have $\mathcal{Q} \cong (Q; a, b)$, where a and b "act" on Q by right multiplication (an *isomorphism* $(D; a, b) \longrightarrow (D'; a', b')$ is a bijective function $\phi : D \to D'$ such that $a\phi = \phi a'$ and $b\phi = \phi b'$). $\mathcal{Q}$ is *reflexible* if $\mathcal{Q}$ is isomorphic to its *mirror pair* $(D; a^{-1}, b^{-1})$. All actions in this paper are right actions.

Let r denote the number of hyperfaces about a hypervertex and s the number of hyperfaces about a hyperedge. As observed in [6], if $\mathcal{Q}$ has 3 or more hyperfaces then $r, s \geq 2$ and one of the following possibilities must occur, either $r \geq 3$ or $s \geq 3$. Up to a $(0,1)$-duality we assume that $r \geq s$. By a σ-duality we mean the duality operation that the permutation $\sigma \in S_3$ induces by interchanging the role of the hypervertices (0-cells), hyperedges (1-cells) and hyperfaces (2-cells); so $(0,1)$-duality just changes hypervertices with hyperedges resulting most of the times a new hypermap.

Let f_1 be a hyperface of $\mathcal{Q}$, v be a hypervertex incident to f_1, e be a hyperedge incident to both v and f_1, and f_2 be the hyperface incident to f_1, v and e. Label the other hyperfaces of $\mathcal{Q}$ as f_3, f_4, ... , f_n. The monodromy group $Mon(\mathcal{Q})$ of a regular oriented hypermap acts regularly on the darts as well as on half of the flags. Fix a dart's root-flag ω (the black flag pictured below) which we identify with the identity of $Mon(\mathcal{Q})$. Acting $Mon(\mathcal{Q})$ on ω, each dart will be marked with a root-flag.

Each element $\gamma \in Mon(\mathcal{Q})$ induces an automorphism φ_γ of $\mathcal{Q}$ by sending each dart $g \in G$ to the dart $\gamma^{-1}g$. The regularity of $\mathcal{Q}$ implies that each automorphism ϕ of $\mathcal{Q}$ is of the form φ_γ for some $\gamma \in G$. Moreover, $Aut(\mathcal{Q}) \cong Q$ and $\mathcal{Q} \cong (Aut(\mathcal{Q}); \varphi_a, \varphi_b)$. Since we have assigned the identity of $Mon(\mathcal{Q})$ to a fixed root-flag of f_1, the automorphism φ_a is a one-step clockwise rotation about the hyperface f_1 while φ_b is a one-step clockwise rotation about the hypervertex v.

The action of $Aut(\mathcal{Q})$ on the hyperfaces induces a permutation of hyperfaces π_g by assigning each hyperface f of $\mathcal{Q}$ to $f\varphi_\gamma$ (functions are written on the right). The automorphisms φ_a and φ_b induce the permutations π_a and π_b. Of course the composition $\varphi_a\varphi_b = \varphi_{ab}$ induces the permutation π_{ab}. For convenience let $A = \pi_a^{-1}$ and $B = \pi_b^{-1}$. By the way f_1, v and e were chosen, the first cycles of B and AB (that is, the cycles containing $1 = f_1$) are (123...) and (12...) respectively. The length of the first cycles of B and AB are r and s respectively. These two permutations must generate a transitive group on $\{1, 2, 3, .., n\}$. Hence the Schreier like diagram[2] induced by B and AB, the $B - AB$ *diagram*, must be connected. A relabelling of f_3, f_4 , ... , f_n produces a new permutation pair $(B', A'B')$ which we call a *relabelling pair*. Relabelling pairs correspond to conjugation pairs $(B^g, (AB)^g)$ by a permutation $g \in S_n$ centralising B and such that $(AB)^g$ sends 1 to 2. Let P be the group generated by A and B, and let $\mathcal{P}$ be the regular oriented hypermap $(P; A^{-1}, B^{-1})$. The permutations A^{-1} and B^{-1} when acting on the right (by right multiplication) give the generators of the monodromy group of $\mathcal{P}$ while when acting on the left (by left multiplication) give the generators of the automorphism group of $\mathcal{P}$. Since $(Aut(\mathcal{Q}); \varphi_a, \varphi_b)$ covers $(P; \pi_a, \pi_b) = (P; A^{-1}, B^{-1})$ then the function $a \mapsto A^{-1}$ and $b \mapsto B^{-1}$ gives rise to a covering from $\mathcal{Q}$ to $\mathcal{P}$. If $(P; A^{-1}, B^{-1})$ is not isomorphic to $(P; A, B)$ then these two hypermaps form a chiral pair. The classification is done up to a chiral pair.

3. Regular oriented hypermaps with 3 and 4 hyperfaces

In this section we complete the classification [6] by analysing the cases that lead to reflexible hypermaps. In Table 3.1 we reprint the list of all possible enumerations of the hyperfaces (through the permutations B, AB and A) of a generic regular hypermap with 3 and 4 hyperfaces, up to a duality and a relabelling of hyperfaces. These permutations determine a regular oriented hypermap $\mathcal{P} = (P; A, B)$ and in this table we also display the associated *H-sequence* of $\mathcal{P}$, that is, a sequence $[l, m, n, V, E, F, |P|]$ formed by the type $(l; m; n)$, the number of hypervertices V, the number of hyperedges E, the number of hyperfaces F and the order $|P|$ of the group generated by A and B.

[2]The diagram whose vertices are 1,...,n, and whose edges reflects the action of A and AB on 1,...,n.

#hyperfaces	Case	B	AB	A	[	l	m	n	V	E	F	$\|P\|$	]
3	**I**	$(1,2,3)$	$(1,2)$	$(2,3)$	[	3	2	2	2	3	3	6	]
3	**II** ✓	$(1,2,3)$	$(1,2,3)$	$()$	[	3	3	1	1	1	3	3	]
4	**III**	$(1,2,3)$	$(1,2)(3,4)$	$(2,3,4)$	[	3	2	3	4	6	4	12	]
4	**IV**	$(1,2,3)$	$(1,2,4)$	$(2,4,3)$	[	3	3	3	4	4	4	12	]
4	**V**	$(1,2,3,4)$	$(1,2)(3,4)$	$(2,4)$	[	4	2	2	2	4	4	8	]
4	**VI** ✓	$(1,2,3,4)$	$(1,2,3,4)$	$()$	[	4	4	1	1	1	4	4	]

Table 3.1: List of all possible enumerations of the 3 and 4 hyperfaces.

Cases II and VI give rise mostly to chiral hypermaps. The chiral ones where classified in [6] so here we classify only the reflexible hypermaps.

Case I. In this case we have $a^n = 1$, $b^3 = a^u$, $(ab)^2 = a^v$ and $b^{-1}ab^{-1} = a^t$, for some n even and $u, v, t \in \{0, \ldots, n-1\}$. Let G be the group generated by a, b subject to these relations and let K be the subgroup generated by a. This has index three in G and so G is partitioned into 3 cosets K, Kb and Kb^2. From the relations we deduce that $Kba^i = Kb$ or Kb^2, according as i is even or odd. Since $Kba^u = Kba^v = Kb$ and $Kba^t = Kb^2$ we find that u and v are even and t is odd. From the second and third relations we derive $b^{-1}a^2b = ba^2b^{-1} = a^{v+t-1}$ which shows that a^2 commutes with b^2, and thus a^2 also commutes with b. Hence $v + t - 1 = 2 \mod n \Leftrightarrow t = 3 - v \mod n$. Then $a^{3-v} = b^{-1}ab^{-1} = b^2ab^2a^{-2u} = ba^{v-1}ba^{-2u} = a^{v-2}baba^{-2u} = a^{2v-3-2u}$ which gives $3v - 2u - 6 = 0 \mod n$. The last relation can then be replaced by $[a^2, b] = 1$ and we have

$$G = \langle a, b | a^n = 1,\ b^3 = a^u,\ (ab)^2 = a^v,\ [a^2, b] = 1 \rangle\,,$$

where n, u and v are even and $3v - 2u - 6 = 0 \mod n$. We now show that these congruencies are enough to describe a group of order $3n$. As G is indexed by 3 even numbers, n, u and v, we rewrite G as $G^{\mathrm{I}}_{(n,u,v)}$. Consider the particular case $(n, u, v) = (n, 0, 2)$, which satisfies the congruency $3v - 2u - 6 = 0 \mod n$. Let now G be $G^{\mathrm{I}}_{(n,0,2)} = \langle a, b | a^n = 1,\ b^3 = 1,\ (ab)^2 = a^2,\ [a^2, b] = 1 \rangle$. Changing generators $\alpha = a$, $\beta = ba^{-v+u+2}$, where u, v are even integers satisfying $3v - 2u - 6 = 0 \mod n$, we get $G^{\mathrm{I}}_{(n,u,v)}$. Hence $G^{\mathrm{I}}_{(n,u,v)}$ has $3n$ elements if and only if G has $3n$ elements. Consider the normal subgroup H of index 2 of G generated by b and a^2. The set $T = \{1, a\}$ is a transversal for H in G. By the Reidmaster-Schreier Rewriting Process, H is freely generated by $A = a^2$ and $B = b$ subject to the relations $A^{\frac{n}{2}} = 1$, $B^3 = A^{\frac{u}{2}}$ and $[A, B] = 1 \Leftrightarrow B^A = A$. Hence H is an abelian metacyclic group of order $3\frac{n}{2}$. Thus $|G| = 2|H| = 3n$ and so $|G^{\mathrm{I}}_{(n,u,v)}| = 3n$.

Case II gives a metacyclic group $G^{\mathrm{II}}_{n,u,t} = \langle a, b \mid a^n = 1, b^3 = a^u, a^b = a^t \rangle$ with $u(t-1) = 0 \mod n$ and $t^3 = 1 \mod n$, and in [6] it was shown that this induces a reflexible hypermap $\mathcal{H}^{\mathrm{II}}_{(n,u)} = (G^{\mathrm{II}}_{n,u,1}; a, b)$ when $t = 1 \mod n$. These two cases show,

Theorem 1. *Any reflexible hypermap with 3 hyperfaces is, up to a duality, isomorphic to* $\mathcal{H}^{\mathrm{I}}_{(n,u,v)} = (G^{\mathrm{I}}_n; a, ba^{u-v+2})$ *for some* n, u *and* v *even such that* $3v - 2u = 6 \mod n$, *where* G^{I}_n *is the group with presentation* $\langle a, b | a^n = 1,\ b^3 = 1,\ (ab)^2 = a^2,\ [a^2, b] = 1\rangle$, *or to* $\mathcal{H}^{\mathrm{II}}_{(n,u)} = (G^{\mathrm{II}}_{n,u,1}; a, b)$ *for some* n *and* u, *where* $G^{\mathrm{II}}_{n,u,1}$ *is the abelian metacyclic group (cyclic* C_{3n} *or a direct product* $C_n \times C_3$*) with presentation* $\langle a, b \mid a^n = 1, b^3 = a^u, a^b = a\rangle$.

The H-sequences of $\mathcal{H}^{\mathrm{I}}_{(n,u,v)}$ and $\mathcal{H}^{\mathrm{II}}_{(n,u)}$ are respectively $[\frac{3n}{(n,u)}, \frac{2n}{(n,v)}, n; (n,u), \frac{3}{2}(n,v), 3; 3n]$ and $[\frac{3n}{(n,u)}, \frac{3n}{(n,u+3)}, n; (n,u), (n, u+3), 3; 3n]$.

Case III. In this case $a^n = 1$, $b^3 = a^u$, $(ab)^2 = a^v$ and $b^{-1}a^2b^{-1} = a^t$, for some $n = 0 \mod 3$ and $u, v, t \in \{0, \ldots, n-1\}$. As bab^{-2} and $b^2a^2b^{-1}$ are elements of $K = \langle a\rangle$, any coset-word Kw can be reduced to one of 4 cosets, K, Kb, Kb^2 and Kb^2a. This means that the oriented monodromy group G of the hypermap corresponding to this case has presentation

$$\langle a, b \mid a^n = 1, b^3 = a^u, (ab)^2 = a^v, b^{-1}a^2b^{-1} = a^t\rangle$$

for some $n = 0 \mod 3$, $u, v, t \in \{0, \ldots, n-1\}$. Since $b^{-1}a^3b = ba^3b^{-1} = a^{v+t-1}$ then b^2 commutes with a^3 and since $b^3 \in Z(G)$, b commutes with a^3. From $Kba = Kb^2$ and $Kba^3 = Kb$ one has $Kba^i = Kb$, Kb^2 or Kba^2 according as $i = 0, 1$ or $2 \mod 3$. Since $Kba^u = Kba^v = Kb$ and $Kba^t = kb^2$ we get $u = 0 \mod 3$, $v = 0 \mod 3$ and $t = 1 \mod 3$. Using the relation $[a^3, b] = 1$ we deduce $b^{-1}a^2b^{-1} = b^{-1}a^{-1}b^{-1}a^3 = a^{4-v}$, that is, the relation 4 can be replaced by $[a^3, b] = 1$. From $b^{a^{-1}b} = b^{-1}aba^{-1}b = b^{-2}a^{v-2}b = a^{v-3}b^{-2}ab = a^{v-u-3}bab = a^{2v-u-4}$ we deduce that $a^u = a^{6v-3u-12}$. Thus

$$G = G^{\mathrm{III}}_{(n,u,v)} = \langle a, b \mid a^n = 1, b^3 = a^u, (ab)^2 = a^v, [a^3, b] = 1\rangle$$

with n, u, $v = 0 \mod 3$ and $4u - 6v + 12 = 0 \mod n$. To show that under these conditions $G_{(n,u,v)}$ has exactly $4n$ elements take the normal subgroup $H = \langle a^3\rangle$ which factors G onto A_4. Using the Schreier transversal $T = \{1, a, a^{-1}, b, b^{-1}, ab, ba, a^{-1}b, ba^{-1}, b^{-1}a, ab^{-1}, ab^{-1}a\}$ of H in G in the Reidmaster-Schreier Rewriting Process we get

$$H = \langle x \mid x^{\frac{n}{3}} = 1, x^{\frac{4u-6v+12}{3}} = 1\rangle\,.$$

Since $\frac{4u-6v+12}{3} = 0 \mod \frac{n}{3}$, H is a cyclic group of order $\frac{n}{3}$ and thus G has order $4n$. Denote by $\mathcal{H}^{\mathrm{III}}_{(n,u,v)}$ the family of reflexible regular oriented hypermaps $(G^{\mathrm{III}}_{(n,u,v)}; a, b)$, where n, u, $v = 0 \mod 3$, $6v - 4u = 12 \mod n$ and a, b generate the group $G^{\mathrm{III}}_{(n,u,v)}$ subject to the relations of the above presentation.

Case IV. In this case we have $a^n = 1$ and $b^{-3}, (ab^{-1})^2, ba^2b \in K = \langle a \rangle$. Thus this is just the ψ-dual of Case III where ψ is the automorphism of the free group $\Delta^+ = F(a,b)$ determined by $a \mapsto a$, $b \mapsto b^{-1}$. Denote by $\mathcal{H}^{\mathrm{IV}}_{(n,u,v)} = D_\psi(\mathcal{H}^{\mathrm{III}}_{(n,u,v)}) = (G^{\mathrm{III}}_{(n,u,v)}; a, b^{-1})$, where a, b generates $G^{\mathrm{III}}_{(n,u,v)}$ subject to the relations that defines this group.

Case V. Here $a^n = 1$, $b^4 = a^u$, $(ab)^2 = a^v$, $b^{-2}ab^2 = a^t$ and $b^{-1}ab^{-1} = a^w$ for some even n and $u, v, t, w \in \{0, \ldots, n-1\}$. As before we let $K = \langle a \rangle$. As $Kba = Kb^{-1}$, $Kb^2a = Kb^2$ and $Kb^{-1}a = Kb$, the index of K in G is at most four, namely $G/_r H = \{K, Kb, Kb^2, Kb^{-1}\}$. Moreover $Kba^i = Kb$ or Kb^{-1}, according as i is even or odd. Thus $Kba^u = Kba^v = Kb$ and $Kba^w = Kb^{-1}$ implies that w is odd and u, v are even. From the third and the last equations we get $b^{-1}a^2b = a^{v+w-1} = ba^2b^{-1}$, thus $a^2 \rightleftharpoons b^2$. Then $b^{-1}ab^{-1} = b^{-2}a^{v-1}b^{-2} = a^v b^{-2}a^{-1}b^{-2} = a^{v-u}b^{-2}a^{-1}b^2 = a^{v-u-t}$ and so the last equation is equivalent to $[a^2, b^2] = 1$. As $w = v - u - t \mod n$, t is odd. Moreover $a^2 = b^{-2}a^2b^2 = a^{2t}$, thus $2(t-1) = 0 \mod n$. Now a^u and a^v are in the centre of G and as $b^{-1}a^{t-1}b = b^{-3}ab^2a^{-1}b = a^{-u}bab^2a^{-1}b = a^{-u+v-1}ba^{-1}b = a^{t-1}$, a^{t-1} is also in the centre of G. From the equality $b^{-1}a^2b = a^{2v-u-t-1}$ we deduce that $a^u = b^{-1}a^ub = a^{(2v-u-t-1)\frac{u}{2}}$, $a^v = b^{-1}a^vb = a^{(2v-u-t-1)\frac{v}{2}}$ and $a^{t-1} = b^{-1}a^{t-1}b = a^{(2v-u-t-1)\frac{t-1}{2}}$. The hypermaps in this case have monodromy group

$$G^{\mathrm{V}}_{(n,u,v,t)} \equiv G = \langle a, b | a^n = 1, b^4 = a^u, (ab)^2 = a^v, b^{-2}ab^2 = a^t, [a^2, b^2] = 1 \rangle,$$

where n, u, v are even, t is odd, $2(t-1) = 0 \mod n$ and $(2v-u-t-3)\frac{u}{2} = (2v-u-t-3)\frac{v}{2} = (2v-u-t-3)\frac{t-1}{2} = 0 \mod n$. For any value of these parameters we always get a group of order $4n$. In fact, the normal closure $H = \overline{\langle a^2, b^2 \rangle}$ in G factors G onto $C_2 \times C_2$. Considering the Schreier transversal $T = \{1, a, b, ab\}$ for H in G and applying the Reidmaster-Schreier Rewriting Process we get $H = \langle A, B \mid A^{\frac{n}{2}} = 1, B^2 = A^{\frac{u}{2}}, [A, B] = 1 \rangle$, which is an abelian metacyclic group of order $\frac{n}{2}2 = n$. Consequently G has order $|G| = 4n$.

Let $\mathcal{H}^{\mathrm{V}}_{(n,u,v,t)}$ denote the reflexible hypermap $(G^{\mathrm{V}}_{(n,u,v,t)}; a, b)$ where a, b generate $G^{\mathrm{V}}_{(n,u,v,t)}$ subject to the above relations.

Recalling [6], case VI gives rise to a metacyclic group $G^{\mathrm{VI}}_{n,u,t} = \langle a, b \mid a^n = 1, b^4 = a^u, bab^{-1} = a^t \rangle$ with $t^4 = 1 \mod n$, and $u(t-1) = 0 \mod n$. This induces a reflexible hypermap $\mathcal{H}^{\mathrm{VI}}_{(n,u,t)} = (G^{\mathrm{VI}}_{n,u,t}; a, b)$ only when $t^2 = 1 \mod n$. This leads to two families of reflexible hypermaps, $\mathcal{H}^{\mathrm{VIa}}_{(n,u)} = \mathcal{H}^{\mathrm{VI}}_{(n,u,1)}$ and $\mathcal{H}^{\mathrm{VIb}}_{(n,u)} = \mathcal{H}^{\mathrm{VI}}_{(n,u,-1)}$, the first of which all its members are abelian while in the second (where $2u = 0 \mod n$) its members are abelian only when $n \leq 2$. This proves,

Theorem 2. *Any reflexible hypermap with 4 hyperfaces is, up to a duality, isomorphic to* $\mathcal{H}^{\mathrm{III}}_{(n,u,v)}$, *or* $\mathcal{H}^{\mathrm{IV}}_{(n,u,v)}$, *for some* n, u, $v = 0 \mod 3$ *and* $6v - 4u = 12 \mod n$, *or* $\mathcal{H}^{\mathrm{V}}_{(n,u,v,t)}$ *for some* n, u, v *even,* t *odd,* $t = 1 \mod \frac{n}{2}$ *and* $(2v - u -$

$t-3)\frac{u}{2} = (2v-u-t-3)\frac{v}{2} = (2v-u-t-3)\frac{t-1}{2} = 0 \mod n$, *or the abelian* $\mathcal{H}^{\mathrm{VIa}}_{(n,u)}$ *for some* n *and* u, *or* $\mathcal{H}^{\mathrm{VIb}}_{(n,u)}$ *for some* n *and* u *such that* $2u = 0 \mod n$.

The H-sequences of these hypermaps are displayed in the following table

$$
\begin{array}{lcl}
\mathcal{H}^{\mathrm{III}}_{(n,u,v)} & : & [\frac{3n}{(n,u)}, \frac{2n}{(n,v)}, n\,;\ \frac{4}{3}(n,u), 2(n,v), 4\,;\ 4n] \\
\mathcal{H}^{\mathrm{IV}}_{(n,u,v)} & : & [\frac{3n}{(n,u)}, \frac{n}{(n,v-u-1)}, n\,;\ \frac{4}{3}(n,u), 4(n,v-u-1), 4\,;\ 4n] \\
\mathcal{H}^{\mathrm{V}}_{(n,u,v,t)} & : & [\frac{4n}{(n,u)}, \frac{2n}{(n,v)}, n\,;\ (n,u), 2(n,v), 4\,;\ 4n] \\
\mathcal{H}^{\mathrm{VIa}}_{(n,u)} & : & [\frac{4n}{(n,u)}, \frac{4n}{(n,u+4)}, n\,;\ (n,u), (n,u+4), 4\,;\ 4n] \\
\mathcal{H}^{\mathrm{VIb}}_{(n,0)} & : & [4, 4, n\,;\ n, n, 4\,;\ 4n] \\
\mathcal{H}^{\mathrm{VIb}}_{(n,\frac{n}{2})} & : & [8, 8, n\,;\ \frac{n}{2}, \frac{n}{2}, 4\,;\ 4n] \quad (n \text{ even})
\end{array}
$$

4. Regular oriented hypermaps with five hyperfaces

The number of hyperfaces fixed by a rotation (as an automorphism) about a hyperface must divide 5 ([26], Corollary 13) and so it must be 1 or 4. This means that the permutation A has support $\{2,3,4,5\}$ or is the identity. Not counting relabelling pairs, and having into account the form of A, we easily find 21 possible pairs (B, AB) determining a connected $B - AB$ diagram and such that $r \geq s$ (Table 4.1), each pair defining a hypermap $\mathcal{P}$.

#	B	AB	A	[	l	m	n	V	E	F	$\lvert P\rvert$	]
1	(1,2,3)	(1,2)(3,4,5)	(2,3,4,5)	[	3	6	4	40	20	30	120	]
2	(1,2,3)	(1,2,4)(3,5)	(2,4,3,5)	[	3	6	4	40	20	30	120	]
3	(1,2,3)(4,5)	(1,2)(3,4)	(2,3,5,4)	[	6	2	4	20	60	30	120	]
4	(1,2,3)(4,5)	(1,2,4)	(2,5,4,3)	[	6	3	4	20	40	30	120	]
5	(1,2,3)(4,5)	(1,2,4)(3,5)	(2,5)(3,4)	[	6	6	2	20	20	60	120	]
6	(1,2,3,4)	(1,2)(3,5)	(2,4,3,5)	[	4	2	4	5	10	5	20	]
7	(1,2,3,4)	(1,2)(4,5)	(2,4,5,3)	[	4	2	4	5	10	5	20	]
8	(1,2,3,4)	(1,2)(3,5,4)	(2,4)(3,5)	[	4	6	2	30	20	60	120	]
9	(1,2,3,4)	(1,2,5)	(2,5,4,3)	[	4	3	4	30	40	30	120	]
10	(1,2,3,4)	(1,2,4,5)	(2,3)(4,5)	[	4	4	2	5	5	10	20	]
11	(1,2,3,4)	(1,2,5,3)	(2,5)(3,4)	[	4	4	2	5	5	10	20	]
12	(1,2,3,4,5)	(1,2)	(2,5,4,3)	[	5	2	4	24	60	30	120	]
13	(1,2,3,4,5)	(1,2)(3,5)	(2,5)(3,4)	[	5	2	2	2	5	5	10	]
14	(1,2,3,4,5)	(1,2)(3,5,4)	(2,5,3,4)	[	5	6	4	24	20	30	120	]
15	(1,2,3,4,5)	(1,2,4)	(2,3)(4,5)	[	5	3	2	12	20	30	60	]
16	(1,2,3,4,5)	(1,2,4)(3,5)	(2,3,4,5)	[	5	6	4	24	20	30	120	]
17	(1,2,3,4,5)	(1,2,4,3)	(2,3,5,4)	[	5	4	4	4	5	5	20	]
18	(1,2,3,4,5)	(1,2,5,3)	(2,4,3,5)	[	5	4	4	24	30	30	120	]
19	(1,2,3,4,5)	(1,2,5,4)	(2,4,5,3)	[	5	4	4	4	5	5	20	]
20	(1,2,3,4,5)	(1,2,3,4,5)	()	[	5	5	1	1	1	5	5	]
21	(1,2,3,4,5)	(1,2,5,4,3)	(2,4)(3,5)	[	5	5	2	12	12	30	60	]

Table 4.1: The 21 cases and the corresponding H-sequences of $\mathcal{P}$.

One observes from this table that most of the hypermaps $\mathcal{P}$ have more than 5 hyperfaces, a situation that cannot occur. This eliminate most of the items in the above table, leaving only 6 possible cases behind, namely the cases 6, 7, 13, 17, 19 and 20.

Looking at the action of $\langle B, AB\rangle$ on the hyperfaces $\{1,2,3,4,5\}$ and taking into account the regularity of $\mathcal{Q}$, the following word relations are common to all cases and are therefore omitted in Table 4.2:

$$a^n = 1\,, \quad \text{where } n = 0 \mod |A|\,;$$
$$b^r = a^u\,, \quad \text{for some } u \in \{0,\ldots,n-1\}\,;$$
$$(ab)^s = a^v\,, \quad \text{for some } v \in \{0,\ldots,n-1\}\,.$$

For each case we derive extra word relations (shown in Table 2). The letters u, v, x, y, z, w, t appearing in these relations are integers in $\{0,...,n-1\}$.

#	B	AB	A	Extra relations for $Mon(\mathcal{Q})$
6	(1,2,3,4)	(1,2)(3,5)	(2,4,3,5)	$b^2ab^{-1} = a^w, b^{-1}aba^{-1}b = a^t, b^{-1}a^2b^2 = a^z$
7	(1,2,3,4)	(1,2)(4,5)	(2,4,5,3)	$b^{-1}ab^2 = a^w, ba^{-1}bab^{-1} = a^t, b^2a^2b^{-1} = a^z$
13	(1,2,3,4,5)	(1,2)(3,5)	(2,5)(3,4)	$b^{-1}ab^{-1} = a^z, b^{-2}ab^{-2} = a^x, b^2ab^2 = a^y$
17	(1,2,3,4,5)	(1,2,4,3)	(2,3,5,4)	$b^{-1}ab^{-2} = a^x, b^{-2}ab = a^y, bab^2 = a^z, b^2ab^{-1} = a^t$
19	(1,2,3,4,5)	(1,2,5,4)	(2,4,5,3)	$b^{-2}ab^{-1} = a^x, bab^{-2} = a^y, b^2ab = a^z, b^{-1}ab^2 = a^t$
20	(1,2,3,4,5)	(1,2,3,4,5)	1	$bab^{-1} = a^t, b^2ab^{-2} = a^x, b^{-1}ab = a^y, b^{-2}ab^2 = a^z$

Table 4.2: The 6 cases with their extra relations.

Before we start with the classification we remark that four of these six regular oriented hypermaps form two chiral pairs. Denote by $\mathcal{P}^i$ the regular oriented hypermap corresponding to item i in table 4.1. In item 6 we have $\mathcal{P}^6 = (P^6; A^{-1}, B^{-1})$ where $B^{-1} = (1,4,3,2)$ and $A^{-1} = (2,5,3,4)$. Relabelling the hyperfaces according to the permutation $(2,4)$, we get $B^{-1} = (1,2,3,4)$, $A^{-1} = (2,4,5,3)$, which are the permutations B and A in line 7, and $\mathcal{P}^6 = (P^6; A, B)$ where $A = (2,4,5,3)$ and $B = (1,2,3,4)$ (line 7). This shows that $P^6 = P^7$. Its chiral pair is

$$chiral(\mathcal{P}^6) = (P^6; A^{-1}, B^{-1}) = (P^7; A^{-1}, B^{-1}) = \mathcal{P}^7\,.$$

It is not difficult to see that $\mathcal{P}^6$ is the toroidal chiral map $\{4,4\}_{2,1}$. Similarly $\mathcal{P}^{17}$ and $\mathcal{P}^{19}$ form a chiral pair. The hypermaps $\mathcal{P}^{13}$ and $\mathcal{P}^{20}$ are easily seen to be reflexible.

4.1. **The Reflexible hypermaps with five hyperfaces.** In this section we analyse the reflexible oriented hypermaps with 5 hyperfaces that appear in the cases 13 and 20. Only two families of reflexible hypermaps will arise and they are exhibited in Theorem 3. In the next section we will see that the remaining cases will give rise to chiral hypermaps.

Theorem 3. *If $\mathcal{H}$ is a reflexible hypermap with 5 hyperfaces (of valency $n > 0$) then, up to a $(0,1)$-duality and an isomorphism, $\mathcal{H}$ is either $\mathcal{H}^{13}_{n,u,v} = (G^{13}_n; a, ba^{u-2v+4})$ for some non-negative even numbers n, u, v (with $u, v < n$) such that $2u - 5v +$*

$10 = 0 \mod n$, *or* $\mathcal{H}^{20}_{n,u} = (G^{20}_{n,u}; a, b)$ *for some non-negative numbers n and u. Here* G^{13}_{n} *is the metacyclic group* $\langle a, b \mid a^n = b^5 = 1, a^{-1}ba = b^{-1}\rangle$ *and* $G^{20}_{n,u}$ *is the abelian metacyclic group* $\langle a, b | a^n = 1, b^5 = a^u, bab^{-1} = a\rangle$*, either a cyclic group* C_{5n} *or a direct product* $C_n \times C_5$.

The H-sequences of $\mathcal{H}^{13}_{(n,u,v)}$ and $\mathcal{H}^{20}_{(n,u)}$ are, respectively,

$$[\tfrac{5n}{(n,u)}, \tfrac{2n}{(n,v)}, n\,;\,(n,u), \tfrac{5}{2}(n,v), 5\,;\,5n] \quad \text{and} \quad [\tfrac{5n}{(n,u)}, \tfrac{2n}{(n,u+5)}, n\,;\,(n,u), (n,u+5), 5\,;\,5n]\,.$$

Proof. The hypermap $\mathcal{H}^{13}_{n,u,v}$ arises from case 13 while $\mathcal{H}^{20}_{n,u}$ arises from case 20. Actually this case gives rise to two families of regular oriented hypermaps with 5 hyperfaces (of valency n), one reflexible $\mathcal{H}^{20}_{n,u}$ and the other chiral $\mathcal{Q}^{20}_{n,u,t}$. We will skip the chiral part here and deal with it later in Theorem 4.

Case 13: Let G be the group with presentation

$$\langle a, b \mid a^n = 1, b^5 = a^u, (ab)^2 = a^v, b^{-1}ab^{-1} = a^z, b^{-2}ab^{-2} = a^x, b^2ab^2 = a^y\rangle\,,$$

where $u, v, z, x, y \in \{0, ..., n-1\}$ and $n = 0 \mod 2$. The monodromy group of $\mathcal{Q}$ is then a factor of G. Let K be the subgroup generated by a. One can see that K divides G into (no more than five) cosets K, Kb, Kb^2, Kb^3 and Kb^4, not necessarily distinct.

Clearly the elements b^5 and $(ab)^2 = (ba)^2$ are central in G. From the 3rd and 4th relations we get $a^{v-1}a^z = a^z a^{v-1} \Leftrightarrow ba^2b^{-1} = b^{-1}a^2b \Leftrightarrow b^2a^2 = a^2b^2$, that is, $a^2 \leftrightharpoons b^2$. Moreover, from the 2nd relation we get $b = a^x b^{-4}$, and so, $ba^2 = a^x b^{-4}a^2 = a^x a^2 b^{-4} = a^2 a^x b^{-4} = a^2 b$, which says that $b \leftrightharpoons a^2$. On the other hand, $Kba = Kb^{-1} = Kb^4$ (3th relation) and $Kba^2 = Kb$. Thus $Kba^i = Kb$ or Kb^4, according as $i = 0$ or $1 \mod 2$. Now, 4th and 2nd relations imply that $Kba^z = Kb^{-1} = Kb^4$, hence $z = 1 \mod 2$. The 5th relation is redundant, in fact, $b^{-2}ab^{-2} = b^{-1}(b^{-1}ab^{-1})b^{-1} = b^{-1}a^z b^{-1} = b^{-1}a^{z-1}ab^{-1} = a^{z-1}b^{-1}ab^{-1} = a^{2z-1}$. Also, $Kb^4a = Kb^{-1}a = Kb$ (from 4th and 2nd relations), hence $Kb^4a^i = Kb^4$ or Kb, according as $i = 0 \mod 2$ or $i = 1 \mod 2$, respectively. Then the 3rd and 2nd relations imply that $Kb^4a^{v-1} = Kb^{-1}a^{v-1} = Kb$, and so $v = 0 \mod 2$. But then the 6th relation is also redundant, $b^2ab^2 = b(bab)b = a^{v-2}bab = a^{2v-3}$.

Looking back at cosets, we also have $Kb^2a = Kb^{-2} = Kb^3$ (6th relation), therefore $Kb^2a^i = Kb^2$ or Kb^3, according as $i = 0$ or $1 \mod 2$, and $Kb^3a = Kb^{-2}a = Kb^2$ which gives $Kb^3a^i = Kb^3$ or Kb^2, according as $i = 0$ or $1 \mod 2$. Hence, the above relations are enough to reduce any word $Kw, w \in F(a, b)$, to one of the cosets, K, Kb, Kb^2, Kb^3 or Kb^4. Thus $|G| \le 5|K| = 5n$.

Now $u = 0 \mod 2$ since $Kb^4a^u = Kb^{-1}a^u = Kb^4$ (2nd relation). From $a^{v-1+z} = ba^2b^{-1} = a^2$ we get $a^{v+z} = a^3 \Leftrightarrow a^z = a^{3-v}$. For any integer $i \ge 0$, $b^iab^i = a^{i(v-2)+1}$. Then $a^{3-v} = b^{-1}ab^{-1} = a^{-u}b^4ab^4a^{-u} = a^{-2u}a^{4v-7} \Leftrightarrow$

$a^{2u-5v+10} = 1$, hence $2u - 5v + 10 = 0 \mod n$. Then

$$G = G^{13}_{n,u,v} = \langle a, b \mid a^n = 1, b^5 = a^u, (ab)^2 = a^v, b^{-1}ab^{-1} = a^{3-v} \rangle$$

with n, u, v even and $2u - 5v + 10 = 0 \mod n$.

The subgroup $F = \langle a^2 \rangle$ is normal in G and $G/F = P = \langle A = Fa, B = Fb \rangle = \langle A, B \mid A^2 = B^5 = (AB)^2 = 1 \rangle = D_5$; this corresponds to the insertion of the relation $a^2 = 1$ in the above presentation. The epimorphism $G \longrightarrow D_5$, mapping a to A and b to B, induces a covering from $\mathcal{H} = \mathcal{H}^{13}_{n,u,v} = (G; a, b)$ to the spherical reflexible map $\mathcal{D}_5$ with two vertices, 5 faces and automorphism group D_5. This shows that $\mathcal{H}$ has at least 5 faces.

The group $G^{13}_{n,0,2} = G_n$ with presentation

$$\langle a, b \mid a^n = b^5 = 1, (ab)^2 = a^2, b^{-1}ab^{-1} = a \rangle = \langle a, b \mid a^n = b^5 = 1, a^{-1}ba = b^{-1} \rangle$$

is clearly a metacyclic group. By changing generators $A = a$, $B = ba^{-2v+u+4}$ for u and v even such that $2u - 5v + 10 = 0 \mod n$, and having into account that a^2 belongs to the center of $G^{13}_{n,0,2}$, we get the presentation

$$\langle A, B \mid A^n = 1, B^5 = A^u, (AB)^2 = A^v, B^{-1}AB^{-1} = A^{3-v} \rangle$$

of $G_{n,u,v}$. Hence for all u, v even such that $2u - 5v + 10 = 0 \mod n$, $G^{13}_{n,u,v} = G_n$ is metacyclic of order $5n$. $\diamond$

Case 20: Let G be the group generated by a and b given by the case 20. As in the above case, the group Q (the monodromy group of $\mathcal{Q}$) is a factor group of G. In this case $K = \langle a \rangle \lhd G$ since $A = 1$, and since $G/K = \langle Kb \rangle$ is also cyclic, G is metacyclic. Hence $G = \langle a, b | a^n = 1, b^5 = a^u, bab^{-1} = a^t \rangle$, for some u, t such that $(t-1)u = 0 \mod n$ and $t^5 = 1 \mod n$. Theorem 8 of [6] says that $\mathcal{Q}' = (G; a, b)$ is reflexible if and only if $t^2 = 1 \mod n$. As $t^5 = 1 \mod n$ then $t^2 = 1 \mod n \Leftrightarrow t = 1 \mod n \Leftrightarrow bab^{-1} = a$, that is, G is abelian. Let $G^{20}_{n,u} := G = \langle a, b | a^n = 1, b^5 = a^u, bab^{-1} = a \rangle$ (the reflexible case) and $Q^{20}_{n,u,t} = \langle a, b | a^n = 1, b^5 = a^u, bab^{-1} = a^t \rangle$, where $(t-1)u = 0 \mod n$, $t^5 = 1 \mod n$ and $t \neq 1 \mod n$ (the chiral case). Both $\mathcal{H}^{20}_{n,u} = (G^{20}_{n,u}; a, b)$ and $\mathcal{Q}^{20}_{n,u,t} = (Q^{20}_{n,u,t}; a, b)$ have 5 hyperfaces of valency n (since a has order n and $\langle a \rangle$ has index 5 in G). Hence $\mathcal{Q} = \mathcal{Q}'$, that is, $\mathcal{Q} \cong \mathcal{H}^{20}_{n,u}$ or $\mathcal{Q}^{20}_{n,u,t}$. $\diamond$

The other cases are easily seen to give chiral hypermaps. The proof of Theorem 3 is finished. $\square$

4.2. **The Chiral hypermaps with five hyperfaces.** Each one of the remaining cases 6, 7, 17 and 19 will give rise to families of chiral hypermaps.

Theorem 4. *If $\mathcal{Q}$ is a chiral hypermap with 5 hyperfaces (of valency n) then, up to a (0,1)-duality, mirror-symmetry and an isomorphism, $\mathcal{Q}$ is either the canonical metacyclic hypermap $\overset{20}{\mathcal{Q}}_{n,u,t} = (\overset{20}{G}_{n,u,t}; a, b)$ for some n, u, t such that $(t-1)u = 0 \mod n$, $t^5 = 1 \mod n$ and $t \neq 1 \mod n$, or $\mathcal{Q}$ is $\overset{6}{\mathcal{Q}}_{n,t} = (G_n; a, ba^{t-1})$, or $\overset{17}{\mathcal{Q}}_{n,t} = (G_n; a^{-1}, ba^{-t})$, for some $n = 0 \mod 4$ and $t = 1 \mod 4$. Here $\overset{20}{Q}_{n,u,t} = \langle a, b | a^n = 1, b^5 = a^u, bab^{-1} = a^t\rangle$ and $G_n = \langle a, b \mid a^n = 1, b^4 = a^4, (ab)^2 = a^4, a^2b = b^2a\rangle$.*

The chirality groups and indices of these hypermaps, shown in the table below, are the last two entries of the H-sequences. In what follows $\gamma = t^4 + t^3 + t^2 + t + 1 + u$.

$\mathcal{Q}$	:	[type ; V, E, F ; $\vert Mon(\mathcal{Q})\vert$; $X(\mathcal{Q})$; κ]
$\overset{20}{\mathcal{Q}}_{n,u,t}$	:	$\left[\frac{5n}{(n,u)}, \frac{5n}{(n,\gamma)}, n\,;\, (n,u), (n,\gamma), 5\,;\, 5n\,;\, \langle a^{t^2-1}\rangle\,;\, \frac{n}{(n,t^2-1)}\right]$
$\overset{6}{\mathcal{Q}}_{n,t}$	:	$\left[\frac{n}{(\frac{n}{4},t)}, \frac{n}{(\frac{n}{2},t+1))}, n\,;\, 5(\frac{n}{4},t), 5\left(\frac{n}{2},t+1\right), 5\,;\, 5n\,;\, \langle [a,b]\rangle\,;\, 5\right]$
$\overset{17}{\mathcal{Q}}_{n,t}$	:	$\left[\frac{5n}{(n,5(t-1))}, \frac{n}{(\frac{n}{4},t)}, n\,;\, (n,5(t-1)), 5\left(\frac{n}{4},t\right), 5\,;\, 5n\,;\, \langle [b,a^{-1}]\rangle\,;\, 5\right]$

Proof. Before we go any further, let us go back to the case 20 where, as we have observed earlier, there is a family of canonical metacyclic chiral hypermaps.

Case 20: As seen in the proof of Theorem 3 (case 20), the group generated by a and b corresponding to this case is the metacyclic group $Q = \langle a, b | a^n = 1, b^5 = a^u, bab^{-1} = a^t\rangle$, where $(t-1)u = 0 \mod n$ and $t^5 = 1 \mod n$. The oriented hypermap $\mathcal{Q} = (Q; a, b)$ is chiral if and only if $t \neq 1 \mod n$, that is, Q is not abelian. By Corollary 9 of [6] the chirality group of $\overset{20}{\mathcal{Q}}_{n,u,t} = (\overset{20}{Q}_{n,u,t}; a, b)$, where $\overset{20}{Q}_{n,u,t} = Q$ with $t \neq 1 \mod n$, is cyclic, given by $X(\overset{20}{\mathcal{Q}}_{n,u,t}) = \langle a^{t^2-1}\rangle$, while its chirality index is given by $\frac{n}{(n,t^2-1)}$. $\diamond$

For the remaining cases 6, 7, 17 and 19 we observe that the cases 7 and 19 correspond to chiral pairs of 6 and 17 respectively. To avoid too much repetition let us fix G to be the group with presentation $\langle a, b | a^n = 1, b^r = a^u, (ab)^s = a^v, \mathcal{R} = 1\rangle$, where $n = 0 \mod |A|$ and $\mathcal{R}$ is the corresponding set of extra relators given in Table 4.2 (the monodromy group of $\mathcal{Q}$ will be a factor group of G) and K the subgroup of G generated by a.

Case 6: $a^n = 1$, $b^4 = a^u$, $(ab)^2 = a^v$, $b^2ab^{-1} = a^w$, $b^{-1}aba^{-1}b = a^t$, $b^{-1}a^2b^2 = a^z$, for some $u, v, w, t, z \in \{0, ..., n-1\}$, with $n = 0 \mod 4$. One may check that K divides G into at most 5 cosets, namely K, Kb, Kb^2, Kb^3 and Kba^2, hence $|G| \leq 5n$. As $(ab)^2 \in Z(G)$ then $(ba)^2 = (ab)^2 \in Z(G)$. Now, relation 3 implies that $Kba = Kb^{-1} = Kb^3$; relation 4 implies that $Kb^2a = Kb$; and relation 5 implies that $Kb^{-1}ab = Kb^{-1}a$. Then $Kba^3 = Kb^{-1}a^2 = Kb^{-1}aba = Kb^{-2}(ba)^2 = K(ba)^2b^{-2} = Kb^{-2} = Kb^2$ and $Kba^4 = Kb^2a = Kb$, thus $Kba^i = Kb, Kb^3, Kba^2$ or Kb^2, according as $i = 0, 1, 2, 3 \mod 4$, respectively. The 6th relation was not used, it must be redundant; in fact, $Kb^{-1}a^2b^2 = K$ is equivalent to $Kb^4 = K$.

Since $Kb^2a = Kb$ then $Kb^2a^i = Kba^{i-1}$. As $Kb^3 = Kb^{-1} = Kb^{-2}a^w = Kb^2a^w = Kba^{w-1}$ and $Kba^t = Kba^{-1}b = Kb^2b = Kb^{-1}$ we have $w = 2 \mod 4$ and $t = 1 \mod 4$. Since $b^4, (ab)^2 \in Z(G)$, powering both sides of the fifth relation by 4 we get $a^u = a^{4t}$, squaring both sides of the fourth relation we get $a^v = a^{2w}$ and combining the fifth and sixth relations we get $a^z = a^{2w-t}$. We have just reduced the six parameters n, w, t, u, v, z to three parameters n, w and t,

$$G = \langle a, b | a^n = 1, b^4 = a^{4t}, (ab)^2 = a^{2w}, b^2ab^{-1} = a^w, b^{-1}aba^{-1}b = a^t \rangle .$$

A further reduction can be done. The equalities $ba^4b^{-1} = (bab)(b^{-1}a^2b^2)b^{-4}(b^2ab^{-1}) = a^{5w-5t-1}$ and $b^{-1}a^4b = b^{-1}a^2b^2b^{-4}b^2ab^{-1}bab = a^{5w-5t-1}$ implies that a^4 commutes with b^2. Then $b^{-1}a^{t-1}b = b^{-1}a^{-1}b^{-1}aba^{-1}bb = a^{-2w+2}ba^{-1}b^{-2}a^{4t} = a^{-3w+4t+2}$ and $ba^{w-2}b^{-1} = bb^2ab^{-1}a^{-2}b^{-1} = b^3abb^{-2}a^{-2}bb^{-2} = b^3aba^{t-2w}b^{-2} = b^2a^{2w-1}a^{t-2w}b^{-2} = a^{t-1}$. Thus $ba^{w-2} = a^{t-1}b = ba^{-3w+4t+2}$, that is, $a^{-4w+4t+4} = 1$. On the other hand, $b^{-1}a^2b^2 = a^{2w-t} \Leftrightarrow b^{-1}a^{-2}b^2 = a^{2w-t-4} \Rightarrow (bab)b^{-1}a^{-2}b^2 = a^{4w-t-5} \Leftrightarrow ba^{-1}b^{-2} = a^{4w-5t-5} \Leftrightarrow a^{-w} = a^{4w-5t-5} \Leftrightarrow a^{5w-5t-5} = 1$. Combining these two relations we get $a^w = a^{t+1}$. Now replacing a^w above we get $ba^4b^{-1} = b^{-1}a^4b = a^4$. Hence a^4 is central in G and

$$G = G^{6}_{n,t} = \langle a, b | a^n = 1, b^4 = a^{4t}, (ab)^2 = a^{2t+2}, b^2ab^{-1} = a^{t+1}, b^{-1}aba^{-1}b = a^t \rangle ,$$

where $n = 0 \mod 4$ and $t = 1 \mod 4$.

Consider the particular case of $n = 4$ and $t = 1$:

$$G_{4,1} = \langle \alpha, \beta | \alpha^4 = 1, \beta^4 = 1, (\alpha\beta)^2 = 1, \beta^2\alpha\beta^{-1} = \alpha^2, \beta^{-1}\alpha\beta\alpha^{-1}\beta = \alpha \rangle.$$

Last equation of this particular case is redundant,

$$\alpha^{-1}\beta^{-1}\alpha\beta\alpha^{-1}\beta = \beta\alpha^2\beta\alpha^{-1}\beta = \beta(\beta^2\alpha\beta^{-1})\beta\alpha^{-1}\beta = \beta^4 = 1 .$$

So the above presentation simplifies to

$$G_{4,1} = \langle \alpha, \beta | \alpha^4 = 1, \beta^4 = 1, (\alpha\beta)^2 = 1, \beta^2\alpha\beta^{-1} = \alpha^2 \rangle$$

and reveals the monodromy group of the toroidal hypermap $\{4, 4\}_{2,1} = \mathcal{P}^6$, with 5 hyperfaces, 5 hypervertices, 10 hyperedges, 20 darts and chirality index 5, see

[1]. As $n = 0 \mod 4$ and $t = 1 \mod 4$ the function $a \to \alpha, b \to \beta$ extends to an epimorphism $G_{n,t} \to G_{4,1}$. Consequently, all oriented hypermaps $\mathcal{Q}_{n,t} = (G_{n,t}, a, b)$ are coverings of $\{4,4\}_{2,1}$ and thus they all have five hyperfaces. On the other hand, adjoining the relation $b = a^t$ to the relations of $G_{n,t}$ we get $C_n = \langle a \mid a^n = 1\rangle$ and an epimorphism $G_{n,t} \longrightarrow C_n$, $a \mapsto a$ and $b \mapsto a^t$. This shows that a in $G_{n,t}$ has order n and hence $|G_{n,t}| = 5n$. Thus $Mon(\mathcal{Q}) = G_{n,t}$.

The chirality group of $\mathcal{Q}$ is the normal closure $X(Q) = \langle b^{-2}a^{-1}ba^{t+1},\ ba^{-1}b^{-1}ab^{-1}a^t\rangle^G$ where $G = G_{n,t}$. Now $b^{-2}a^{-1}ba^{t+1} = b^{-1}a^{-1}ba = [b,a]$ and $[b,a] = (b^{-1}a^{-1})ba = aba^{-2t-2}ba = ab^2aa^{-2t-2} = a(b^2ab^{-1})ba^{-2t-2} = aa^{t+1}ba^{-2t-2} = a^{-t}b$. On the other hand, $ba^{-1}b^{-1}ab^{-1}a^t = (b^{-3}a^{4t})a^{-1}b^{-1}ab^{-1}(b^{-1}aba^{-1}b) =$ $b^{-2}(b^{-1}a^{-1}b^{-1})a(b^{-2}a^{4t})aba^{-1}b = b^{-2}a^{-2t}(b^2ab^{-1})b^2a^{-1}b = b^{-2}a^{-2t}a^{t+1}b^2a^{-1}b =$ $b^{-2}a^{-t+1}b^2a^{-1}b = a^{-t}b$. Hence $X(Q) = \langle [b,a]\rangle^G$. Let $X = [b,a]$. Table below shows the conjugates X^d for $d \in \{a, a^2, a^3, a^4, b, b^2, b^3, b^4\}$,

d	a	a^2	a^3	a^4	b	b^2	b^3	b^4
X^d	Y	X^{-1}	Y^{-1}	X	Y	X^{-1}	Y^{-1}	X

where $Y = [b, a^{-1}]$. From the 4th relation we derive $[b^{-1}, a^{-1}] = b^{-1}a^t$ (thus $[a^{-1}, b^{-1}] = a^{-t}b = [b,a] = X$) and from the 5th relation we deduce $Y = [b, a^{-1}] = a^tb^{-1}$. Then $X^2 = [a^{-1}, b^{-1}][b,a] = aba^{-1}b^{-2}a^{-1}ba = ba^{-1}b^{-2}ba = [b^{-1}, a]$. But $X^2 = [b,a][a^{-1},b^{-1}] = a^{-t}(bab)a^{-1}b^{-1} = a^vb^{-1} = [b,a^{-1}]$, hence $[b^{-1},a] = [b,a^{-1}]$, or equivalently, $[a,b^{-1}] = [a^{-1},b]$. Thus $Y = [b,a^{-1}] = [b,a]^2 = X^2$ and hence $\mathcal{Q}$ has cyclic chirality group generated by $[a,b]$. Since $[b,a]^3 = [a,b]^2$, $\mathcal{Q}$ has chirality index $k = 5$. $\diamond$

Case 17: $G = \langle a, b | a^n = 1, b^5 = a^u, (ab)^4 = a^v, b^{-1}ab^{-2} = a^x, b^{-2}ab = a^y, bab^2 = a^z, b^2ab^{-1} = a^t\rangle$ for some $u, v, x, y, z, t \in \{0, ..., n-1\}$, with $n = 0 \mod 4$. The cosets K, Kb, Kb^2, $Kb^3 = Kb^{-2}$ and $Kb^4 = Kb^{-1}$ is a complete set of right cosets. As $Kba = Kb^{-2}$, $Kba^2 = Kb^{-2}a = Kb^{-1}$, $Kba^3 = Kb^{-1}a = Kb^2$ and $Kba^4 = Kb^2a = Kb$ then $Kba^i = Kb, Kb^{-2}, Kb^{-1}$ or Kb^2, according as $i = 0,1,2,3 \mod 4$. Since $Kb^{-2} = Kba^x$, $Kb = Kb^2a^y = Kba^{3+y}$, $Kb^2 = Kb^{-1}a^z = Kba^{z+2}$ and $Kb^{-1} = Kb^{-2}a^t = Kba^{t+1}$ we conclude that $x, y, z, t = 1 \mod 4$.

Now b^5 and $(ab)^4 \in Z(G)$. Since $A^4 = 1$ the subgroup generated by a^4 is normal in G. The two equalities $ba^4b^{-1} = bab^2b^{-2}abb^{-1}ab^{-2}b^2ab^{-1} = a^{x+y+z+t}$ and $b^{-1}a^4b = b^{-1}ab^{-2}b^2ab^{-1}bab^2b^{-2}ab = a^{x+y+z+t}$ shows that a^4 commutes with b^2. Since a^4 commutes with b^5 then also a^4 commutes with $b = b^5b^{-4}$ and so $a^4 \in Z(G)$.

Equation $b^2ab^{-1} = a^t$ is equivalent to $b(ba)b^{-1} = a^t$. Powering by 4 we get $a^v = a^{4t}$. Because $a^{4t-1} = (bab)abab = a^z(b^{-1}ab)ab = a^va^zb^5b^{-2}ab = a^{z+x+u+y}$ we have $a^u = a^{4t-x-z-y-1}$. Since $a^{2x} = b^{-1}ab^{-2}b^{-1}ab^{-2} = b^{-1}aa^{-u}b^2ab^{-2} =$

$b^{-1}aa^{-u}a^tbb^{-2} \Leftrightarrow a^{2x+y+z} = b^{-1}a^{-u+t+1}b^{-1}(bab^2)(b^{-2}ab) = b^{-1}a^{-u+t+3}b = a^{-u+t+3}$ we have $a^x = a^{4-3t}$. As $a^{x+1} = b^{-1}a(b^{-2}a) = b^{-1}aa^yb^{-1} \Leftrightarrow a^{x+y+z+1} = b^{-1}a^{y+1}b^{-1}(bab^2)(b^{-2}ab) = b^{-1}a^{y+3}b = a^{y+3}$ we have $a^z = a^{2-x} = a^{3t-2}$. Finally, from $a^{x+t+z+y} = a^4$ we get $a^y = a^{2-t}$ and so $a^u = a^{5(t-1)}$. Therefore G is determined by two parameters n and t satisfying $n = 0 \mod 4$ and $t = 1 \mod 4$:

$$G = \langle a, b \mid a^n = 1, b^5 = a^{5(t-1)}, (ab)^4 = a^{4t}, b^{-1}ab^{-2} = a^{4-3t}, b^{-2}ab = a^{2-t}, bab^2 = a^{3t-2}, b^2ab^{-1} = a^t \rangle .$$

Adding $[a^4, b] = 1$ we get $G = \langle a, b \mid a^n = 1, b^5 = a^{5(t-1)}, [a^4, b] = 1, b^2ab^{-1} = a^t \rangle = G_{n,t}$.

The $(0, 2)$-dual of the particular case $n = 4$ and $t = 1$ yields a metacyclic group of order 20

$$\langle \alpha, \beta | \alpha^5 = 1, \beta^4 = 1, \beta\alpha\beta^{-1} = \alpha^2 \rangle.$$

This is the monodromy group of a chiral hypermap with 4 hyperfaces, 5 hypervertices, 5 hyperedges and chirality index $k = 5$ see [6]. Thus $\mathcal{M} = (G_{4,1}; \alpha, \beta)$ is a chiral hypermap of order 20 with five hyperfaces. As $n = 0 \mod 4$ and $t = 1 \mod 4$ the function $a \to \alpha, b \to \beta$ extends to an isomorphism $G_{n,t} \to G_{4,1}$. Consequently each $\mathcal{Q}_{n,t} = (G_{n,t}; a, b)$ is a covering of $\mathcal{M}$ and therefore has five hyperfaces. Adjoining the relation $b = a^{t-1}$ to the above presentation of $G_{n,t}$ we get a cyclic group C_n and an epimorphism $G_{n,t} \longrightarrow C_n$ given by $a \mapsto a$, $b \mapsto a^{t-1}$. Thus $|a| = n$ and $|G_{n,t}| = 5n$, and therefore $Mon(\mathcal{Q}) = G_{n,t}$.

The chirality group of $\mathcal{Q} = \mathcal{Q}_{n,t}$ is the normal closure $X(\mathcal{Q}) = \langle X \rangle^G$, where

$$X = b^{-2}a^{-1}ba^t = b^{-2}a^{-1}b^3ab^{-1} = b^{-2}a^{-1}a^{2-t}b^3 = b^{-2}a^{1-t}b^3 = a^{1-t}b = [b, a^{-1}]$$

and $G = G_{n,t}$. Notice that $a^{1-t} \in Z(G)$. Looking at the conjugates X^Θ for $\theta \in \{a, a^2, a^3, a^4, b\}$ (table below)

Θ	a	a^2	a^3	a^4	b
X^Θ	X^{-2}	X^{-1}	X^2	X	X

one sees that $X(\mathcal{Q})$ is cyclic and generated by $X = a^{1-t}b$. Since X has order 5 ($X^5 = a^{5-5t}b^5 = 1$ and $X^i \neq 1$ for $0 < i < 5$), $\mathcal{Q}$ has chirality index $k = 5$. $\diamond$

Let G_n be the group $\overset{6}{G}_{n,1} = \langle a, b \mid a^n = 1, b^4 = a^4, (ab)^2 = a^4, a^2b = b^2a \rangle$. One easily computes that $(ba^{-1})^2 = a^{-1}b$. This shows that $(ba^{-1})^a = (ba^{-1})^b = (ba^{-1})^2$ and so ba^{-1} generates a normal subgroup. Hence G_n is metacyclic. From the covering $\overset{6}{\mathcal{Q}}_{n,1} \longrightarrow \overset{6}{\mathcal{P}}$ one sees that ba^{-1} has order at least 5. On the other hand, $(ba^{-1})^4 = (a^{-1}b)^2 = ab^{-1}$ and so ba^{-1} has order 5. Changing generators $a' = a$ and $b' = ba^{t-1}$ we get $\overset{6}{G}_{n,t}$ and changing generators $a' = a^{-1}$ and $b' = ba^{-t}$ we get $\overset{17}{G}_{n,t}$.

Finally, if $\mathcal{Q}_{n,t}^{6}$ and $\mathcal{Q}_{n,t}^{17}$ were canonical metacyclic then also $\mathcal{P}^{6}$ and $\mathcal{P}^{17}$ would be canonical metacyclic. But a quick checking shows that these are not canonical metacyclic.

The proof of Theorem 4 is finished. □

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