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CATEGORICALLY-ALGEBRAIC DUALITIES

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Abstract. The paper introduces a new technique for producing topological representations of algebraic structures called *categorically-algebraic* (*catalg*) *dualities*. Based on our recent results, generalizing the famous duality for bounded distributive lattices of H. Priestley and developed in the framework of catalg topology (subsuming both the crisp and the fuzzy approaches), the theory incorporates not only the representation theories of H. Priestley and M. Stone, but also *natural dualities* of D. Clark and B. Davey, bringing into light their catalg properties and serving as a tool for generating new topological representation theorems for algebraic structures. We apply the emerging theory to investigate the relations between topological representations of a given variety and its reduct, already considered by several researchers under the name of *piggyback dualities*. The results obtained are illustrated by the examples of *J-distributive lattices* of A. Petrovich and ¬-*lattices* of S. Celani, providing a better insight into their properties.

1. Introduction

The important question of relation between algebra and topology has always occupied mathematician's mind. One of the most important steps in the developments was done in the mid-30's of the last century by the famous representation theorems of M. Stone for *Boolean algebras* [75] and *distributive lattices* [76], and L. Pontrjagin for *abelian groups* [51], which opened a truly novel topological outlook on the well-known algebraic concepts. Transforming algebraic problems, stated in an abstract symbolic language, into their dual topological ones, where geometric intuition comes to our help, the new machinery induced many researchers to consider topological counterparts of different algebraic structures.

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The well-known representation theorem for *monadic Boolean algebras* of P. Halmos [34], which is a useful tool in the realm of algebraic logic coined *monadic* [35], serves as a good example.

While the representation of Boolean algebras has been appreciated right from the start, the respective machinery for distributive lattices appeared to be less satisfactory. In 1970 H. Priestley [53] presented another approach in her celebrated duality theorem, which combined the results of M. Stone for Boolean algebras and G. Birkhoff [5] for finite distributive lattices. The crucial point of her setting was the enrichment of the representing topological space with a partial order (getting the so-called *Priestley space*) that simplified the framework dramatically, making it more application-friendly. Equipped with the new machinery (called *Priestley duality*), working algebraists constructed a plethora of topological representations of structures based on distributive lattices [8, 9, 10, 12, 14, 26, 30, 49]. Soon it became clear that the above-mentioned results share a common background which, expressed in the language of category theory, brought into light a powerful theory of *natural dualities* [15, 16, 18, 20, 52]. Its main interest lies in establishing a dual isomorphism between a quasi-variety generated by a finite algebra (a finite set of finite algebras) and a particular category of structured (actually, enriched) topological spaces called *topological quasi-variety*. The technique is based on the concept of the so-called *schizophrenic object*, which has two personalities: algebraic and topological, the former (resp. latter) generating the algebraic (resp. topological) quasi-variety in question. The approach provides a common framework for many existing representation theories (e.g., the above-mentioned results of H. Priestley and M. Stone fit perfectly into the new setting), on one hand, and serves as a useful tool for generating new ones, on the other.

The notion of (*L*-)*fuzzy set* of L. A. Zadeh [78] and J. A. Goguen [28] brought a fresh challenge into the theory of topological representations. The new area of mathematics called *fuzzy* necessitated a new setting for well-known notions, based on the procedure of *fuzzification* [28]. In particular, numerous attempts were made to provide a fuzzy version of the above-mentioned representation theorems of M. Stone and H. Priestley. The most successful ones are due to S. E. Rodabaugh [57, 59], who considered the representation theories of M. Stone, applying his new *point-set lattice-theoretic* (*poslat*) technique. The results obtained not only generalized the classical theory, but in some cases uniquely streamlined it via two explicit evaluation maps. Motivated by the achievements, we provided their generalizations [73, 74], relying on our own *categorically-algebraic* (*catalg*) approach (the word "categorically" stems from "category theory"). The crucial difference from the ideas of S. E. Rodabaugh was the use of an arbitrary variety of algebras instead of a fixed category of lattices of particular kind (mostly, semi-quantales, currently popular in the fuzzy community), in which the fuzzification in question was being made. This brought our theory more inline with the above-mentioned natural dualities of D. Clark and B. Davey [15].

The case of Priestley duality appeared to be more difficult to attack. Up to now, there has been no fruitful fuzzification of the machinery, suitable for applications. In [70] we tried to fill in the gap, introducing a catalg framework for the duality in question. The theory presented was based on two important steps. On the first one, we showed sufficient conditions for an adjunction to exist between the dual category of a variety of algebras and the category of (catalg) topological spaces enriched in a variety of relational structures [17]. On the second step, we singled out particular subcategories (the so-called *spatial algebras* and *sober topological spaces*), the restriction to which of the obtained adjunction provided an equivalence. The setting was based on the well-known sobriety-spatiality approach to the representation theorems of M. Stone promoted by P. T. Johnstone, A. Pultr, S. Vickers, *etc.* [39, 54, 77], and also involved certain aspects of the theory of natural dualities (the notion of schizophrenic object was generalized to construct the adjunction in question; the structured topological spaces, however, were truncated to relational ones). The underlying machinery was borrowed from our previous research on the Stone dualities [73, 74], relying on the concept of *powerset operator*, coming from a generalization of the classical *image* and *preimage* operators induced by a map.

Soon afterwards, it appeared that the obtained theory provides a common framework not only for both the Priestley and the Stone representation theorems, but also incorporates the above-mentioned natural dualities (the category of structured topological spaces is, in fact, a particular instance of the category of relational ones; moreover, our schizophrenic object needs to be neither finite nor have the discrete topology), bringing into light their catalg properties, the study of which is indispensable for their development. This paper presents the emerging approach, calling it the theory of *catalg dualities*. Its most crucial property is applicability to both crisp and fuzzy topologies (providing a fuzzification of natural dualities), that extends considerably the field of potential applications and makes another step towards our ultimate goal of erasing the border between traditional and fuzzy mathematics. The theory also underlines the advantage of our catalg approach over the poslat one of S. E. Rodabaugh [56], the latter one, tied to lattices, being unable to switch to arbitrary algebras.

Looking closely at Priestley duality (which served as a motivation for our theory), a working algebraist will easily notice that it is not the result itself, but the amount of representation theorems based on it, that constitutes its importance in mathematics. As an example, recall topological representations of *Q-distributive lattices* [12], *J-distributive lattices* [49], *¬-lattices* [8, 9], (*weak-*) *quasi-Stone algebras* [10, 26], *complex algebras* [30], *etc.* All of them rely on the fact that the algebraic structure in question has a bounded distributive lattice as a reduct and therefore, Priestley duality is at hand. Additional operations on the lattices are then compensated by certain relations on their respective Priestley spaces, in such a way that the underlying duality for distributive lattices can be lifted to the new setting providing the desired topological representation. As was already mentioned, the gateway to topology makes the solution of some problems much easier. For example, it is possible to characterize congruences and thus, to get an insight into simple and subdirectly irreducible algebras [8, 27, 30, 49]. Moreover, R. Cignoli [13] uses the duality to construct free Q-distributive lattices from bounded distributive lattices, whereas H. Gaitán [26] does the same job for quasi-Stone algebras. In this paper, we show a catalg framework for the above-mentioned procedures. More particularly, given two varieties of algebras $\mathbf{C}$ and $\mathbf{C}'$ such that $\mathbf{C}'$ is a reduct of $\mathbf{C}$, we investigate thoroughly the possibilities of obtaining a topological representation for $\mathbf{C}$ (resp. $\mathbf{C}'$) from that for $\mathbf{C}'$ (resp. $\mathbf{C}$). The results obtained are illustrated by the examples of J-distributive lattices of A. Petrovich [49] and $\neg$-lattices of S. Celani [8], underlying almost all of the above-mentioned derived representation theorems. It is important to notice that the problem of reducts has already been considered in the theory of natural dualities under the name of *piggyback dualities* [15, 52]. The crucial difference of our approach from the already considered in the literature is the lack of an explicit construction of the duality in question in terms of particular algebras and topological spaces. Motivated by the essence of category theory itself, we provide instead a common catalg framework for the machinery employed, leaving it for the researcher to find its concrete realization in each particular case. The situation is similar to that of *adjoint situation* [36], which is defined in the abstract language of categories, without providing an explicit way of its obtainment (apart from general existence conditions) in every specific case.

The paper uses both category theory and algebra, relying more on the former. The necessary categorical background can be found in [1, 36, 45, 46]. For the notions of universal algebra we recommend [7, 17, 31, 46]. Although we tried to make the paper as much self-contained as possible, some details are still omitted and left to the reader.

2. Categorically-algebraic topology

In this section we recall from [70] basic concepts of *categorically-algebraic* (*catalg*) *topology* (see also [67, 68, 69, 66]). The approach is motivated (but is destined to replace) by the currently so popular *point-set lattice-theoretic* (*poslat*) one, introduced by S. E. Rodabaugh [56] and developed by P. Eklund, C. Guido, U. Höhle, T. Kubiak, A. Šostak and the initiator himself [24, 32, 33, 37, 42, 43, 58]. The main advantage of the new setting is the fact that the catalg framework ultimately erases the border between the traditional and the fuzzy developments, producing a theory which underlines the algebraic essence of the whole (not only fuzzy) mathematics and thus, propagating algebra as the main driving force of modern exact sciences.

The cornerstone of our approach is the notion of *algebra*. The structure is to be thought of as a set with a family of operations defined on it, satisfying certain identities, e.g., semigroup, monoid, group and also complete lattice, frame, quantale. In case of *finitary algebras*, i.e., those induced by a set of finitary operations, there are (at least) two ways of describing the resulting entities [7, 17, 31]. The categorical one uses the concept of *variety*, i.e., a class of algebras closed under homomorphic images, subalgebras and direct products. The algebraic one is based on the notion of *equational class*, namely, providing a set of identities and taking precisely those algebras which satisfy all of them. The well-known HSP-theorem of G. Birkhoff [4] says that varieties and equational classes coincide. Motivated by the algebraic structures used in fuzzy topology (where unions are usually represented as joins), this paper includes infinitary cases as well, extending the categorical approach of varieties to cover its needs, and leaving aside the infinitary algebraic machineries of *equationally-definable class* [46] and *equational category* [44, 64].

Definition 1.

- Let $\Omega = (n_\lambda)_{\lambda\in\Lambda}$ be a (possibly proper) class of cardinal numbers. An Ω-*algebra* is a pair $(A, (\omega_\lambda^A)_{\lambda\in\Lambda})$, which consists of a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$, called n_λ-*ary operations* on A. An Ω-*homomorphism* $(A, (\omega_\lambda^A)_{\lambda\in\Lambda}) \xrightarrow{\varphi} (B, (\omega_\lambda^B)_{\lambda\in\Lambda})$ is a map $A \xrightarrow{\varphi} B$ making the diagram

$$\begin{array}{ccc} A^{n_\lambda} & \xrightarrow{\varphi^{n_\lambda}} & B^{n_\lambda} \\ {\scriptstyle \omega_\lambda^A}\downarrow & & \downarrow{\scriptstyle \omega_\lambda^B} \\ A & \xrightarrow[\varphi]{} & B \end{array}$$

  commute for every $\lambda \in \Lambda$. $\mathbf{Alg}(\Omega)$ is the construct of Ω-algebras and Ω-homomorphisms, with the underlying functor denoted by $|-|$.
- Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps. A *variety of* Ω-*algebras* is a full subcategory of $\mathbf{Alg}(\Omega)$ closed under the formation of products, $\mathcal{M}$-subobjects (subalgebras) and $\mathcal{E}$-quotients (homomorphic images). The objects (resp. morphisms) of a variety are called *algebras* (resp. *homomorphisms*).
- Let $\mathbf{A}$ be a variety of Ω-algebras and let Ω' be a subclass of Ω. An Ω'-*reduct* of $\mathbf{A}$ is a pair $(\|-\|, \mathbf{B})$, where $\mathbf{B}$ is a variety of Ω'-algebras and $\mathbf{A} \xrightarrow{\|-\|} \mathbf{B}$ is a concrete functor.

The concept can be illustrated by several examples, all of which (except the last one) are currently rather popular in fuzzy topology [61, 63], due to the fact that their induced categories of fuzzified structures are topological over their ground categories. The last item in the list was motivated by our interest in *closure spaces* and their interrelationships with *state property systems* [2, 3, 72], introduced as the basic mathematical structure in the Geneva-Brussels approach to foundations of physics and modeling an arbitrary physical system by means of its set of states, its set of properties, and a relation of "actuality of a certain property for a certain state". A catalg modification of the notion has been developed by us in [72].

Definition 2.

- Given $\Xi \in \{\bigvee, \bigwedge\}$, a Ξ-*semilattice* is a partially ordered set having arbitrary Ξ. $\mathbf{CSLat}(\Xi)$ is the variety of Ξ-semilattices.
- A *semi-quantale* (*s-quantale*) is a $\bigvee$-semilattice equipped with a binary operation $\otimes$ (*multiplication*). **SQuant** is the variety of s-quantales.
- An s-quantale is called *unital* (*us-quantale*) provided that its multiplication has the unit $\mathfrak{1}$. **USQuant** is the variety of us-quantales.
- An s-quantale is called *distributive* (*ds-quantale*) provided that its multiplication distributes across finite $\vee$ from both sides. **DSQuant** is the variety of ds-quantales.
- An s-quantale is called *DeMorgan* provided that it is equipped with an order-reversing involution $(-)'$. **DmSQuant** is the variety of DeMorgan s-quantales.
- A *quantale* is an s-quantale whose multiplication is associative and distributes across $\bigvee$ from both sides. **Quant** is the variety of quantales.
- A *semi-frame* (*s-frame*) is a us-quantale whose multiplication and unit are $\wedge$ and $\top$ respectively. **SFrm** is the variety of s-frames.
- A *frame* is an s-frame which is a quantale. **Frm** is the variety of frames.
- A *closure semilattice* (*c-semilattice*) is a $\bigwedge$-semilattice, with the singled out bottom element $\bot$. **CSL** is the variety of c-semilattices.

The reader should be aware that our concept of ds-quantale is a stronger version of *ordered s-quantale* of [61, 63], where monotonicity instead of $\vee$-distributivity is postulated. The new notion was motivated by the definition of variety, i.e., its closure under homomorphic images that fails in the weaker case. Also notice the simple facts: $\mathbf{CSLat}(\bigvee)$ is a reduct of **SQuant**; **SQuant** is a reduct of **USQuant**, **DSQuant** and **DmSQuant**; **DSQuant** is a reduct of **Quant**; **USQuant** is a reduct of **SFrm**; **UQuant** is a reduct of **Frm**; $\mathbf{CSLat}(\bigwedge)$ is a reduct of **CSL**.

For the sake of convenience, from now on we use the following notations (see, e.g., [23, 58, 61] for the motivation). An arbitrary variety is denoted by **A**, **B**, **C**, *etc.* (sometimes with indices). The categorical dual of a variety **A** is denoted by **LoA** (the "**Lo**" comes from "localic"), whose objects (resp. morphisms) are called *localic algebras* (resp. *homomorphisms*). Following the already accepted notations of [39], the dual of **Frm** is denoted by **Loc**, whose objects are called *locales*. Given a localic algebra A, $\mathbf{S}_A$ stands for the subcategory of **LoA** with the only morphism 1_A. To distinguish maps (or, more generally, morphisms) and homomorphisms, the former are denoted by f, g, h (α, β, γ in case of fuzzy sets), reserving φ, ψ, ϕ for the latter. Given a homomorphism φ, the respective localic one is denoted by φ^{op} and vice versa.

The second crucial notion of our approach is a mixture of *powerset theories* of [61, Definition 3.5] (see also [60, 63]) and *topological theories* of [1, Exercise 22B].

Definition 3. A *variety-based backward powerset theory* (*vbp-theory*) in a category $\mathbf{X}$ (the *ground category* of the theory) is a functor $\mathbf{X} \xrightarrow{P} \mathbf{LoA}$.

The intuition for the new concept comes from the so-called *image* (resp. *preimage*) operators [61], well-known for every working mathematician. Recall that given a set map $X \xrightarrow{f} Y$, there exist the maps $\mathcal{P}(X) \xrightarrow{f^{\rightarrow}} \mathcal{P}(Y)$ (resp. $\mathcal{P}(Y) \xrightarrow{f^{\leftarrow}} \mathcal{P}(X)$) with $f^{\rightarrow}(S) = \{f(x) \,|\, x \in S\}$ (resp. $f^{\leftarrow}(T) = \{x \,|\, f(x) \in T\}$). The latter operator can be extended to a more general setting.

Example 4. Given a variety $\mathbf{A}$, every subcategory $\mathbf{C}$ of **LoA** induces a functor $\mathbf{Set} \times \mathbf{C} \xrightarrow{(-)^{\leftarrow}} \mathbf{LoA}$ defined by $((X, A) \xrightarrow{(f,\varphi)} (Y, B))^{\leftarrow} = A^X \xrightarrow{((f,\varphi)^{\leftarrow})^{op}} B^Y$ with $(f, \varphi)^{\leftarrow}(\alpha) = \varphi^{op} \circ \alpha \circ f$. Considered as a vbp-theory $\mathcal{S}^{\mathbf{C}}_{\mathbf{A}}$, the functor was used in our former approach to catalg topology of, e.g., [73, 74], incorporating at the same time a multitude of important subcases, some of which are listed below.

(1) $\mathbf{Set} \times \mathbf{S_2} \xrightarrow{\mathcal{P}=(-)^{\leftarrow}} \mathbf{LoCBool}$ with **CBool** the variety of *complete Boolean algebras* (complete, complemented, distributive lattices) and $\mathbf{2} = \{\bot, \top\}$, provides the above-mentioned preimage operator.

(2) $\mathbf{Set} \times \mathbf{S}_I \xrightarrow{\mathcal{Z}=(-)^{\leftarrow}_I} \mathbf{DmLoc}$ with $I = [0, 1]$ the unit interval, provides the fixed-basis fuzzy approach of L. A. Zadeh [78].

(3) $\mathbf{Set} \times \mathbf{S}_L \xrightarrow{\mathcal{G}_1=(-)^{\leftarrow}_L} \mathbf{Loc}$ provides the fixed-basis L-fuzzy approach of J. A. Goguen [28]. The setting was changed to $\mathbf{Set} \times \mathbf{S}_L \xrightarrow{\mathcal{G}_2=(-)^{\leftarrow}_L} \mathbf{LoUQuant}$ in [29]. The machinery can be generalized to an arbitrary variety $\mathbf{A}$ and the theory $\mathcal{S}^{\mathbf{S}_A}_{\mathbf{A}}$, uniting the previous items in one common fixed-basis framework.

(4) $\mathbf{Set} \times \mathbf{C} \xrightarrow{\mathcal{R}^{\mathbf{C}}_1=(-)^{\leftarrow}} \mathbf{DmLoc}$ with $\mathbf{C}$ a subcategory of **DmLoc**, gives the variable-basis poslat approach of S. E. Rodabaugh [55]. The setting was generalized to $\mathbf{Set} \times \mathbf{C} \xrightarrow{\mathcal{R}^{\mathbf{C}}_2=(-)^{\leftarrow}} \mathbf{LoUSQuant}$ in [61].

(5) $\mathbf{Set} \times \mathbf{FuzLat} \xrightarrow{\mathcal{E}=(-)^{\leftarrow}} \mathbf{FuzLat}$ provides the variable-basis approach of P. Eklund [24], motivated by those of S. E. Rodabaugh [55] and B. Hutton [38]. Notice that **FuzLat** is the dual of the variety **HUT** of completely distributive DeMorgan frames called *Hutton algebras* [58].

Two important points should be mentioned at once. Firstly, the topic of the paper restricts us to the ground categories of the form $\mathbf{Set} \times \mathbf{LoA}$. In [67, 71] more general categories come into account, motivated by *generalized topology* of M. Demirci [21, 22] and *non-commutative topology* of C. J. Mulvey and J. W. Pelletier [47, 48]. Secondly, Example 4 deals with the preimage operator, leaving the image one aside. The reason is that the current fuzzifications of the map are $\bigvee$-dependant (e.g., having the form of $(f_A^{\rightarrow}(\alpha))(y) = \bigvee_{f(x)=y} \alpha(x)$ in the fixed-basis case), whereas a general variety may lack even a partial order. An additional restriction on the variety in question (the existence of categorical *biproducts* [36]) allows one to restore the full framework [68].

The next concept is a modification of *composite topological theories* of [67]. The crucial change is that they are no longer the powerset theories in question (before going forward, the reader is advised to recall the construction of *product of categories* [36]). To avoid unnecessary complications (touched in [67]), from now on, "set-indexed" means "indexed by a *non-empty* set".

Definition 5. Let $\mathbf{X}$ be a category and let $\mathcal{T}_I = ((P_i, (\| - \|_i, \mathbf{B}_i)))_{i \in I}$ be a set-indexed family, where for every $i \in I$, $\mathbf{X} \xrightarrow{P_i} \mathbf{LoA}_i$ is a vbp-theory in the category $\mathbf{X}$ and $(\| - \|_i, \mathbf{B}_i)$ is a reduct of $\mathbf{A}_i$. A *composite variety-based topological theory* (*cvt-theory*) *in* $\mathbf{X}$ *induced by* $\mathcal{T}_I$ is the functor $\mathbf{X} \xrightarrow{T_I = \langle \| - \|_i^{op} \circ P_i \rangle_I} \prod_{i \in I} \mathbf{LoB}_i$ defined by commutativity of the diagram

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{P_j} & \mathbf{LoA}_j \\ {\scriptstyle T_I} \downarrow & & \downarrow {\scriptstyle \|-\|_j^{op}} \\ \prod_{i \in I} \mathbf{LoB}_i & \xrightarrow[\pi_j]{} & \mathbf{LoB}_j \end{array}$$

for every $j \in I$ (π_j is the respective projection).

Since a cvt-theory is completely determined by the respective family $\mathcal{T}_I$, we use occasionally the notation $((P_i, \mathbf{B}_i))_{i \in I}$ instead of T_I. A cvt-theory induced by a singleton family is denoted by T. We also employ the shorter T_i for $\|-\|_i^{op} \circ P_i$. All preliminaries on their places, we are ready to introduce catalg topology (notice that the use of the standard image operator in the definition of continuity was motivated by purely esthetic reasons).

Definition 6. Let T_I be a cvt-theory in the category $\mathbf{X}$. $\mathbf{CTop}(T_I)$ is the concrete category over $\mathbf{X}$, whose objects (called *composite variety-based topological spaces*) are pairs $(X,(\tau_i)_{i\in I})$ with X an $\mathbf{X}$-object and τ_i a subalgebra of $T_i(X)$ for every $i \in I$ ($(\tau_i)_{i\in I}$ is called *composite variety-based topology* on X), and whose morphisms $(X,(\tau_i)_{i\in I}) \xrightarrow{f} (Y,(\sigma_i)_{i\in I})$ are those $\mathbf{X}$-morphisms $X \xrightarrow{f} Y$ that satisfy $((T_i f)^{op})^{\rightarrow}(\sigma_i) \subseteq \tau_i$ for every $i \in I$ (called *composite continuity*). The underlying functor to the ground category $\mathbf{X}$ is defined by $|(X,(\tau_i)_{i\in I}) \xrightarrow{f} (Y,(\sigma_i)_{i\in I})| = X \xrightarrow{f} Y$.

For the sake of simplicity, $\mathbf{CTop}(T)$ is denoted by $\mathbf{Top}(T)$. The new concept was motivated by the multitude of approaches to topology in the fuzzy community. Our main purpose was to provide a common unifying framework suitable for exploring interrelations between different topological settings. The machinery employed was inspired by *(bi)topological theories* of S. E. Rodabaugh [61, 62] and T. Kubiak [42].

Example 7. The following provides a short list of examples illustrating the notion of catalg topology, to give the feeling of their abundance and the fruitfulness of the new unifying framework.

(1) $\mathbf{Top}((\mathcal{P},\mathbf{Frm}))$ is isomorphic to the category $\mathbf{Top}$ of topological spaces and continuous maps.
(2) $\mathbf{Top}((\mathcal{P},\mathbf{CSL}))$ is isomorphic to the category $\mathbf{Cls}$ of closure spaces and continuous maps studied by D. Aerts *et al.* [2, 3].
(3) $\mathbf{CTop}(((\mathcal{P},\mathbf{Frm}))_{i\in\{1,2\}})$ is isomorphic to the category $\mathbf{BiTop}$ of bitopological spaces and bicontinous maps [41].
(4) $\mathbf{Top}((\mathcal{Z},\mathbf{Frm}))$ is isomorphic to the category I-$\mathbf{Top}$ of fixed-basis fuzzy topological spaces introduced by C. L. Chang [11].
(5) $\mathbf{Top}((\mathcal{G}_2,\mathbf{UQuant}))$ is isomorphic to the category L-$\mathbf{Top}$ of fixed-basis L-fuzzy topological spaces of J. A. Goguen [29].
(6) $\mathbf{Top}((\mathcal{R}_i^{\mathbf{C}},\mathbf{USQuant}))$ is isomorphic to the category $\mathbf{C}$-$\mathbf{Top}_i$ ($i \in \{1,2\}$) for variable-basis poslat topology of S. E. Rodabaugh [55, 61].
(7) $\mathbf{CTop}(((\mathcal{R}_1^{\mathbf{S}_L},\mathbf{Frm}))_{i\in\{1,2\}})$ is isomorphic to the category L-$\mathbf{BiTop}$ of fixed-basis L-bitopological spaces of T. Kubiak [42].
(8) $\mathbf{CTop}(((\mathcal{R}_2^{\mathbf{S}_L},\mathbf{USQuant}))_{i\in\{1,2\}})$ is isomorphic to the category L-$\mathbf{BiTop}$ of fixed-basis L-bitopological spaces of S. E. Rodabaugh [62].
(9) $\mathbf{Top}((\mathcal{E},\mathbf{Frm}))$ is isomorphic to the category $\mathbf{FUZZ}$ for variable-basis poslat topology of P. Eklund [24], motivated by those of S. E. Rodabaugh [55] and B. Hutton [38].
(10) $\mathbf{Top}((\mathcal{S}_{\mathbf{A}}^{\mathbf{S}_Q},\mathbf{A}))$ (resp. $\mathbf{Top}((\mathcal{S}_{\mathbf{A}}^{\mathbf{LoA}},\mathbf{A}))$) is isomorphic to the fixed- (resp. variable-) basis category Q-$\mathbf{Top}$ (resp. $\mathbf{LoA}$-$\mathbf{Top}$) used in our former approach to catalg topologies [74] (resp. [73]).

Notice the frequent use of the variety **USQuant**, whose objects are proposed by S. E. Rodabaugh [61] as the basic mathematical structure for doing poslat topology upon, since the axioms of s-quantales constitute the minimum allowing the obtained categories for topology (and these include many well-known categories) to be topological over their ground categories. The claim was justified through our catalg approach in [69].

The reader may be well aware of the important result of classical topology stating that continuity of a map can be checked on the elements of a subbase (for a full discussion of the (categorical) role of subbase in topology see [58]). It appears that the result can be readily extended to our current setting.

Definition 8.

- Let $\mathbf{A}$ be a variety of Ω-algebras and let $\Omega' \subseteq \Omega$ be a (possibly) subclass. Given an algebra A and a subset $S \subseteq A$, $\langle S\rangle_{\Omega'}$ denotes the smallest Ω'-subreduct of A containing S ($\langle S\rangle_{\Omega}$ is shortened to $\langle S\rangle$).
- Given a cvt-theory $\mathbf{X} \xrightarrow{T} \mathbf{LoB}$ with $\mathbf{B}$ an Ω'-reduct of $\mathbf{A}$, a subclass $\Omega'' \subseteq \Omega'$ and a $\mathbf{Top}(T)$-space (X, τ), a subset $S \subseteq T(X)$ is an Ω''-*base* for τ provided that $\tau = \langle S\rangle_{\Omega''}$. Ω'-bases are called *subbases*.

The next example provides the intuition for the new concept, justifying its fruitfulness.

Example 9. In the category $\mathbf{C}$-$\mathbf{Top}_2$, $\{\bigvee\}$-bases (resp. $\{\bigvee, \otimes, 1\}$-bases) are well-known bases (resp. subbases) of poslat topology as defined in, e.g., [62]. **Top** gives the classical definition of base (resp. subbase), where elements of the topology are unions of (resp. unions of finite intersections of) elements of the base (resp. subbase).

Lemma 10. *Let $A_1 \xrightarrow{\varphi} A_2$ be a homomorphism of a variety $\mathbf{A}$ of Ω-algebras and let $\Omega' \subseteq \Omega$.*
(1) For every Ω'-subreduct B of A_2, $\varphi^{\leftarrow}(B)$ is an Ω'-subreduct of A_1.
(2) For every subset $S \subseteq A_1$, $\varphi^{\rightarrow}(\langle S\rangle_{\Omega'}) = \langle \varphi^{\rightarrow}(S)\rangle_{\Omega'}$.

Proof. Since Item (1) is easy, we show Item (2). $S \subseteq (\varphi^{\leftarrow} \circ \varphi^{\rightarrow})(S) \subseteq \varphi^{\leftarrow}(\langle \varphi^{\rightarrow}(S)\rangle_{\Omega'})$ gives $\langle S\rangle_{\Omega'} \subseteq \varphi^{\leftarrow}(\langle \varphi^{\rightarrow}(S)\rangle_{\Omega'})$ by Item (1) and therefore $\varphi^{\rightarrow}(\langle S\rangle_{\Omega'}) \subseteq (\varphi^{\rightarrow} \circ \varphi^{\leftarrow})(\langle \varphi^{\rightarrow}(S)\rangle_{\Omega'}) \subseteq \langle \varphi^{\rightarrow}(S)\rangle_{\Omega'}$. Conversely, $S \subseteq \langle S\rangle_{\Omega'}$ gives $\varphi^{\rightarrow}(S) \subseteq \varphi^{\rightarrow}(\langle S\rangle_{\Omega'})$ and thus $\langle \varphi^{\rightarrow}(S)\rangle_{\Omega'} \subseteq \langle \varphi^{\rightarrow}(\langle S\rangle_{\Omega'})\rangle_{\Omega'} = \varphi^{\rightarrow}(\langle S\rangle_{\Omega'})$. □

Corollary 11. *Let T_I be a cvt-theory in a category $\mathbf{X}$ and let $(X, (\tau_i)_{i\in I})$, $(Y, (\sigma_i)_{i\in I})$ be $\mathbf{CTop}(T_I)$-spaces with $\sigma_i = \langle S_i\rangle_{\Omega''_i}$ for every $i \in I$. An $\mathbf{X}$-morphism $X \xrightarrow{f} Y$ is continuous iff $((T_i f)^{op})^{\rightarrow}(S_i) \subseteq \tau_i$ for every $i \in I$.*

Proof. To show the sufficiency, use Lemma 10(2) in the following: $((T_i f)^{op})^{\rightarrow}(\sigma_i) = ((T_i f)^{op})^{\rightarrow}(\langle S_i\rangle_{\Omega''_i}) = \langle ((T_i f)^{op})^{\rightarrow}(S_i)\rangle_{\Omega''_i} \subseteq \langle \tau_i\rangle_{\Omega''_i} = \tau_i$. □

Corollary 11 provides a generalization of the above-mentioned achievement for subbases in various settings (cf., e.g., Theorem 3.2.6 of [58]). In particular, the result incorporates directly the cases of classical subbase and base (no additional calculation is required as in, e.g., [25, Proposition 1.4.1]).

It may be well-known to the reader that **Top** has products of objects (Cartesian products of the underlying sets with the initial topology induced by projections). In conclusion of the section, we show that the result holds in a more general setting of catalg topologies.

Lemma 12. *Let T_I be a cvt-theory in a category* $\mathbf{X}$ *and let* $\mathbf{X}$ *have products. Then the category* $\mathbf{CTop}(T_I)$ *has concrete products.*

Proof. Given a set-indexed family $((X_j, (\tau_{j_i})_{i\in I}))_{j\in J}$ of $\mathbf{CTop}(T_I)$-spaces, the desired product is given by $((\prod_{k\in J} X_k, (\prod_{k\in J} \tau_{k_i})_{i\in I}) \xrightarrow{\pi_j} (X_j, (\tau_{j_i})_{i\in I}))_{j\in J}$, where $(\prod_{k\in J} X_k \xrightarrow{\pi_j} X_j)_{j\in J}$ is the product of $(X_j)_{j\in J}$ in the category $\mathbf{X}$ (implying concreteness) and $\prod_{k\in J} \tau_{k_i} = \langle \bigcup_{j\in J}((T_i\pi_j)^{op})^{\rightarrow}(\tau_{j_i})\rangle$ for every $i \in I$. □

A simple application of Lemma 12 runs as follows (recall the category Q-**Top** of Example 7(10)).

Corollary 13. *The category Q-***Top** *has concrete products.*

Notice that the respective result for the category **LoA-Top** does not necessarily hold, since in general **A** may lack coproducts (cf., e.g., the variety **CLat** of complete lattices [1, Exercise 10S]).

3. Categorically-algebraic dualities

In this section we provide a catalg approach to the theory of *natural dualities* developed by D. Clark and B. Davey in [15]. The cited theory is based on the concept of the so-called *schizophrenic object*, i.e., a finite set M equipped with two structures: algebraic (providing an algebra M_A) and topological (assumed to be discrete) with some additional enrichment consisting of finitary total and partial operations as well as finitary relations (providing a *structured topological space* M_T). Under suitable conditions, the theory developed produces a dual equivalence (the so-called *natural duality*) between the algebraic quasi-variety generated by M_A and the appropriately defined topological quasi-variety obtained from M_T.

The approach of this paper differs from the above-mentioned one in several respects. The most important points are underlined below.

- Every requirement of finiteness on the structures in question is dropped.
- Topological enrichment is reduced to a family of relations, incorporating both total and partial operations as their particular kinds.

- Arbitrary topologies on the set M are allowed, extending the framework considerably.
- Algebraic (resp. topological) quasi-varieties are replaced with the notion of *spatiality* (resp. *sobriety*), producing an equivalence between the categories of spatial algebras and sober spaces in the sense of P. T. Johnstone [39] (see also [54, 77]).
- The category **Top** is replaced with the category Q-**Top** of Example 7(10), providing catalg fuzzification.

Having an informal description of what follows in hand, we turn to the explicit construction of the promised machinery. The results obtained come from [70], where we restricted our attention to the case of H. Priestley duality for bounded distributive lattices [53]. It is the purpose of this section to present the achievements in a more general light of natural dualities.

3.1. **Underlying adjunction.** For the sake of simplicity, we reduce the topological setting to the fixed-basis category $\mathbf{Top}((\mathcal{S}_{\mathbf{A}}^{\mathbf{S}_Q}, \mathbf{B}))$, denoted by $Q_{\mathbf{B}}$-**Top**, with the prefix "$Q_{\mathbf{B}}$" added to the respective topological stuff, e.g., "$Q_{\mathbf{B}}$-space", "$Q_{\mathbf{B}}$-topology", "$Q_{\mathbf{B}}$-continuity" (we also use the notation τ_X, to underline the set, a given $Q_{\mathbf{B}}$-topology τ is referring to), leaving the (composite) variable-basis generalization to the subsequent developments of the topic. On the next step, we show an analogue of the aforesaid structured topological space, suitable for our framework and motivated by *relational structures* of [17, Chapter V] (the reader should recall Definition 1).

Definition 14.

- Let $\Sigma = (m_\upsilon)_{\upsilon \in \Upsilon}$ be a (possibly proper) class of cardinal numbers. A Σ-*structure* is a pair $(R,(\varpi_\upsilon^R)_{\upsilon\in\Upsilon})$, which consists of a set R and a family of subsets $\varpi_\upsilon^R \subseteq R^{m_\upsilon}$, called m_υ-*ary relations* on R. A Σ-*homomorphism* $(R,(\varpi_\upsilon^R)_{\upsilon\in\Upsilon}) \xrightarrow{f} (S,(\varpi_\upsilon^S)_{\upsilon\in\Upsilon})$ is a map $R \xrightarrow{f} S$ such that $(f^{m_\upsilon})^{\rightarrow}(\varpi_\upsilon^R) \subseteq \varpi_\upsilon^S$ for every $\upsilon \in \Upsilon$. $\mathbf{Rel}(\Sigma)$ is the construct of Σ-structures and Σ-homomorphisms, with the underlying functor denoted by $|-|$.
- Let $\mathcal{R}$ be the class of all Σ-homomorphisms $R \xrightarrow{f} S$ such that for every $\upsilon \in \Upsilon$ and every $\langle r_i\rangle_{m_\upsilon} \in R^{m_\upsilon}$, $\langle f(r_i)\rangle_{m_\upsilon} \in \varpi_\upsilon^S$ implies $\langle r_i\rangle_{m_\upsilon} \in \varpi_\upsilon^R$, and let $\mathcal{M}$ (resp. $\mathcal{E}$) be the subclass of $\mathcal{R}$ with injective (resp. surjective) underlying maps. A *variety of* Σ-*structures* is a full subcategory of $\mathbf{Rel}(\Sigma)$ closed under the formation of products, $\mathcal{M}$-subobjects and $\mathcal{E}$-quotients. The objects (resp. morphisms) of a variety are called *structures* (resp. *homomorphisms*).

- Let $\mathbf{R}$ be a variety of Σ-structures and let Σ' be a subclass of Σ. A Σ'-*reduct* of $\mathbf{R}$ is a pair $(\|-\|, \mathbf{S})$, where $\mathbf{S}$ is a variety of Σ'-structures and $\mathbf{R} \xrightarrow{\|-\|} \mathbf{S}$ is a concrete functor.

From now on, varieties of Σ-structures are denoted by $\mathbf{R}$, $\mathbf{S}$, $\mathbf{T}$, *etc.* To avoid ambiguity with varieties of Ω-algebras, we add the letter "r" (stemming from "relational") to their name, i.e., *r-variety*, referring to their objects (resp. morphisms) as *r-structures* (resp. *r-homomorphisms*).

Example 15. The construct **Pos** of partially ordered sets and order-preserving maps is an r-variety induced by the category $\mathbf{Rel}(2)$, whose signature consists of a single binary relation.

Example 16. Given a category $\mathbf{Alg}(\Omega)$, define $\Sigma = (n_\lambda + 1)_{\lambda \in \Lambda}$ and get the category $\mathbf{Rel}(\Sigma)$. There exists a concrete functor $\mathbf{Alg}(\Omega) \xrightarrow{V} \mathbf{Rel}(\Sigma)$ defined by $V((A_1, (\omega_\lambda^{A_1})_{\lambda \in \Lambda}) \xrightarrow{\varphi} (A_2, (\omega_\lambda^{A_2})_{\lambda \in \Lambda})) = (A_1, (\varpi_\lambda^{A_1})_{\lambda \in \Lambda}) \xrightarrow{\varphi} (A_2, (\varpi_\lambda^{A_2})_{\lambda \in \Lambda})$ where $\varpi_\lambda^{A_j} = \operatorname{Grph} \omega_\lambda^{A_j} = \{\langle \langle a_i \rangle_{n_\lambda}, a \rangle \,|\, \omega_\lambda^{A_j}(\langle a_i \rangle_{n_\lambda}) = a\} \subseteq A_j^{n_\lambda} \times A_j$ for every $\lambda \in \Lambda$ and every $j \in \{1, 2\}$. The image under V of a variety $\mathbf{A}$ (denoted by $V^{\to}(\mathbf{A})$) is closed under the formation of products, but may miss the closure under $\mathcal{M}$-subobjects and $\mathcal{E}$-quotients (every subset (resp. quotient set) of a Σ-structure gives rise to its $\mathcal{M}$-subobject (resp. $\mathcal{E}$-quotient), that almost never holds for algebras). The smallest r-variety containing $V^{\to}(\mathbf{A})$ (which exists since the family of r-varieties of the same signature is closed under (possibly class-indexed) intersections) gives the one corresponding to $\mathbf{A}$.

It is important to keep in mind that unlike the case of algebras, a bijective r-homomorphism is *not* necessarily an r-isomorphism. Due to the lack of space, we will not dwell upon other properties of r-varieties, modifying the category $Q_{\mathbf{B}}$-**Top** with their help instead.

Definition 17. Given an r-variety $\mathbf{R}$, $Q_{\mathbf{B}}$-**RTop** is the category, whose objects (called *r-$Q_{\mathbf{B}}$-spaces*) are pairs (R, τ) with R an r-structure and $(|R|, \tau)$ a $Q_{\mathbf{B}}$-space, and whose morphisms $(R, \tau) \xrightarrow{f} (S, \sigma)$ are those $Q_{\mathbf{B}}$-continuous maps $(|R|, \tau) \xrightarrow{f} (|S|, \sigma)$ that are also r-homomorphisms (called *r-$Q_{\mathbf{B}}$-morphisms*). The underlying functor to the ground category $Q_{\mathbf{B}}$-**Top** is defined by $|(R, \tau) \xrightarrow{f} (S, \sigma)| = (|R|, \tau) \xrightarrow{f} (|S|, \sigma)$.

In the language of enriched category theory of G. M. Kelly [40], $Q_{\mathbf{B}}$-**RTop** is nothing else than the category $Q_{\mathbf{B}}$-**Top** enriched in an r-variety $\mathbf{R}$.

Example 18. The category **Top** enriched in the r-variety **Pos** yields the category **PoTop** of partially-ordered topological spaces and order-preserving continuous maps.

The reader may remember that dualities of [15] consist of a dual equivalence between the algebraic quasi-variety $\boldsymbol{\mathcal{A}}$ generated by M_A (algebraic personality of the schizophrenic object in question) and the respective topological quasi-variety $\mathfrak{X}$ induced by M_T (topological personality of M). We propose the following generalization of the machinery, motivated by the approach to the Stone representation theories of P. T. Johnstone [39].

Step 1.: Replace $\mathfrak{X}$ (resp. $\boldsymbol{\mathcal{A}}$) with $Q_{\mathbf{B}}$-**RTop** (resp. a variety $\mathbf{C}$).

Step 2.: Construct two functors $Q_{\mathbf{B}}\text{-}\mathbf{RTop} \xrightarrow{E} \mathbf{LoC}$ and $\mathbf{LoC} \xrightarrow{D} Q_{\mathbf{B}}\text{-}\mathbf{RTop}$ with D a right adjoint to E.

Step 3.: Single out particular subcategories of $Q_{\mathbf{B}}$-**RTop** (resp. **LoC**), the restriction to which of the adjunction obtained provides an equivalence.

Having outlined briefly the forthcoming developments, we proceed to the explicit construction.

We begin by fixing a variety $\mathbf{C}$, with the ultimate goal to construct a functor $Q_{\mathbf{B}}\text{-}\mathbf{RTop} \xrightarrow{E} \mathbf{LoC}$. Two preliminary concepts are necessary for the completion of the task. The first one is a modification of the notion of reduct, already encountered by the reader in the paper (Definitions 1, 14).

Definition 19. An *r-reduct* of a variety $\mathbf{C}$ is a pair $(\| - \|, \mathbf{S})$, where $\mathbf{S}$ is an r-variety and $\mathbf{C} \xrightarrow{\|-\|} \mathbf{S}$ is a concrete functor. An r-reduct is *algebraic* provided that for every algebra C and every $\upsilon \in \Upsilon_{\mathbf{S}}$, $\varpi_{\upsilon}^{\|C\|}$ is a subalgebra of $C^{m_{\upsilon}}$.

Notice the notation $\Upsilon_{\mathbf{S}}$ for the signature of $\mathbf{S}$ in Definition 19, which will be used frequently in the subsequent developments. Also important is the fact that Definition 19 never assumes any connection between the signatures of $\mathbf{C}$ and $\mathbf{S}$. The next example gives the intuition for the new concept.

Example 20. The functor $\mathbf{DSQuant} \xrightarrow{\|-\|} \mathbf{Pos}$ defined by $\|(A \xrightarrow{\varphi} B)\| = (A, \leqslant) \xrightarrow{\varphi} (B, \leqslant)$ produces an algebraic r-reduct. Similarly, $(\| - \|, \mathbf{Pos})$ is an r-reduct of **SQuant** which is not algebraic.

To get the intuition for the second concept recall that sometimes topologies are defined on sets already equipped with an algebraic structure. It is natural then to ask for some compatibility between algebra and topology. One of the simplest requirements is the continuity of the algebraic operations. The next definition shows a catalg modification of the concept (recall that $Q_{\mathbf{B}}$-**Top** has concrete products by Corollary 13).

Definition 21. A $Q_{\mathbf{B}}$-*continuous algebra* is a pair (D, τ), where D is an algebra of some variety $\mathbf{D}$ and $(|D|, \tau)$ is a $Q_{\mathbf{B}}$-space such that $|D|^{n_{\lambda}} \xrightarrow{\omega_{\lambda}^{D}} |D|$ is $Q_{\mathbf{B}}$-continuous for every $\lambda \in \Lambda_{\mathbf{D}}$.

The next lemmas suggest two important examples of continuous algebras.

Lemma 22. *Let $\Omega_{\mathbf{B}}$ induce the structure of* **SFrm** *on* $\|Q\|$ *and let* $\mathbf{D}$ *be a finitary variety. Every* $\mathbf{D}$*-algebra* D *equipped with the* discrete $Q_{\mathbf{B}}$-topology $\tau^d = Q^{|D|}$ *provides a* $Q_{\mathbf{B}}$*-continuous algebra* (D, τ^d).

Proof. Given $\lambda \in \Lambda_{\mathbf{D}}$, we show that $|D|^{n_\lambda}$ has the discrete $Q_{\mathbf{B}}$-topology. Every $(d, q) \in |D| \times |Q|$ gives rise to a map $D \xrightarrow{\alpha_d^q} Q$ defined by $\alpha_d^q(e) = q$, if $e = d$; otherwise, $\alpha_d^q(e) = \bot$. Given $\beta \in Q^{|D|^{n_\lambda}}$, every $\langle d_i \rangle_{n_\lambda} \in |D|^{n_\lambda}$ induces a map (n_λ is finite) $\alpha_{\langle d_i \rangle_{n_\lambda}}^{\beta(\langle d_i \rangle_{n_\lambda})} = \wedge_{i \in n_\lambda}(\alpha_{d_i}^{\beta(\langle d_i \rangle_{n_\lambda})} \circ \pi_i) = \wedge_{i \in n_\lambda}(\pi_i)_Q^{\leftarrow}(\alpha_{d_i}^{\beta(\langle d_i \rangle_{n_\lambda})}) \in \tau_{|D|^{n_\lambda}}$ ($\Omega_{\mathbf{SFrm}} \subseteq \Omega_{\mathbf{B}}$ on $\|Q\|$), yielding $\beta = \bigvee_{\langle d_i \rangle_{n_\lambda} \in |D|^{n_\lambda}} \alpha_{\langle d_i \rangle_{n_\lambda}}^{\beta(\langle d_i \rangle_{n_\lambda})} \in \tau_{|D|^{n_\lambda}}$. □

Corollary 23. *In the framework of the category* **Top**, *the lattice* $\mathbf{2} = \{\bot, \top\}$ *equipped with the discrete topology* $\tau^d = \{\varnothing, \{\bot\}, \{\top\}, \mathbf{2}\}$ *provides a continuous algebra.*

Lemma 24. *Let* $\mathbf{D}$ *be a variety such that* $\Omega_{\mathbf{D}} \subseteq \Omega_{\mathbf{B}}$ *and let* D *be a* $\mathbf{D}$*-algebra with an* $\Omega_{\mathbf{D}}$*-homomorphism* $D \xrightarrow{\varphi} Q$. *The* Sierpinski $Q_{\mathbf{B}}$-topology $\tau^s = \langle \varphi \rangle$ *on* D *provides a* $Q_{\mathbf{B}}$*-continuous algebra* (D, τ^s).

Proof. Given $\lambda \in \Lambda_{\mathbf{D}}$, $(\omega_\lambda^D)_Q^{\leftarrow}(\varphi) = \varphi \circ \omega_\lambda^D = \omega_\lambda^Q \circ \varphi^{n_\lambda} = \omega_\lambda^{Q^{|D|^{n_\lambda}}}(\langle \varphi \circ \pi_i \rangle_{n_\lambda}) = \omega_\lambda^{Q^{|D|^{n_\lambda}}}(\langle (\pi_i)_Q^{\leftarrow}(\varphi) \rangle_{n_\lambda}) \in \tau_{|D|^{n_\lambda}}$. The desired result now follows from Corollary 11. □

Corollary 25. *In the framework of the category* **Top**, *the frame* $\mathbf{2} = \{\bot, \top\}$ *equipped with the Sierpinski topology* $\tau^s = \{\varnothing, \{\top\}, \mathbf{2}\}$ *provides a continuous algebra.*

All preliminaries on their places, we proceed to the definition of the desired functor $Q_{\mathbf{B}}$-**RTop** $\xrightarrow{E}$ **LoC**. Following the line of schizophrenic object of D. Clark and B. Davey [15], we fix a $\mathbf{C}$-algebra $\mathbb{C}$ and equip it with a $Q_{\mathbf{B}}$-topology $\boldsymbol{\delta}$. It appears that sufficient conditions for the existence of the functor in question can be formulated as follows (notice that $\boldsymbol{\delta}$ is never assumed to be discrete):

($\mathcal{R}$) $\mathbf{R}$ is an algebraic r-reduct of $\mathbf{C}$.

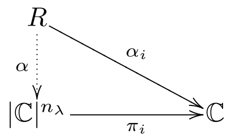
($\mathcal{C}$) $(\mathbb{C}, \boldsymbol{\delta})$ is a $Q_{\mathbf{B}}$-continuous algebra.

The next lemma constructs the functor explicitly (notice the use of the vbp-theory $\mathcal{S}_{\mathbf{C}}^{\mathbf{S}_{\mathbb{C}}}$ in the action on morphisms).

Lemma 26. *If* ($\mathcal{R}$), ($\mathcal{C}$) *hold, then there exists a functor* $Q_{\mathbf{B}}$-**RTop** $\xrightarrow{E}$ **LoC** *given by* $E((R, \tau) \xrightarrow{f} (S, \sigma)) = Q_{\mathbf{B}}$-**RTop**$(R, \|\mathbb{C}\|) \xrightarrow{(f_{\mathbb{C}}^{\leftarrow})^{op}} Q_{\mathbf{B}}$-**RTop**$(S, \|\mathbb{C}\|)$.

Proof. It will be enough to check the correctness of E on both objects and morphisms. For the first claim, we show that $Q_{\mathbf{B}}$-**RTop**$(R, \|\mathbb{C}\|)$ is a subalgebra of $\mathbb{C}^{|R|}$. Fix $\lambda \in \Lambda_{\mathbf{C}}$ and $\alpha_i \in Q_{\mathbf{B}}$-**RTop**$(R, \|\mathbb{C}\|)$ for $i \in n_\lambda$. Given

$\upsilon \in \Upsilon_{\mathbf{R}}$ and $\langle r_j\rangle_{m_\upsilon} \in \varpi_\upsilon^R$, $\langle \alpha_i(r_j)\rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{C}\|}$ for every $i \in n_\lambda$ and therefore $\langle(\omega_\lambda^{\mathbb{C}^{|R|}}(\langle\alpha_i\rangle_{n_\lambda}))(r_j)\rangle_{m_\upsilon} = \langle\omega_\lambda^{\mathbb{C}}(\langle\alpha_i(r_j)\rangle_{n_\lambda})\rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{C}\|}$, yielding $\omega_\lambda^{\mathbb{C}^{|R|}}(\langle\alpha_i\rangle_{n_\lambda}) \in \mathbf{R}(R, \|\mathbb{C}\|)$. To show that the map is also $Q_{\mathbf{B}}$-continuous, notice that the existence of products of $Q_{\mathbf{B}}$-spaces (Corollary 13) induces a $Q_{\mathbf{B}}$-continuous map $R \xrightarrow{\alpha} |\mathbb{C}|^{n_\lambda}$ making the diagram



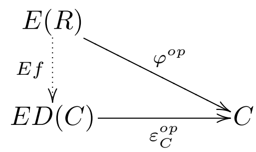
commute for every $i \in n_\lambda$. The desired $Q_{\mathbf{B}}$-continuity follows then from the facts that $\omega_\lambda^{\mathbb{C}^{|R|}}(\langle\alpha_i\rangle_{n_\lambda}) = \omega_\lambda^{\mathbb{C}} \circ \alpha$ and $\omega_\lambda^{\mathbb{C}}$ is $Q_{\mathbf{B}}$-continuous.

To show correctness of E on morphisms, notice that given some $\alpha \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(S, \|\mathbb{C}\|)$, $f_{\mathbb{C}}^{\leftarrow}(\alpha) = \alpha \circ f \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R, \|\mathbb{C}\|)$ since both α and f are r-$Q_{\mathbf{B}}$-morphisms. The rest follows from the fact that the assignment $(-)^{\leftarrow}$ at the beginning of Example 4 defines a functor, or, more particularly, given $\lambda \in \Lambda_{\mathbf{C}}$ and $\alpha_i \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(S, \|\mathbb{C}\|)$ for $i \in n_\lambda$, $f_{\mathbb{C}}^{\leftarrow}(\omega_\lambda^{\mathbb{C}^{|S|}}(\langle\alpha_i\rangle_{n_\lambda})) = \omega_\lambda^{\mathbb{C}^{|S|}}(\langle\alpha_i\rangle_{n_\lambda}) \circ f = \omega_\lambda^{\mathbb{C}^{|R|}}(\langle\alpha_i \circ f\rangle_{n_\lambda}) = \omega_\lambda^{\mathbb{C}^{|R|}}(\langle f_{\mathbb{C}}^{\leftarrow}(\alpha_i)\rangle_{n_\lambda})$. □

It is important to notice that algebraicity of **R** w.r.t. **C** was exploited on $\mathbb{C}$ only.

Theorem 27. *If* $(\mathcal{R})$, $(\mathcal{C})$ *hold, then* $Q_{\mathbf{B}}\text{-}\mathbf{RTop} \xrightarrow{E} \mathbf{LoC}$ *has a right adjoint.*

Proof. We show that every localic algebra C has a E-co-universal arrow, i.e., a localic homomorphism $ED(C) \xrightarrow{\varepsilon_C^{op}} C$ such that every localic homomorphism $E(R) \xrightarrow{\varphi^{op}} C$ has a unique r-$Q_{\mathbf{B}}$-morphism $R \xrightarrow{f} D(C)$ making the diagram



commute.

Let the underlying set of $D(C)$ be $\mathbf{C}(C, \mathbb{C})$. Given $\upsilon \in \Upsilon_{\mathbf{R}}$ and $\langle\varphi_j\rangle_{m_\upsilon} \in (\mathbf{C}(C, \mathbb{C}))^{m_\upsilon}$ let $\langle\varphi_j\rangle_{m_\upsilon} \in \varpi_\upsilon^{\mathbf{C}(C,\mathbb{C})}$ iff $\langle\varphi_j(c)\rangle_{m_\upsilon} \in \varpi_\upsilon^{\|C\|}$ for every $c \in C$ (pointwise relational structure induced on $\mathbf{C}(C, \mathbb{C})$ by the product $\|\mathbb{C}\|^{|C|}$). Given $c \in C$ and $\alpha \in \boldsymbol{\delta}$, define $\mathbf{C}(C, \mathbb{C}) \xrightarrow{t_{c\alpha}} Q$ by $t_{c\alpha}(\varphi) = \alpha \circ \varphi(c) = ev_c((\varphi_Q^{\leftarrow})(\alpha))$ and set $\tau = \langle\{t_{c\alpha} \,|\, c \in C,\ \alpha \in \boldsymbol{\delta}\}\rangle$. It follows that $D(C)$ is an r-$Q_{\mathbf{B}}$-space (notice the use of closure of r-varieties under products and subobjects). The desired map $C \xrightarrow{\varepsilon_C} (ED(C) = Q_{\mathbf{B}}\text{-}\mathbf{RTop}(\mathbf{C}(C, \mathbb{C}), \|\mathbb{C}\|))$ is now given by $\varepsilon_C(c) = ev_c$.

Two points are the subject to verification at once. Firstly, $\varepsilon_C(c)$ should be in $ED(C)$ for every $c \in C$. Given $\upsilon \in \Upsilon_{\mathbf{R}}$ and $\langle \varphi_j \rangle_{m_\upsilon} \in \varpi_\upsilon^{D(C)}$, $\langle (\varepsilon_C(c))(\varphi_j) \rangle_{m_\upsilon} = \langle \varphi_j(c) \rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{C}\|}$ and therefore $\varepsilon_C(c)$ is an r-homomorphism. To show $Q_{\mathbf{B}}$-continuity, notice that given $\alpha \in \boldsymbol{\delta}$, $((\varepsilon_C(c))_Q^{\leftarrow}(\alpha))(\varphi) = \alpha \circ \varphi(c) = t_{c\alpha}(\varphi)$ for every $\varphi \in \mathbf{C}(C, \mathbb{C})$, yields $(\varepsilon_C(c))_Q^{\leftarrow}(\alpha) = t_{c\alpha} \in \tau$ and use Corollary 11. Secondly, ε_C should be a homomorphism. Given $\lambda \in \Lambda_{\mathbf{C}}$ and $c_i \in C$ for $i \in n_\lambda$, $(\varepsilon_C(\omega_\lambda^C(\langle c_i \rangle_{n_\lambda})))(\varphi) = \varphi(\omega_\lambda^C(\langle c_i \rangle_{n_\lambda})) = \omega_\lambda^{\mathbb{C}}(\langle \varphi(c_i) \rangle_{n_\lambda}) = \omega_\lambda^{\mathbb{C}}(\langle (\varepsilon_C(c_i))(\varphi) \rangle_{n_\lambda}) = (\omega_\lambda^{ED(C)}(\langle \varepsilon_C(c_i) \rangle_{n_\lambda}))(\varphi)$ for every $\varphi \in \mathbf{C}(C, \mathbb{C})$.

It remains to show that ε_C^{op} has the properties of an E-co-universal arrow. Given a localic homomorphism $E(R) \xrightarrow{\varphi^{op}} C$, define $R \xrightarrow{f} D(C)$ by $(f(r))(c) = (\varphi(c))(r)$. To check that $f(r)$ is a homomorphism, notice that given $\lambda \in \Lambda_{\mathbf{C}}$ and $c_i \in C$ for $i \in n_\lambda$, $f(r)(\omega_\lambda^C(\langle c_i \rangle_{n_\lambda})) = (\varphi(\omega_\lambda^C(\langle c_i \rangle_{n_\lambda})))(r) = (\omega_\lambda^{\mathbb{C}^{|R|}}(\langle \varphi(c_i) \rangle_{n_\lambda}))(r) = \omega_\lambda^{\mathbb{C}}(\langle (\varphi(c_i))(r) \rangle_{n_\lambda}) = \omega_\lambda^{\mathbb{C}}(\langle (f(r))(c_i) \rangle_{n_\lambda})$. To verify that f is an r-homomorphism, notice that given $\upsilon \in \Upsilon_{\mathbf{R}}$ and $\langle r_j \rangle_{m_\upsilon} \in \varpi_\upsilon^R$, $\langle (f(r_j))(c) \rangle_{m_\upsilon} = \langle (\varphi(c))(r_j) \rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{C}\|}$ for every $c \in C$, yields $\langle f(r_j) \rangle_{n_\lambda} \in \varpi_\upsilon^{D(C)}$. To show $Q_{\mathbf{B}}$-continuity, take any $t_{c\alpha} \in \tau$ and get $(f_Q^{\leftarrow}(t_{c\alpha}))(r) = (\alpha \circ f(r))(c) = (\alpha \circ \varphi(c))(r) = ((\varphi(c))_Q^{\leftarrow}(\alpha))(r)$ for every $r \in R$ and therefore $f_Q^{\leftarrow}(t_{c\alpha}) = (\varphi(c))_Q^{\leftarrow}(\alpha) \in \tau_R$. It is time to use Corollary 11 again.

Equality $\varepsilon_C^{op} \circ Ef = \varphi^{op}$ comes the fact that for $c \in C$, $((Ef)^{op} \circ \varepsilon_C(c))(r) = (\varepsilon_C(c) \circ f)(r) = (f(r))(c) = (\varphi(c))(r)$ for every $r \in R$. Suppose $R \xrightarrow{g} D(C)$ is another r-$Q_{\mathbf{B}}$-morphism with $\varepsilon_C^{op} \circ Eg = \varphi^{op}$. Given $r \in R$ and $c \in C$, $(g(r))(c) = (\varepsilon_C(c))(g(r)) = (g_{\mathbb{C}}^{\leftarrow}(\varepsilon_C(c)))(r) = (((Eg)^{op} \circ \varepsilon_C)(c))(r) = (\varphi(c))(r) = (f(r))(c)$. □

Corollary 28. *If* $(\mathcal{R})$, $(\mathcal{C})$ *hold, then there exists an adjoint situation* $(\eta, \varepsilon) : E \dashv D : \mathbf{LoC} \to Q_{\mathbf{B}}$*-*$\mathbf{RTop}$.

Proof. We use the standard scheme of obtaining an adjunction from the existence of co-universal arrows [1]. Given a localic homomorphism $C_1 \xrightarrow{\varphi^{op}} C_2$, $D(C_1 \xrightarrow{\varphi^{op}} C_2) = D(C_1) \xrightarrow{D\varphi^{op}} D(C_2)$, with $D\varphi^{op}$ defined by commutativity of the diagram

$$\begin{array}{ccc} ED(C_1) & \xrightarrow{\varepsilon_{C_1}^{op}} & C_1 \\ {\scriptstyle ED\varphi^{op}} \downarrow & & \downarrow {\scriptstyle \varphi^{op}} \\ ED(C_2) & \xrightarrow[\varepsilon_{C_2}^{op}]{} & C_2 \end{array}$$

and therefore $D\varphi^{op} = \varphi_{\mathbb{C}}^{\leftarrow}$. Given an r-$Q_{\mathbf{B}}$-space R, $R \xrightarrow{\eta_R} (DE(R) = \mathbf{C}(Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R, \|\mathbb{C}\|), \mathbb{C}))$ is defined by commutativity of the diagram

$$\begin{array}{ccc} E(R) & & \\ {\scriptstyle E\eta_R}\,\vdots & \searrow^{1_{E(R)}} & \\ EDE(R) & \xrightarrow[\varepsilon^{op}_{E(R)}]{} & E(R) \end{array}$$

and therefore $(\eta_R(r))(f) = f(r)$. □

It is worthwhile to underline once more that the action on morphisms of the obtained adjoint situation is based on the functor $\mathbf{Set} \times \mathbf{S}_{\mathbb{C}} \xrightarrow{(-)^{\leftarrow}_{\mathbb{C}}} \mathbf{LoC}$, which in general is different from the underlying cvt-theory $\mathbf{Set} \times \mathbf{S}_Q \xrightarrow{\|-\| \circ (-)^{\leftarrow}_{Q}} \mathbf{LoB}$ of the category $Q_{\mathbf{B}}$-$\mathbf{Top}$.

3.2. **Catalg sobriety and spatiality.** Having succeeded in the construction of the desired adjunction (or a *preduality* in terms of [15]), we proceed to the last stage of our plan, i.e., to singling out particular subcategories of $Q_{\mathbf{B}}$-$\mathbf{RTop}$ (resp. $\mathbf{LoC}$) such that the restriction to them of the adjunction produces an equivalence. Simple as it looks, the task has the drawback of the (potential) multitude of the possible solutions. There is, however, a "maximal" equivalence between a pair of full subcategories induced by an adjunction [50] and that will be our choice. For convenience of the reader, we start with the necessary categorical preliminaries.

Lemma 29. *Let $(\eta, \varepsilon) : F \dashv G : \mathbf{A} \to \mathbf{X}$ be an adjunction and let $\bar{\mathbf{A}}$ (resp. $\bar{\mathbf{X}}$) be the full subcategory of $\mathbf{A}$ (resp. $\mathbf{X}$) of those objects A (resp. X) for which $FG(A) \xrightarrow{\varepsilon_A} A$ (resp. $X \xrightarrow{\eta_X} GF(X)$) is an isomorphism in $\mathbf{A}$ (resp. $\mathbf{X}$).*

(1) There exists the restriction $(\bar{\eta}, \bar{\varepsilon}) : \bar{F} \dashv \bar{G} : \bar{\mathbf{A}} \to \bar{\mathbf{X}}$ which is an equivalence, maximal in the sense that every other equivalence $(\bar{\bar{\eta}}, \bar{\bar{\varepsilon}}) : \bar{\bar{F}} \dashv \bar{\bar{G}} : \bar{\bar{\mathbf{A}}} \to \bar{\bar{\mathbf{X}}}$ provides subcategories $\bar{\bar{\mathbf{A}}}$ (resp. $\bar{\bar{\mathbf{X}}}$) of $\bar{\mathbf{A}}$ (resp. $\bar{\mathbf{X}}$).

(2) An $\mathbf{A}$-object A is in $\bar{\mathbf{A}}$ iff $A \cong F(X)$ for some $\mathbf{X}$-object X such that η_X is an $\mathbf{X}$-epimorphism.

(3) An $\mathbf{X}$-object X is in $\bar{\mathbf{X}}$ iff $X \cong G(A)$ for some $\mathbf{A}$-object A such that ε_A is an $\mathbf{A}$-monomorphism.

(4) Let $\mathbf{X}_e$ be the full subcategory of $\mathbf{X}$ of all objects X such that η_X is an $\mathbf{X}$-epimorphism. The full embedding $\bar{\mathbf{X}} \xhookrightarrow{M_{\bar{\mathbf{X}}}} \mathbf{X}_e$ has a left adjoint $\mathbf{X}_e \xrightarrow{GF} \bar{\mathbf{X}}$.

(5) Let $\mathbf{A}_m$ be the full subcategory of $\mathbf{A}$ of all objects A such that ε_A is an $\mathbf{A}$-monomorphism. The full embedding $\bar{\mathbf{A}} \xhookrightarrow{M_{\bar{\mathbf{A}}}} \mathbf{A}_m$ has a right adjoint $\mathbf{A}_m \xrightarrow{FG} \bar{\mathbf{A}}$.

Proof. *Ad* (1). Is is enough to show the existence of the restrictions $\bar{\mathbf{A}} \xrightarrow{\bar{G}} \bar{\mathbf{X}}$ and $\bar{\mathbf{X}} \xrightarrow{\bar{F}} \bar{\mathbf{A}}$. Given $A \in \mathcal{O}b(\bar{\mathbf{A}})$, commutativity of the diagram

$$\begin{array}{ccc} G(A) & \xrightarrow{\eta_{G(A)}} & GFG(A) \\ & \searrow^{1_{G(A)}} & \downarrow{\scriptstyle G\varepsilon_A} \\ & & G(A) \end{array}$$

and the assumption on A yield, $\eta_{G(A)}$ is an $\mathbf{X}$-isomorphism. The case of $\bar{F}$ is similar.

Ad (2). For the necessity notice that $A \in \mathcal{O}b(\bar{\mathbf{A}})$ implies $FG(A) \xrightarrow{\varepsilon_A} A$ is an isomorphism and $G(A) \in \mathcal{O}b(\bar{\mathbf{X}})$ by Item (1). It follows that $\eta_{G(A)}$ is an isomorphism and therefore an epimorphism. For the sufficiency let $A \xrightarrow{\varphi} F(X)$ be the isomorphism in question. On the first step, we show that $\varepsilon_{F(X)}$ is an isomorphism. The first part of the result follows from commutativity of the diagram

$$\begin{array}{ccc} F(X) & & \\ {\scriptstyle F\eta_X}\downarrow & \searrow^{1_{F(X)}} & \\ FGF(X) & \xrightarrow[\varepsilon_{F(X)}]{} & F(X) \end{array}$$

Moreover, it implies $F\eta_X \circ \varepsilon_{F(X)} \circ F\eta_X = F\eta_X \circ 1_{F(X)} = 1_{FGF(X)} \circ F\eta_X$. Since left adjoint functors preserve epimorphisms, $F\eta_X$ is an epimorphism and therefore $F\eta_X \circ \varepsilon_{F(X)} = 1_{FGF(X)}$, yielding the desired result. Since ε is a natural transformation, the diagram

$$\begin{array}{ccc} FG(A) & \xrightarrow{\varepsilon_A} & A \\ {\scriptstyle FG\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ FGF(X) & \xrightarrow[\varepsilon_{F(X)}]{} & F(X) \end{array}$$

commutes and therefore $\varepsilon_A = \varphi^{-1} \circ \varepsilon_{F(X)} \circ FG\varphi$, the morphism on the right being an isomorphism.

Ad (3). Dual to *Ad* (2).

Ad (4). Given an $\mathbf{X}_e$-object X, $F(X)$ is in $\bar{\mathbf{A}}$ by Item (2) and therefore $GF(X)$ is in $\bar{\mathbf{X}}$ by Item (1). It follows that $X \xrightarrow{\eta_X} E_{\bar{\mathbf{X}}}GF(X)$ is an $M_{\bar{\mathbf{X}}}$-universal arrow for X.

Ad (5). Dual to *Ad* (4). □

Applying Lemma 29 to the adjunction of Corollary 28, we get the category $\overline{\mathbf{LoC}}$ (resp. $\overline{Q_{\mathbf{B}}\text{-}\mathbf{RTop}}$) and the desired equivalence seems to be in hand. It appears, however, that in the current setting, the categories in question have a more explicit description, motivated by the respective one of P. T. Johnstone [39]. We begin with the case of the category $\overline{Q_{\mathbf{B}}\text{-}\mathbf{RTop}}$, which requires an additional definition.

Definition 30. An r-$Q_{\mathbf{B}}$-space (R,τ) is called

- $r_{\mathbb{C}}$-$Q_{\mathbf{B}}$-T_0 provided that
  (1) every distinct $r,s\in R$ have an r-$Q_{\mathbf{B}}$-morphism $R\xrightarrow{f}\|\mathbb{C}\|$ such that $f(r)\neq f(s)$;
  (2) given $\upsilon\in\Upsilon_{\mathbf{R}}$ and $\langle r_j\rangle_{m_\upsilon}\in R^{m_\upsilon}$, if $\langle f(r_j)\rangle_{m_\upsilon}\in\varpi_\upsilon^{\|\mathbb{C}\|}$ for every r-$Q_{\mathbf{B}}$-morphism $R\xrightarrow{f}\|\mathbb{C}\|$, then $\langle r_j\rangle_{m_\upsilon}\in\varpi_\upsilon^{R}$.
- $r_{\mathbb{C}}$-$Q_{\mathbf{B}}$-S_0 provided that
  (1) every homomorphism $Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R,\|\mathbb{C}\|)\xrightarrow{\varphi}\mathbb{C}$ has some $r\in R$ such that $\varphi(f)=f(r)$ for every r-$Q_{\mathbf{B}}$-morphism $R\xrightarrow{f}\|\mathbb{C}\|$;
  (2) $\tau=\langle\{\alpha\circ f\,|\,f\in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R,\|\mathbb{C}\|),\ \alpha\in\boldsymbol{\delta}\}\rangle$.
- $r_{\mathbb{C}}$-$Q_{\mathbf{B}}$-*sober* provided that it is both r-$Q_{\mathbf{B}}$-T_0 and r-$Q_{\mathbf{B}}$-S_0.

The first and the last items of Definition 30 were suggested by the classical topological notions of T_0 *separation axiom* (every two distinct points have an open set containing only one of them) and *sobriety* (every irreducible closed subset is the closure of a unique point), whereas the middle one is a modified version of the respective notion of [57, Definition 5.3]. The intuition for the new concepts is given by the following example from Priestley duality [19].

Example 31. A **PoTop**-space $(X,\leqslant,\tau)$ is called *totally order-disconnected* provided that for every $x,y\in X$ such that $x\not\leqslant y$, there exists a *clopen* (closed and open) *up-set* $U\subseteq X$ ($z\in U$ and $z\leqslant w$ yield $w\in U$) such that $x\in U$ and $y\notin U$. Given the lattice $\mathbf{2}=\{\bot,\top\}$ equipped with the discrete topology, a **PoTop**-space X is $r_{\mathbf{2}}$-T_0 iff X is totally order-disconnected (**PoTop**-morphisms $X\xrightarrow{f}\mathbf{2}$ are in one-to-one correspondence with clopen up-sets $U\subseteq X$).

Lemma 32. *An r-$Q_{\mathbf{B}}$-space R is $r_{\mathbb{C}}$-$Q_{\mathbf{B}}$-sober iff η_R is an isomorphism.*

Proof. For the necessity, we show that η_R is bijective and its inverse η_R^{-1} is an r-$Q_{\mathbf{B}}$-morphism. Since R is $\mathrm{r}_{\mathbb{C}}$-$Q_{\mathbf{B}}$-T_0, Item (1) and the definition of η_R in Corollary 28 imply its injectivity. Similarly, $\mathrm{r}_{\mathbb{C}}$-$Q_{\mathbf{B}}$-S_0 implies surjectivity and therefore η_R is bijective. To show that η_R^{-1} is an r-homomorphism, fix $\upsilon\in\Upsilon_{\mathbf{R}}$ and $\langle\varphi_j\rangle_{m_\upsilon}\in\varpi_\upsilon^{DE(R)}$. By the bijectivity of η_R, $\varphi_j=\eta_R(r_j)$ for every $j\in m_\upsilon$. Given an r-$Q_{\mathbf{B}}$-morphism $R\xrightarrow{f}\|\mathbb{C}\|$, $\varpi_\upsilon^{\|\mathbb{C}\|}\ni\langle\varphi_j(f)\rangle_{m_\upsilon}=\langle(\eta_R(r_j))(f)\rangle_{m_\upsilon}=\langle f(r_j)\rangle_{m_\upsilon}$ and therefore $\langle\eta_R^{-1}(\varphi_j)\rangle_{m_\upsilon}=\langle r_j\rangle_{m_\upsilon}\in\varpi_\upsilon^{R}$ by Item (2) of $\mathrm{r}_{\mathbb{C}}$-$Q_{\mathbf{B}}$-T_0. To show $Q_{\mathbf{B}}$-continuity of η_R^{-1}, notice that given $f\in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R,\|\mathbb{C}\|)$ and $\alpha\in\boldsymbol{\delta}$, $((\eta_R^{-1})^{\leftarrow}_{Q}(\alpha\circ f))(\eta_R(r))=\alpha\circ f(r)=\alpha\circ(\eta_R(r))(f)=t_{f\alpha}(\eta_R(r))$ for every

$r \in R$ and therefore $(\eta_R^{-1})_Q^{\leftarrow}(\alpha \circ f) = t_{f\alpha} \in \tau_{DE(R)}$. Corollary 11 yields the desired result.

For the sufficiency, notice that bijectivity of η_R implies Item (1) of both $\mathrm{r}_{\mathbb{C}}$-$Q_{\mathbf{B}}$-T_0 and $\mathrm{r}_{\mathbb{C}}$-$Q_{\mathbf{B}}$-S_0. To show Item (2) of $\mathrm{r}_{\mathbb{C}}$-$Q_{\mathbf{B}}$-T_0, notice that given $\upsilon \in \Upsilon_{\mathbf{R}}$ and $\langle r_j \rangle_{m_\upsilon} \in R^{m_\upsilon}$ such that $\langle f(r_j) \rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{C}\|}$ for every r-$Q_{\mathbf{B}}$-morphism $R \xrightarrow{f} \|\mathbb{C}\|$, $\langle (\eta_R(r_j))(f) \rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{C}\|}$ for every $f \in E(R)$, and therefore $\langle \eta_R(r_j) \rangle_{m_\upsilon} \in \varpi_\upsilon^{DE(R)}$. It follows that $\langle r_j \rangle_{m_\upsilon} = \langle \eta_R^{-1} \circ \eta_R(r_j) \rangle_{m_\upsilon} \in \varpi_\upsilon^R$ since η_R^{-1} is an r-homomorphism. To show Item (2) of $\mathrm{r}_{\mathbb{C}}$-$Q_{\mathbf{B}}$-S_0, notice that the inclusion "$\supseteq$" follows from the fact that every f in question is $Q_{\mathbf{B}}$-continuous. Notice as well that given $f \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R, \|\mathbb{C}\|)$ and $\alpha \in \boldsymbol{\delta}$, $((\eta_R)_Q^{\leftarrow}(t_{f\alpha}))(r) = \alpha \circ (\eta_R(r))(f) = \alpha \circ f(r)$ for every $r \in R$ and therefore $\alpha \circ f = (\eta_R)_Q^{\leftarrow}(t_{f\alpha})$. On the other hand, $((\eta_R^{-1})_Q^{\leftarrow})^{\rightarrow}(\tau_R) \subseteq \tau_{DE(R)}$ by $Q_{\mathbf{B}}$-continuity of η_R^{-1} and therefore $\tau_R = ((1_R)_Q^{\leftarrow})^{\rightarrow}(\tau_R) = ((\eta_R^{-1} \circ \eta_R)_Q^{\leftarrow})^{\rightarrow}(\tau_R) = ((\eta_R)_Q^{\leftarrow} \circ (\eta_R^{-1})_Q^{\leftarrow})^{\rightarrow}(\tau_R) = ((\eta_R)_Q^{\leftarrow})^{\rightarrow} \circ ((\eta_R^{-1})_Q^{\leftarrow})^{\rightarrow}(\tau_R) \subseteq ((\eta_R)_Q^{\leftarrow})^{\rightarrow}(\tau_{DE(R)}) = ((\eta_R)_Q^{\leftarrow})^{\rightarrow}(\langle\{t_{f\alpha} \,|\, f \in E(R), \alpha \in \boldsymbol{\delta}\}\rangle) = \langle\{(\eta_R)_Q^{\leftarrow}(t_{f\alpha}) \,|\, f \in E(R),\ \alpha \in \boldsymbol{\delta}\}\rangle = \langle\{\alpha \circ f \,|\, f \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R, \|\mathbb{C}\|),\ \alpha \in \boldsymbol{\delta}\}\rangle$ using functorial properties of the image (resp. preimage) operators $(-)^{\rightarrow}$ (resp. $(-)_Q^{\leftarrow}$) and Item (2) of Lemma 10. □

Corollary 33. $\overline{Q_{\mathbf{B}}\text{-}\mathbf{RTop}}$ *is the full subcategory* $Q_{\mathbf{B}}$-$\mathbb{C}$**RSob** *of* $Q_{\mathbf{B}}$-**RTop** *comprising precisely the* $\mathrm{r}_{\mathbb{C}}$-$Q_{\mathbf{B}}$*-sober r-*$Q_{\mathbf{B}}$*-spaces.*

Having characterized the category $\overline{Q_{\mathbf{B}}\text{-}\mathbf{RTop}}$, we do the same job for $\overline{\mathbf{LoC}}$.

Definition 34. A **LoC**-object C is called $\mathrm{r}_{\mathbb{C}}$-$Q_{\mathbf{B}}$*-spatial* provided that

(1) every distinct $c, d \in C$ have some homomorphism $C \xrightarrow{\varphi} \mathbb{C}$ such that $\varphi(c) \neq \varphi(d)$;

(2) every r-$Q_{\mathbf{B}}$-morphism $\mathbf{C}(C, \mathbb{C}) \xrightarrow{f} \|\mathbb{C}\|$ has some $c \in C$ such that $f(\varphi) = \varphi(c)$ for every homomorphism $C \xrightarrow{\varphi} \mathbb{C}$.

Lemma 35. *A* **LoC***-object* C $\mathrm{r}_{\mathbb{C}}$-$Q_{\mathbf{B}}$*-spatial iff* ε_C *is an isomorphism.*

Proof. The result follows from the definition of ε_C in Theorem 27 and the fact that bijective homomorphisms are isomorphisms. □

Corollary 36. $\overline{\mathbf{LoC}}$ *is the full subcategory* $Q_{\mathbf{B}}$-$\mathbb{C}$**RSpat** *of* **LoC** *comprising precisely the* $\mathrm{r}_{\mathbb{C}}$-$Q_{\mathbf{B}}$*-spatial localic algebras.*

Corollaries 28, 33, 36 and Lemma 29 imply the main result of the section, which provides a generalization of the respective one for the Stone representation theories.

Theorem 37. *Suppose* $(\mathcal{R})$*,* $(\mathcal{C})$ *hold.*

(1) There exists an equivalence $(\bar{\eta}, \bar{\varepsilon}) : \bar{E} \dashv \bar{D} : Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSpat} \to Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSob}$.

(2) The full embedding $Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSob} \xhookrightarrow{M_{Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSob}}} Q_{\mathbf{B}}\text{-}\mathbf{RTop}_e$ *has a left adjoint* $Q_{\mathbf{B}}\text{-}\mathbf{RTop}_e \xrightarrow{DE} Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSob}$.

(3) The full embedding $Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSpat} \xhookrightarrow{M_{Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSpat}}} \mathbf{LoC}_m$ *has a right adjoint* $\mathbf{LoC}_m \xrightarrow{ED} Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSpat}$.

By analogy with the case of the Stone representation theories considered by P. T. Johnstone [39], $Q_{\mathbf{B}}\text{-}\mathbf{RTop}_e \xrightarrow{DE} Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSob}$ is called the *$r_{\mathbb{C}}$-$Q_{\mathbf{B}}$-soberification functor.*

3.3. **Composite setting.** The reader should be aware of the fact that there exists no direct generalization of Theorem 37 for the framework of composite topology, providing an equivalence involving a subcategory of some product category $\prod_{i \in I} \mathbf{LoC}_i$. To show the sticking point, we provide a possible approach to such a modification.

Start with the category $\mathbf{CTop}((\mathcal{S}_{\mathbf{A}_i}^{\mathbf{S}_{Q_i}}, \mathbf{B}_i)_{i \in I})$, for the sake of convenience (as well as to fit the just considered framework) denoted by $(Q_{i\mathbf{B}_i})_I\text{-}\mathbf{CTop}$. Relational enrichment requires then a more rigid formulation, where a common underlying set for a family of relational structures should be stated explicitly.

Definition 38. Given a family of r-varieties $(\mathbf{R}_i)_{i \in I}$, $(Q_{i\mathbf{B}_i})_I\text{-}\mathbf{CRTop}$ is the category, whose objects (called *r-$(Q_{i\mathbf{B}_i})_I$-spaces*) are triples $(X, (R_i)_{i \in I}, (\tau_i)_{i \in I})$, where $(R_i)_{i \in I}$ is a $\prod_{i \in I} \mathbf{R}_i$-object and $(X, (\tau_i)_{i \in I})$ is a $(Q_{i\mathbf{B}_i})_I$-space with $|R_i| = X$ for every $i \in I$, and whose morphisms $(X, (R_i)_{i \in I}, (\tau_i)_{i \in I}) \xrightarrow{f} (Y, (S_i)_{i \in I}, (\sigma_i)_{i \in I})$ are $(Q_{i\mathbf{B}_i})_I$-continuous maps $(X, (\tau_i)_{i \in I}) \xrightarrow{f} (Y, (\sigma_i)_{i \in I})$ such that $(R_i)_{i \in I} \xrightarrow{(f)_{i \in I}} (S_i)_{i \in I}$ is a $\prod_{i \in I} \mathbf{R}_i$-morphism (called *r-$(Q_{i\mathbf{B}_i})_I$-morphisms*). The underlying functor to the ground category $(Q_{i\mathbf{B}_i})_I\text{-}\mathbf{CTop}$ is defined by $|(X, (R_i)_{i \in I}, (\tau_i)_{i \in I}) \xrightarrow{f} (Y, (S_i)_{i \in I}, (\sigma_i)_{i \in I})| = (X, (\tau_i)_{i \in I}) \xrightarrow{f} (Y, (\sigma_i)_{i \in I})$.

On the next step, we fix a family $(\mathbf{C}_i)_{i \in I}$ of varieties and the respective set of schizophrenic objects $(\mathbb{C}_i)_{i \in I}$, each equipped with a topology $\boldsymbol{\delta}_i$. Moreover, we stipulate an additional coherence condition, namely, the existence of a set X such that $|\mathbb{C}_i| = X$ for every $i \in I$. Requirement $(\mathfrak{R})$ is converted to the family $((\mathfrak{R}_i))_{i \in I}$. On the other hand, requirement $(\mathfrak{C})$ uses the following modification of Definition 21.

Definition 39. A *$(Q_{i\mathbf{B}_i})_I$-continuous family of algebras* is a triple $(X, (D_i)_{i \in I}, (\tau_i)_{i \in I})$, where $(X, (\tau_i)_{i \in I})$ is a $(Q_{i\mathbf{B}_i})_I$-space and for every $i \in I$, (D_i, τ_i) is a $Q_{i\mathbf{B}_i}$-continuous algebra (in the sense of Definition 21) of some variety $\mathbf{D}_i$ such that $|D_i| = X$.

Modification of Lemma 26 in the new setting is straightforward (the explicit details are left to the reader).

Lemma 40. *If* $((\mathcal{R}_i))_{i\in I}$, $(\mathcal{C})$ *hold, then there exists a functor* $(Q_{i\mathbf{B}_i})_I$-**CRTop** $\xrightarrow{E_I} \prod_{i\in I}\mathbf{LoC}_i$ *given by*

$$E_I((X,(R_i)_{i\in I},(\tau_i)_{i\in I}) \xrightarrow{f} (Y,(S_i)_{i\in I},(\sigma_i)_{i\in I})) =$$
$$(Q_{i\mathbf{B}_i}\text{-}\mathbf{RTop}(R_i,\|\mathbb{C}_i\|))_{i\in I} \xrightarrow{((f^{\leftarrow}_{\mathbb{C}_i})^{op})_{i\in I}} (Q_{i\mathbf{B}_i}\text{-}\mathbf{RTop}(S_i,\|\mathbb{C}_i\|))_{i\in I}.$$

It is the result of Theorem 27 that causes the main problem. An attentive reader will recall that the right adjoint to E was based on the hom-set $\mathbf{C}(C,\mathbb{C})$ for a given localic algebra C. Our framework will translate the single hom-set into a family $(\mathbf{C}_i(C_i,\mathbb{C}_i))_{i\in I}$, which should produce an r-$(Q_{i\mathbf{B}_i})_I$-space. The sticking point is the requirement of Definition 38 on the common underlying set of the elements of the family obtained. The question on whether E_I has a right adjoint for I having more than one element is still open.

4. Beyond the framework

We have already noticed in Introduction that it is not the topological duality itself, but its consequences that constitute its real worth. In particular, there are numerous procedures for obtaining new representations from the already existing ones. In the following, we provide a catalg foundations for some of them.

4.1. **Representations induced by subcategories.** The reader is probably aware that the classical Stone representation theorems are consequences of the equivalence **Sob** $\sim$ **Spat** for the variety **Frm** of frames [39]. The variety **BDLat** of bounded distributive lattices is dually equivalent to the subcategory of **Spat** comprising *coherent locales*, the image of which under the equivalence in question is the category of *coherent spaces*, providing the famous Stone representation theorem for distributive lattices. Since Boolean algebras constitute a subcategory **Bool** of **BDLat**, one obtains the second Stone representation theorem, singling out a particular subcategory of coherent spaces consisting of the *Stone* (compact, Hausdorff, totally disconnected) ones. The following shows catalg foundations for the procedure, yielding a simple (but extremely useful) machinery for obtaining new dualities from old. For the sake of flexibility, we take a rather general standpoint from the beginning of Section 3.2.

Definition 41. A subcategory **S** of a category **X** is said to be *strongly isomorphism-closed* in **X** provided that given an **S**-morphism $S_1 \xrightarrow{f} S_2$ and two **X**-isomorphisms $X_1 \xrightarrow{g} S_1$ and $S_2 \xrightarrow{h} X_2$, $X_1 \xrightarrow{h\circ f\circ g} X_2$ is an **S**-morphism.

Notice that given a category, every its full isomorphism-closed [1] subcategory is strongly isomorphism-closed, but not vise versa (the fullness fails).

Definition 42. Given a functor $\mathbf{X} \xrightarrow{F} \mathbf{Y}$ and a subcategory $\mathbf{S}$ of $\mathbf{Y}$, $F^{\leftarrow}(\mathbf{S})$ is the subcategory of $\mathbf{X}$ of all morphisms f such that Ff is an $\mathbf{S}$-morphism.

Lemma 43. *Let* $(\bar{\eta}, \bar{\varepsilon}) : \bar{F} \dashv \bar{G} : \bar{\mathbf{A}} \to \bar{\mathbf{X}}$ *be an equivalence and let* $\mathfrak{A}$ *be a strongly isomorphism-closed subcategory of* $\bar{\mathbf{A}}$*. If* $\mathfrak{X} = \bar{F}^{\leftarrow}(\mathfrak{A})$*, then there exists the restriction* $\mathfrak{A} \xrightarrow{\bar{G}} \mathfrak{X}$*, providing the equivalence* $(\bar{\bar{\eta}}, \bar{\bar{\varepsilon}}) : \bar{\bar{F}} \dashv \bar{\bar{G}} : \mathfrak{A} \to \mathfrak{X}$*.*

Proof. Given $A \xrightarrow{\varphi} B$ in $\mathfrak{A}$, $\bar{F}\bar{G}\varphi = \bar{\varepsilon}_B^{-1} \circ \varphi \circ \bar{\varepsilon}_A$ yields $\bar{F}\bar{G}\varphi$ is an $\mathfrak{A}$-morphism and thus, $\bar{G}\varphi$ lies in $\mathfrak{X}$. □

Notice that the proposed machinery can be reversed, in the sense that given a strongly isomorphism-closed subcategory $\mathfrak{X}$ of $\bar{\mathbf{X}}$, one obtains the category $\mathfrak{A} = \bar{G}^{\leftarrow}(\mathfrak{X})$ and the equivalence $\mathfrak{X} \sim \mathfrak{A}$. Applying the new concepts to our setting, we get the following result.

Corollary 44. *If* $\mathfrak{A}$ *is a strongly isomorphism-closed subcategory of the category* $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat}$ *and* $\mathfrak{T} = \bar{E}^{\leftarrow}(\mathfrak{A})$*, then there exists the equivalence* $(\bar{\bar{\eta}}, \bar{\bar{\varepsilon}}) : \bar{\bar{E}} \dashv \bar{\bar{D}} : \mathfrak{A} \to \mathfrak{T}$*. In particular, if* $\mathbf{D}$ *is a variety such that* $\mathbf{LoD} \sim \mathfrak{A}$*, then* $\mathbf{LoD} \sim \mathfrak{T}$*.*

4.2. **Representations induced by reducts.** In the previous section we considered the case, when a new representation is induced by a particular subcategory of the variety in question. A more common occurrence, however, is to have a reduct instead of a subcategory. More particularly, fix a variety $\mathbf{C}$ (resp. $\mathbf{C}'$) and its algebraic r-reduct $(\| - \|, \mathbf{R})$ (resp. $(\| - \|, \mathbf{R}')$). Moreover, assume that $\mathbf{C}'$ (resp. $\mathbf{R}'$) is a reduct of $\mathbf{C}$ (resp. $\mathbf{R}$) such that the diagram

$$(1) \qquad \begin{array}{ccc} \mathbf{C} & \xrightarrow{\|-\|} & \mathbf{C}' \\ {\scriptstyle \|-\|}\downarrow & & \downarrow{\scriptstyle \|-\|} \\ \mathbf{R} & \xrightarrow[\|-\|]{} & \mathbf{R}' \end{array}$$

commutes. The problem of the previous section translates into the new setting as follows: given a catalg duality for $\mathbf{C}$ and $\mathbf{R}$ (resp. $\mathbf{C}'$ and $\mathbf{R}'$), is it possible to obtain a duality for $\mathbf{C}'$ and $\mathbf{R}'$ (resp. $\mathbf{C}$ and $\mathbf{R}$). It is the purpose of the next three subsections to give a partial answer to the problem.

4.2.1. *From variety to its reduct.* Suppose there exists an equivalence $(\bar{\eta}, \bar{\varepsilon}) : \bar{E} \dashv \bar{D} : Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat} \to Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSob}$ based on $\mathbf{C}$ and $\mathbf{R}$. This subsection investigates the question on whether some parts of it can be used in the setting of $\mathbf{C}'$ and $\mathbf{R}'$. To begin with, notice that the new framework satisfies requirement $(\mathcal{R})$. Moreover, $\|\mathbb{C}\|$ (for the sake of shortness denoted by $\mathbb{C}'$) is a $\mathbf{C}'$-algebra, and since $(\mathbb{C}, \boldsymbol{\delta})$ is $Q_{\mathbf{B}}$-continuous, $(\mathbb{C}', \boldsymbol{\delta})$ must be as well. By Corollary 28, there exists an adjoint situation $(\eta', \varepsilon') : E' \dashv D' : \mathbf{LoC}' \to Q_{\mathbf{B}}$-$\mathbf{R}'\mathbf{Top}$. A relation between the adjunctions obtained is established by the functor $Q_{\mathbf{B}}$-$\mathbf{RTop} \xrightarrow{\|-\|} Q_{\mathbf{B}}$-$\mathbf{R}'\mathbf{Top}$

given by $\|(R,\tau) \xrightarrow{f} (S,\sigma)\| = (\|R\|,\tau) \xrightarrow{\|f\|} (\|S\|,\sigma)$, which provides two (in general, non-commutative) diagrams:

$$(2)\qquad \begin{array}{ccc} Q_{\mathbf{B}}\text{-}\mathbf{RTop} & \xrightarrow{E} & \mathbf{LoC} \\ \|-\| \downarrow & & \downarrow \|-\|^{op} \\ Q_{\mathbf{B}}\text{-}\mathbf{R}'\mathbf{Top} & \xrightarrow[E']{} & \mathbf{LoC}' \end{array} \qquad \begin{array}{ccc} \mathbf{LoC} & \xrightarrow{D} & Q_{\mathbf{B}}\text{-}\mathbf{RTop} \\ \|-\|^{op} \downarrow & & \downarrow \|-\| \\ \mathbf{LoC}' & \xrightarrow[D']{} & Q_{\mathbf{B}}\text{-}\mathbf{R}'\mathbf{Top}. \end{array}$$

It appears that the non-commutativity in question can be replaced by a suitable 2-cell structure [6]:

$$(3)\qquad Q_{\mathbf{B}}\text{-}\mathbf{RTop} \underset{\|-\|^{op}E}{\overset{E'\|-\|}{\Downarrow\alpha}} \mathbf{LoC}' \qquad \mathbf{LoC} \underset{D'\|-\|^{op}}{\overset{\|-\|D}{\Downarrow\beta}} Q_{\mathbf{B}}\text{-}\mathbf{R}'\mathbf{Top},$$

where $(\|E(R)\| = \|Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R,\|\mathbb{C}\|)\|) \xrightarrow{\alpha_R} (E'(\|R\|) = Q_{\mathbf{B}}\text{-}\mathbf{R}'\mathbf{Top}(\|R\|,\|\mathbb{C}'\|))$ is given by $\alpha_R(f) = f$ and $(\|D(C)\| = \|\mathbf{C}(C,\mathbb{C})\|) \xrightarrow{\beta_C} (D'(\|C\|) = \mathbf{C}'(\|C\|,\mathbb{C}'))$ is defined by $\beta_C(\varphi) = \varphi$ (notice that we have omitted some $(-)^{op}$ indexing for the sake of clearness). Moreover, straightforward computations show that the following diagrams commute (e.g., for the left one, notice that $((E'\beta_C)^{op} \circ \varepsilon'_{\|C\|}(c))(\varphi) = (\varepsilon'_{\|C\|}(c)) \circ \beta_C(\varphi) = \varphi(c) = (\|\varepsilon_C\|(c))(\varphi) = (\alpha_{D(C)} \circ \|\varepsilon_C\|(c))(\varphi)$ for every $C \in \mathcal{O}b(\mathbf{LoC})$, $c \in C$ and $\varphi \in \|\mathbf{C}(C,\mathbb{C})\|$):

$$(4)\qquad \begin{array}{ccc} E'\|-\|D & \xrightarrow{E'\beta} & E'D'\|-\|^{op} \\ \alpha D \downarrow & & \downarrow \varepsilon'\|-\|^{op} \\ \|-\|^{op}ED & \xrightarrow[\|-\|^{op}\varepsilon]{} & \|-\|^{op} \end{array} \qquad \begin{array}{ccc} \|-\| & \xrightarrow{\|-\|\eta} & \|-\|DE \\ \eta'\|-\| \downarrow & & \downarrow \beta E \\ D'E'\|-\| & \xrightarrow[D'\alpha]{} & D'\|-\|^{op}E. \end{array}$$

It appears that there exists a nice relation between sobriety (resp. spatiality) of both settings.

Lemma 45. *For an* $r_{\mathbb{C}}$-$Q_{\mathbf{B}}$*-sober space* (R,τ)*, the following are equivalent:*
(1) $(\|R\|,\tau)$ *is* $r_{\mathbb{C}'}$-$Q_{\mathbf{B}}$*-sober;*
(2) $\eta'_{\|R\|}$ *is surjective.*

Proof. Since the implication (1) $\Rightarrow$ (2) is clear, it will be enough to show the converse one. We will use Definition 30 for the purpose. By the right-hand rectangle of Diagram (4), $D'\alpha_R \circ \eta'_{\|R\|} = \beta_{E(R)} \circ \|\eta_R\|$. Since both $\beta_{E(R)}$ and $\|\eta_R\|$ are injective, $\eta'_{\|R\|}$ must be as well and therefore $\eta'_{\|R\|}$ is bijective. Thus, Item (1) of both $\mathrm{r}_{\mathbb{C}'}$-$Q_{\mathbf{B}}$-T_0 and $\mathrm{r}_{\mathbb{C}'}$-$Q_{\mathbf{B}}$-S_0 hold. To show Item (2) of $\mathrm{r}_{\mathbb{C}'}$-$Q_{\mathbf{B}}$-T_0, notice that given $\upsilon \in \Upsilon_{\mathbf{R}'}$ and $\langle r_j \rangle_{m_\upsilon} \in R^{m_\upsilon}$, if $\langle f(r_j) \rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{C}'\|}$ for every $f \in Q_{\mathbf{B}}\text{-}\mathbf{R}'\mathbf{Top}(\|R\|, \|\mathbb{C}'\|)$, then $\langle f(r_j) \rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{C}\|}$ for every $f \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R, \|\mathbb{C}\|)$ and therefore $\langle r_j \rangle_{m_\upsilon} \in \varpi_\upsilon^{\|R\|}$ by the assumption. Item (2) of $\mathrm{r}_{\mathbb{C}'}$-$Q_{\mathbf{B}}$-S_0 follows from $\tau = \langle \{\alpha \circ f \,|\, f \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R, \|\mathbb{C}\|),\ \alpha \in \boldsymbol{\delta}\} \rangle \subseteq \langle \{\alpha \circ f \,|\, f \in Q_{\mathbf{B}}\text{-}\mathbf{R}'\mathbf{Top}(\|R\|, \|\mathbb{C}'\|),\ \alpha \in \boldsymbol{\delta}\} \rangle \subseteq \tau$. □

Lemma 46. *For an $\mathrm{r}_{\mathbb{C}}$-$Q_{\mathbf{B}}$-spatial localic algebra C, the following are equivalent:*
(1) $\|C\|$ is $\mathrm{r}_{\mathbb{C}'}$-$Q_{\mathbf{B}}$-spatial;
(2) $\varepsilon'_{\|C\|}$ is surjective.

Proof. The implication (1) $\Rightarrow$ (2) being clear, we show the converse one. By the left-hand rectangle of Diagram (4), $\alpha_{D(C)} \circ \|\varepsilon_C\| = (E'\beta_C)^{op} \circ \varepsilon'_{\|C\|}$. Since both $\alpha_{D(C)}$ and $\|\varepsilon_C\|$ are injective, $\varepsilon'_{\|C\|}$ must be as well and therefore $\varepsilon'_{\|C\|}$ is bijective. □

Define $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSob}_s$ (resp. $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat}_s$) to be the full subcategory of $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSob}$ (resp. $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat}$) of all r-$Q_{\mathbf{B}}$-spaces (R, τ) (resp. localic algebras C) such that $\eta'_{\|R\|}$ (resp. $\varepsilon'_{\|C\|}$) is surjective.

Lemma 47. *There exist the restrictions $Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSob}_s \xrightarrow{\|-\|} Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Sob}$, $Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSpat}_s \xrightarrow{\|-\|} Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Spat}$ of the functors $Q_{\mathbf{B}}\text{-}\mathbf{RTop} \xrightarrow{\|-\|} Q_{\mathbf{B}}\text{-}\mathbf{R}'\mathbf{Top}$, $\mathbf{LoC} \xrightarrow{\|-\|^{op}} \mathbf{LoC}'$.*

Proof. Follows from Lemmas 45 and 46. □

The reader should notice the important point that (in general) there is no restriction of the functors $Q_{\mathbf{B}}\text{-}\mathbf{RTop} \xrightarrow{\|-\|} Q_{\mathbf{B}}\text{-}\mathbf{R}'\mathbf{Top}$ (resp. $\mathbf{LoC} \xrightarrow{\|-\|^{op}} \mathbf{LoC}'$) to the categories $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSob}$, $Q_{\mathbf{B}}$-$\mathbb{C}'\mathbf{R}'\mathbf{Sob}$ (resp. $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat}$, $Q_{\mathbf{B}}$-$\mathbb{C}'\mathbf{R}'\mathbf{Spat}$).

4.2.2. *From reduct to its generating variety through an algebraic r-reduct.* With Diagram (1) in mind, suppose there exists an equivalence $(\bar{\eta}', \bar{\varepsilon}') : \bar{E}' \dashv \bar{D}' : Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Spat} \to Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Sob}$ based on $\mathbf{C}'$ and $\mathbf{R}'$. The question is how it relates to the setting of $\mathbf{C}$ and $\mathbf{R}$. Unlike the just considered framework, the current one needs some additional requirements:

- ($\mathcal{A}$) There exists a $\mathbf{C}$-algebra $\mathbb{C}$ such that $\|\mathbb{C}\| = \mathbb{C}'$.
- ($\mathcal{T}$) There exists a $Q_{\mathbf{B}}$-topology $\boldsymbol{\delta}$ on $\mathbb{C}$ such that
  - (1) $(\mathbb{C}, \boldsymbol{\delta})$ is a $Q_{\mathbf{B}}$-continuous algebra;
  - (2) $\boldsymbol{\delta}' \subseteq \boldsymbol{\delta}$.

By Corollary 28, there exists an adjoint situation $(\eta, \varepsilon) : E \dashv D : \mathbf{LoC} \to Q_{\mathbf{B}}\text{-}\mathbf{RTop}$. A relation between the adjunctions in question is established again by the functor $Q_{\mathbf{B}}\text{-}\mathbf{RTop} \xrightarrow{\|-\|} Q_{\mathbf{B}}\text{-}\mathbf{R}'\mathbf{Top}$, providing (non-commutative) Diagram (2). The 2-cell structure of Diagram (3) is guaranteed by Item (2) of ($\mathcal{T}$).

Straightforward computations provide Diagram (4). Moreover, there exists a relation between sobriety (resp. spatiality) of both settings.

Lemma 48. *For an $r_{\mathbb{C}}$-$Q_{\mathbf{B}}$-sober space (R, τ), the following are equivalent:*
(1) $(\|R\|, \tau)$ *is* $r_{\mathbb{C}'}$-$Q_{\mathbf{B}}$*-sober;*
(2) *(i)* $\eta'_{\|R\|}$ *is surjective;*
(ii) $\langle\{\alpha \circ f \mid f \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R, \|\mathbb{C}\|),\ \alpha \in \boldsymbol{\delta}\}\rangle \subseteq \langle\{\alpha \circ f \mid f \in Q_{\mathbf{B}}\text{-}\mathbf{R}'\mathbf{Top}(\|R\|, \|\mathbb{C}'\|),\ \alpha \in \boldsymbol{\delta}'\}\rangle$.

Proof. Use the machinery of the proof of Lemma 45. Notice that Item (ii) of (2) holds in case of $\boldsymbol{\delta} \subseteq \boldsymbol{\delta}'$ (and therefore $\boldsymbol{\delta} = \boldsymbol{\delta}'$ by Item (2) of $(\mathfrak{T})$). □

Lemma 49. *For an $r_{\mathbb{C}}$-$Q_{\mathbf{B}}$-spatial localic algebra C, the following are equivalent:*
(1) $\|C\|$ *is* $r_{\mathbb{C}'}$-$Q_{\mathbf{B}}$*-spatial;*
(2) $\varepsilon'_{\|C\|}$ *is surjective.*

Proof. Use the proof of Lemma 46. □

By analogy with the previous section, one can define the categories $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSob}_s$ and $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat}_s$, the former one having an additional condition on its objects induced by Lemma 48. Lemmas 48, 49 yield the restriction $Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSob}_s \xrightarrow{\|-\|} Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Sob}$ (resp. $Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSpat}_s \xrightarrow{\|-\|} Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Spat}$) of the functor $Q_{\mathbf{B}}\text{-}\mathbf{RTop} \xrightarrow{\|-\|} Q_{\mathbf{B}}\text{-}\mathbf{R}'\mathbf{Top}$ (resp. $\mathbf{LoC} \xrightarrow{\|-\|^{op}} \mathbf{LoC}'$). The reader should notice that again (in general) $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSob}_s$ (resp. $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat}_s$) could not be changed to $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSob}$ (resp. $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat}$).

4.2.3. *From reduct to its generating variety through a non-algebraic r-reduct.* The last subsection dealt with a relation between dualities for a given variety and its reduct. Motivated by various representations in the literature, this subsection considers a more general setting, presenting the just mentioned problem in a different light. Consider once more Diagram (1) and suppose that $(\| - \|, \mathbf{R})$ is an r-reduct of $\mathbf{C}$ which is not necessarily algebraic. Due to the assumption, it may be not possible to obtain an equivalence of the type $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat} \sim Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSob}$. On the other hand, numerous examples (see Introduction) clearly show that a $\mathbf{C}'$-$\mathbf{R}'$-equivalence $(\bar{\eta}', \bar{\varepsilon}') : \bar{E}' \dashv \bar{D}' : Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Spat} \to Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Sob}$ can provide a $\mathbf{C}$-$\mathbf{R}$-representation theorem. In the following, we show a catalg approach to the challenge. Start by introducing two additional categories, serving as a cornerstone of the approach (notice that we reverse slightly the setting of Definition 17 and consider the category $Q_{\mathbf{B}}$-$\mathbf{RTop}$ as the r-variety $\mathbf{R}$ enriched in the category $Q_{\mathbf{B}}$-$\mathbf{Top}$, with the respective underlying functor to $\mathbf{R}$ defined by $|(R, \tau) \xrightarrow{f} (S, \sigma)| = R \xrightarrow{f} S$).

Definition 50. $\mathbb{T}$ is the category, whose objects are triples (R, τ, C), where (R, C) is in $\mathbf{R} \times \mathbf{LoC}$ and $(\|R\|, \tau)$ is in $Q_{\mathbf{B}}$-$\mathbb{C}'\mathbf{R}'\mathbf{Sob}$ such that $\|C\| = E'(\|R\|, \tau)$, and whose morphisms $(R_1, \tau_1, C_1) \xrightarrow{f} (R_2, \tau_2, C_2)$ are $Q_{\mathbf{B}}$-$\mathbf{R}'\mathbf{Top}$-morphisms $(\|R_1\|, \tau_1) \xrightarrow{f} (\|R_2\|, \tau_2)$ such that $\|C_1\| \xrightarrow{E'f} \|C_2\|$ is a $\mathbf{LoC}$-morphism.

Definition 51. $\mathbb{A}$ is the category, whose objects are pairs (C, R), where (C, R) is in $\mathbf{LoC} \times \mathbf{R}$ and $\|C\|$ is in $Q_{\mathbf{B}}$-$\mathbb{C}'\mathbf{R}'\mathbf{Spat}$ such that $\|R\| = |D'(\|C\|)|$ (notice the use of the aforesaid underlying functor to $\mathbf{R}'$), and whose morphisms $(C_1, R_1) \xrightarrow{\varphi} (C_2, R_2)$ are $\mathbf{C}$-morphisms $C_1 \xrightarrow{\varphi} C_2$.

On the first step, we construct a functor $\mathbb{T} \xrightarrow{E} \mathbb{A}^{op}$. Given a $\mathbb{T}$-object (R, τ, C), $(\|R\|, \tau) \xrightarrow{\eta'_{\|R\|}} (D'E'(\|R\|, \tau) = D'(\|C\|))$ is a $Q_{\mathbf{B}}$-$\mathbf{R}'\mathbf{Top}$-isomorphism and thus, a bijective map. Given $\upsilon \in \Upsilon_{\mathbf{R}} \backslash \Upsilon_{\mathbf{R}'}$ and $\langle r_j \rangle_{m_\upsilon} \in R^{m_\upsilon}$, let $\langle \eta'_{\|R\|}(r_j) \rangle_{m_\upsilon} \in \varpi_\upsilon^{|D'(\|C\|)|}$ iff $\langle r_j \rangle_{m_\upsilon} \in \varpi_\upsilon^R$, and obtain an $\mathbf{R}$-isomorphism $R \xrightarrow{\eta'_{\|R\|}} |D'(\|C\|)|$. That gives an $\mathbf{R}$-object $\hat{R}$ such that $\|\hat{R}\| = |D'(\|C\|)|$ (notice that R and $\hat{R}$ have different underlying sets). The considerations, backed by the definition of $\mathbb{T}$-morphisms, suggest the next lemma.

Lemma 52. *There exists a functor* $\mathbb{T} \xrightarrow{E} \mathbb{A}^{op}$, $E((R_1, \tau_1, C_1) \xrightarrow{f} (R_2, \tau_2, C_2)) = (C_1, \hat{R}_1) \xrightarrow{E'f} (C_2, \hat{R}_2)$.

On the second step, we obtain a functor $\mathbb{A}^{op} \xrightarrow{D} \mathbb{T}$. Given an $\mathbb{A}^{op}$-morphism $(C_1, R_1) \xrightarrow{\varphi} (C_2, R_2)$, $\|C_i\| \xrightarrow{\varepsilon'_{\|C_i\|}} (E'D'(\|C_i\|) = E'(\|R_i\|, \tau_i))$ is a $\mathbf{LoC}'$-isomorphism and thus, a bijective map. Given $\lambda \in \Lambda_{\mathbf{C}} \backslash \Lambda_{\mathbf{C}'}$ and $\langle c_j \rangle_{n_\lambda} \in C^{n_\lambda}$,
let $\omega_\lambda^{E'(\|R_i\|, \tau_i)}(\langle \varepsilon'_{\|C_i\|}(c_j) \rangle_{n_\lambda}) = \varepsilon'_{\|C_i\|}(\omega_\lambda^{C_i}(\langle c_j \rangle_{n_\lambda}))$, and obtain a $\mathbf{C}$-isomorphism $C_i \xrightarrow{\varepsilon'_{\|C_i\|}} E'(\|R_i\|, \tau_i)$. That gives a $\mathbf{C}$-algebra $\hat{C}_i$ such that $\|\hat{C}_i\| = E'(\|R_i\|, \tau_i)$ (notice again that C_i and $\hat{C}_i$ have different underlying sets). Moreover, commutativity of the diagram

$$\begin{array}{ccc}
(\|\hat{C}_1\| = E'D'(\|C_1\|)) & \xrightarrow{(\varepsilon'_{\|C_1\|})^{op}} & C_1 \\
{\scriptstyle E'D'\varphi} \downarrow & & \downarrow {\scriptstyle \varphi} \\
(\|\hat{C}_2\| = E'D'(\|C_2\|)) & \xrightarrow[(\varepsilon'_{\|C_2\|})^{op}]{} & C_2
\end{array}$$

and our definition of $\hat{C}_i$ imply $E'D'\varphi = ((\varepsilon'_{\|C_2\|})^{op})^{-1} \circ \varphi \circ (\varepsilon'_{\|C_1\|})^{op}$, the right-hand side of the equality being a $\mathbf{LoC}$-morphism. Altogether, one gets the following result.

Lemma 53. *There exists a functor* $\mathbb{A}^{op} \xrightarrow{D} \mathbb{T}$, $D((C_1, R_1) \xrightarrow{\varphi} (C_2, R_2)) = (R_1, \tau_1, \hat{C}_1) \xrightarrow{D'\varphi} (R_2, \tau_2, \hat{C}_2)$.

Having the functors in hand, we proceed to constructing two natural transformations. Given a $\mathbb{T}$-object (R, τ, C), define $(R, \tau, C) \xrightarrow{\eta_{(R,\tau,C)}} (DE(R, \tau, C) = (\hat{R}, \tau_{|D'(\|C\|)|}, \hat{C}))$ by $\eta_{(R,\tau,C)}(r) = \eta'_{\|R\|}(r)$. It follows that $(\|R\|, \tau) \xrightarrow{\eta_{(R,\tau,C)}} ((\|\hat{R}\|, \tau_{|D'(\|C\|)|}) = D'E'(\|R\|, \tau))$ is a $Q_{\mathbf{B}}$-**RTop**-isomorphism. Moreover, it appears that a stronger result holds.

Lemma 54. *$(R, \tau, C) \xrightarrow{\eta_{(R,\tau,C)}} DE(R, \tau, C)$ is a $\mathbb{T}$-isomorphism.*

Proof. Following Definition 50, it will be enough to show that $\|C\| \xrightarrow{E'\eta_{(R,\tau,C)}} \|\hat{C}\|$ is a **LoC**-isomorphism. Consider the following commutative triangle:

$$\begin{array}{ccc} (\|C\| = E'(\|R\|, \tau)) & \xrightarrow{1_{E'(\|R\|,\tau)}} & \\ \downarrow{\scriptstyle E'\eta_{(R,\tau,C)} = E'\eta'_{(\|R\|,\tau)}} & & \\ (\|\hat{C}\| = E'(\|\hat{R}\|, \tau_{|D'(\|C\|)|}) = E'D'(\|C\|) = E'D'E'(\|R\|, \tau)) & \xrightarrow[(\varepsilon'_{\|C\|})^{op} = (\varepsilon'_{E'(\|R\|,\tau)})^{op}]{} & (\|C\| = E'(\|R\|, \tau)). \end{array}$$

By the construction of E and D, $(\varepsilon'_{\|C\|})^{op}$ is a **LoC**-isomorphism and then $E'\eta_{(R,\tau,C)}$ must be as well. □

Corollary 55. *$1_{\mathbb{T}} \xrightarrow{\eta} DE$ is a natural isomorphism.*

Proof. The statement in question follows from the fact that both D and E as well as η are based on the functors D' and E' as well as the natural transformation η'. □

The second natural transformation can be obtained equally easy. Given an $\mathbb{A}^{op}$-object (C, R), define $(C, R) \xrightarrow{\varepsilon_{(C,R)}} (ED(C, R) = (\hat{C}, \hat{R}))$ by $\varepsilon_{(C,R)}(c) = \varepsilon'_{\|C\|}(c)$. It follows that $C \xrightarrow{\varepsilon_{(C,R)}} (\|\hat{C}\| = E'D'(\|C\|))$ is a **C**-isomorphism. Moreover, similar to Corollary 55, one can show the following lemma.

Lemma 56. *$EQ \xrightarrow{\varepsilon} 1_{\mathbb{A}^{op}}$ is a natural isomorphism.*

It is possible now to state the first important result of this section.

Theorem 57. *There exists an equivalence $(\eta, \varepsilon) : E \dashv D : \mathbb{A}^{op} \to \mathbb{T}$.*

Proof. Follows from Lemmas 52, 53, 56 and Corollary 55. □

Theorem 57, being interesting by itself, gives rise to a procedure of obtaining new dualities from old, running as follows. Let $\mathcal{F}_T$ (resp. $\mathcal{F}_A$) be a set of axioms (in the obvious sense; for a particular example, see the next section) which can be satisfied by $Q_{\mathbf{B}}$-**RTop**-spaces (resp. **LoC**-algebras). Satisfaction relation for an r-$Q_{\mathbf{B}}$-space (R, τ) (resp. localic algebra C) will be denoted by $(R, \tau) \models \mathcal{F}_T$ (resp. $C \models \mathcal{F}_A$). Define

- $\mathcal{T} = \{(R, \tau) \in \mathcal{O}b(Q_{\mathbf{B}}\text{-}\mathbf{RTop}) \mid (R, \tau) \models \mathcal{F}_T \text{ and } (\|R\|, \tau) \in \mathcal{O}b(Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Sob})\}$;

- $\mathcal{A} = \{C \in \mathcal{O}b(\mathbf{LoC}) \mid C \models \mathcal{F}_A \text{ and } \|C\| \in \mathcal{O}b(Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Spat})\}$.

and suppose there exist two maps (notice that both their domains and codomains can be proper classes):

- $\mathcal{T} \xrightarrow{F_T} \mathcal{O}b(\mathbf{C})$ such that $\|F_T(R,\tau)\| = E'(\|R\|,\tau)$;
- $\mathcal{A} \xrightarrow{F_A} \mathcal{O}b(\mathbf{R})$ such that $\|F_A(C)\| = |D'(\|C\|)|$.

The new definitions give rise to a particular subcategory $\bar{\mathbb{T}}$ (resp. $\bar{\mathbb{A}}$) of $\mathbb{T}$ (resp. $\mathbb{A}$).

Definition 58. $\bar{\mathbb{T}}$ is the full subcategory of $\mathbb{T}$ of all triples (R,τ,C) such that $(R,\tau) \in \mathcal{T}$ and $C = F_T(R,\tau)$.

Definition 59. $\bar{\mathbb{A}}$ is the full subcategory of $\mathbb{A}$ of all pairs (C,R) such that $C \in \mathcal{A}$ and $R = F_A(C)$.

We would like to restrict the equivalence $\mathbb{A}^{op} \sim \mathbb{T}$ of Theorem 57 to the new setting and therefore introduce the following requirements:

($\mathcal{C}_T$) If $(R,\tau) \in \mathcal{T}$, then $F_T(R,\tau) \models \mathcal{F}_A$.
($\mathcal{C}_A$) If $C \in \mathcal{A}$, then $(F_A(C), \tau_{|D'(\|C\|)|}) \models \mathcal{F}_T$.
($\mathcal{I}_T$) If $(R,\tau) \in \mathcal{T}$, then $\|R\| \xrightarrow{\eta'_{\|R\|}} \|F_A F_T(R,\tau)\|$ is an $\mathbf{R}$-isomorphism.
($\mathcal{I}_A$) If $C \in \mathcal{A}$, then $\|C\| \xrightarrow{\varepsilon'_{\|C\|}} \|F_T F_A(C)\|$ is a $\mathbf{C}$-isomorphism.

Lemma 60. *There exist the restrictions $\bar{\mathbb{T}} \xrightarrow{\bar{E}} \bar{\mathbb{A}}^{op}$ and $\bar{\mathbb{A}}^{op} \xrightarrow{\bar{D}} \bar{\mathbb{T}}$ of the functors $\mathbb{T} \xrightarrow{E} \mathbb{A}^{op}$ and $\mathbb{A}^{op} \xrightarrow{D} \mathbb{T}$.*

Proof. Given $(R,\tau,F(R,\tau))$ in $\bar{\mathbb{T}}$, $E(R,\tau,F(R,\tau)) = (F_T(R,\tau),\hat{R})$, where $\|R\| \xrightarrow{\eta'_{\|R\|}} \|\hat{R}\|$ is an $\mathbf{R}$-isomorphism. Since $|\hat{R}| = |D'E'(\|R\|,\tau)| = |F_A F_T(R,\tau)|$ (recall that $|-|$ denotes the underlying set of the structure in question), ($\mathcal{I}_T$) implies $\hat{R} = F_A F_T(R,\tau)$ and thus, $(F_T(R,\tau),\hat{R}) = (F_T(R,\tau), F_A F_T(R,\tau))$ is in $\bar{\mathbb{A}}^{op}$ by ($\mathcal{C}_T$). On the other hand, given $(C,F_A(C))$ in $\bar{\mathbb{A}}^{op}$, $D(C,F_A(C)) = (F_A(C), \tau_{|D'(\|C\|)|}, \hat{C})$, where $\|C\| \xrightarrow{\varepsilon'_{\|C\|}} \|\hat{C}\|$ is a $\mathbf{C}$-isomorphism. Since $|\hat{C}| = |E'D'(\|C\|)| = |F_T F_A(C)|$, ($\mathcal{I}_A$) yields $\hat{C} = F_T F_A(C)$ and thus, $(F_A(C), \tau_{|D'(\|C\|)|}, \hat{C}) = (F_A(C), \tau_{|D'(\|C\|)|}, F_T F_A(C))$ is in $\bar{\mathbb{T}}$ by ($\mathcal{C}_A$). □

Corollary 61. *There exists an equivalence $(\bar{\eta},\bar{\varepsilon}) : \bar{E} \dashv \bar{D} : \bar{\mathbb{A}}^{op} \to \bar{\mathbb{T}}$.*

To bring more clarity in the machinery developed, we add two more definitions.

Definition 62. $\mathcal{T}Q_{\mathbf{B}}$-$\mathbf{RTop}$ is the category, whose objects are the elements of $\mathcal{T}$, and whose morphisms $(R_1,\tau_1) \xrightarrow{f} (R_2,\tau_2)$ are those $Q_{\mathbf{B}}$-$\mathbf{R}'\mathbf{Top}$-morphisms $(\|R_1\|,\tau_1) \xrightarrow{f} (\|R_2\|,\tau_2)$ for which $\|F_T(R_1,\tau_1)\| \xrightarrow{E'f} \|F_T(R_2,\tau_2)\|$ is a $\mathbf{LoC}$-morphism.

Definition 63. $\mathbf{LoC}_{\mathcal{A}}$ is the full subcategory of $\mathbf{LoC}$, whose objects are the elements of $\mathcal{A}$.

It is possible now to state the second important result of this section.

Theorem 64. *There exists an equivalence* $(\bar{\eta}, \bar{\varepsilon}) : \bar{E} \dashv \bar{D} : \mathbf{LoC}_{\mathcal{A}} \to \mathcal{T}Q_{\mathbf{B}}\text{-}\mathbf{RTop}$.

Proof. It is easy to see that $\mathbf{LoC}_{\mathcal{A}}$ (resp. $\mathcal{T}Q_{\mathbf{B}}\text{-}\mathbf{RTop}$) is isomorphic to $\bar{\mathbb{A}}^{op}$ (resp. $\bar{\mathbb{T}}$). □

The reader should be aware that the category $\mathcal{T}Q_{\mathbf{B}}\text{-}\mathbf{RTop}$ (resp. $\mathbf{LoC}_{\mathcal{A}}$) is not a subcategory of $Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Sob}$ (resp. $Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Spat}$). More precisely, the following diagrams commute:

(5)

$$\begin{array}{ccc} \mathbf{LoC}_{\mathcal{A}} & \xrightarrow{\bar{D}} & \mathcal{T}Q_{\mathbf{B}}\text{-}\mathbf{RTop} \\ {\scriptstyle \|-\|^{op}}\downarrow & & \downarrow{\scriptstyle \|-\|} \\ Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Spat} & \xrightarrow[D']{} & Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Sob} \end{array} \qquad \begin{array}{ccc} \mathcal{T}Q_{\mathbf{B}}\text{-}\mathbf{RTop} & \xrightarrow{\bar{E}} & \mathbf{LoC}_{\mathcal{A}} \\ {\scriptstyle \|-\|}\downarrow & & \downarrow{\scriptstyle \|-\|^{op}} \\ Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Sob} & \xrightarrow[E']{} & Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Spat}, \end{array}$$

where $\mathcal{T}Q_{\mathbf{B}}\text{-}\mathbf{RTop} \xrightarrow{\|-\|} Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Sob}$ is given by $\|(R_1, \tau_1) \xrightarrow{f} (R_2, \tau_2)\|$ $= (\|R_1\|, \tau_1) \xrightarrow{f} (\|R_2\|, \tau_2)$. Also notice that we do not give an explicit description of the axioms $\mathcal{F}_T$ (resp. $\mathcal{F}_A$) and their induced maps F_T (resp. F_A), the duality is based upon. Our goal (motivated by category theory itself) is to provide a common framework for such procedures, and that stands in contrast to *piggyback dualities* of D. Clark and B. Davey [15], where the authors try to show an explicit construction of the duality in question.

The reader has probably noticed (the fact was underlined by our notations as well) that the category $\mathbf{LoC}_{\mathcal{A}}$ is a (full) subcategory of $\mathbf{LoC}$, whereas $\mathcal{T}Q_{\mathbf{B}}\text{-}\mathbf{RTop}$ (in general) is not a subcategory of $Q_{\mathbf{B}}\text{-}\mathbf{RTop}$ since its morphisms are just $\mathbf{R}'$-morphisms. Concrete examples (see the next section) show that often the category obtained is indeed a subcategory of $Q_{\mathbf{B}}\text{-}\mathbf{RTop}$, the result, however, being rather dependant on the particular setting employed. In our current general one, it is possible to state the following lemma.

Lemma 65. *Given a* $\mathcal{T}Q_{\mathbf{B}}\text{-}\mathbf{RTop}$*-morphism* $(R_1, \tau_1) \xrightarrow{f} (R_2, \tau_2)$, *equivalent are:*

(1) $\|R_1\| \xrightarrow{f} \|R_2\|$ *is an* $\mathbf{R}$*-morphism;*
(2) $\|F_A F_T(R_1, \tau_1)\| \xrightarrow{D'E'f} \|F_A F_T(R_2, \tau_2)\|$ *is an* $\mathbf{R}$*-morphism.*

Proof. The statement follows from commutativity of the diagram

$$\begin{array}{ccc} \|R_1\| & \xrightarrow{\eta'_{\|R_1\|}} & \|F_A F_T(R_1,\tau_1)\| \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle D'E'f} \\ \|R_2\| & \xrightarrow[\eta'_{\|R_2\|}]{} & \|F_A F_T(R_2,\tau_2)\| \end{array}$$

and requirement $(\mathcal{I}_T)$. □

With the just obtained result in view, an additional requirement seems to be advisable:

($\mathcal{H}$) Given $(R_1,\tau_1),(R_2,\tau_2) \in \mathcal{T}$ and a $Q_{\mathbf{B}}$-$\mathbf{R'Top}$-morphism $(\|R_1\|,\tau_1) \xrightarrow{f} (\|R_2\|,\tau_2)$ with the property that $\|F_T(R_1,\tau_1)\| \xrightarrow{E'f} \|F_T(R_2,\tau_2)\|$ is a **LoC**-morphism, $\|F_A F_T(R_1,\tau_1)\| \xrightarrow{D'E'f} \|F_A F_T(R_2,\tau_2)\|$ is an **R**-morphism.

The new assumption allows one to make a modification in the definition of the category $\mathcal{T}Q_{\mathbf{B}}$-**RTop**.

Definition 66. $Q_{\mathbf{B}}$-$\mathbf{RTop}_{\mathcal{T}}$ is the subcategory of $Q_{\mathbf{B}}$-**RTop**, with objects those of $\mathcal{T}Q_{\mathbf{B}}$-**RTop**, and morphisms $(R_1,\tau_1) \xrightarrow{f} (R_2,\tau_2)$ having the property of $\|F_T(R_1,\tau_1)\| \xrightarrow{E'f} \|F_T(R_2,\tau_2)\|$ being in **LoC**.

Lemma 67. *If* ($\mathcal{H}$) *holds, then the categories* $\mathcal{T}Q_{\mathbf{B}}$-**RTop** *and* $Q_{\mathbf{B}}$-$\mathbf{RTop}_{\mathcal{T}}$ *are isomorphic.*

As an immediate consequence of Lemma 67 and Theorem 64, one gets the main result of this section.

Theorem 68. *There exists an equivalence* $\mathbf{LoC}_{\mathcal{A}} \sim Q_{\mathbf{B}}$-$\mathbf{RTop}_{\mathcal{T}}$.

In other words, we have obtained an equivalence between particular subcategories of **LoC** and $Q_{\mathbf{B}}$-**RTop** ($\mathbf{LoC}_{\mathcal{A}}$ and $Q_{\mathbf{B}}$-$\mathbf{RTop}_{\mathcal{T}}$), based entirely on a catalg duality in the framework of $\mathbf{LoC}'$ and $Q_{\mathbf{B}}$-$\mathbf{R'Top}$.

5. Examples of representations

In the previous sections we presented a catalg approach to natural dualities in the sense of D. Clark and B. Davey [15]. This section illustrates the obtained machinery by the famous representation theorem for bounded distributive lattices of H. Priestley [53] and its application to topological representations of *J-distributive lattices* of A. Petrovich [49] and ¬*-lattices* of S. Celani [8], providing a better insight into their properties. The case of distributive lattices has already been considered by us in [70] and will be recalled here once more for the convenience of the reader, whereas its application to the structures of A. Petrovich and S. Celani was motivated by the results on relations between topological representations of a given variety and its reduct, dealt extensively upon in the previous section.

5.1. **Representation theorem for distributive lattices of H. Priestley.** Start with the classical vbp-theory $\mathcal{P}$ (Example 4(1)) and obtain the category **Top** of topological spaces and continuous maps (Example 7(1)), where $Q = \mathbf{2}$ and $\mathbf{B} = \mathbf{Frm}$. Enrichment of **Top** in the r-variety **Pos** (Example 15) provides the category **PoTop** of partially-ordered topological spaces and order-preserving, continuous maps (Example 18). Choose the variety **BLat** of bounded lattices as the required one **C**, which has the algebraic r-reduct $(\| - \|, \mathbf{Pos})$ given by $\|(A \xrightarrow{\varphi} B)\| = (A, \leqslant) \xrightarrow{\varphi} (B, \leqslant)$. The lattice $\mathbf{2} = \{\bot, \top\}$ with the discrete topology $\tau^d = \{\varnothing, \{\bot\}, \{\top\}, \mathbf{2}\}$ produces a continuous algebra (Corollary 23) and therefore take $(\mathbb{C}, \boldsymbol{\delta}) = (\mathbf{2}, \tau^d)$. Since requirements $(\mathcal{R})$, $(\mathcal{C})$ are satisfied, Corollary 28 gives the adjoint situation $(\eta, \varepsilon) : E \dashv D : \mathbf{LoBLat} \to \mathbf{PoTop}$, the explicit form of which is as follows:

- $\mathbf{PoTop} \xrightarrow{E} \mathbf{LoBLat}$ is defined by $E(X) = (\mathcal{COU}(X), \cap, \cup, \varnothing, X)$ and $Ef = (f^{\leftarrow})^{op}$, where $\mathcal{COU}(X)$ is the set of clopen up-sets of X (Example 31);
- $\mathbf{LoBLat} \xrightarrow{D} \mathbf{PoTop}$ is defined by $D(C) = (\mathcal{PF}(C), \subseteq, \tau)$ and $D\varphi = (\varphi^{op})^{\leftarrow}$, where $\mathcal{PF}(C)$ is the set of *prime filters* of C ($c_1 \vee c_2 \in F$ implies $c_1 \in F$ or $c_2 \in F$) and $\tau = \langle \{\rho_c \,|\, c \in C\} \cup \{\hat{\rho}_c \,|\, c \in C\} \rangle$, with $F \in \rho_c$ (resp. $F \in \hat{\rho}_c$) iff $c \in F$ (resp. $c \notin F$);
- $C \xrightarrow{\varepsilon} ED(C)$ is defined by $\varepsilon_C(c) = \rho_c$;
- $X \xrightarrow{\eta_X} DE(X)$ is defined by $\eta_X(x) = \{U \in \mathcal{COU}(X) \,|\, x \in U\}$.

The obtained framework is that of Priestley duality, with the exception of the target categories, i.e., **BLat** (resp. **PoTop**) instead of the variety **BDLat** of bounded distributive lattices (resp. the category **PrSpc** of *Priestley* (compact, totally order-disconnected) *spaces*). Theorem 37 gives the equivalence $(\bar{\eta}, \bar{\varepsilon}) : \bar{E} \dashv \bar{D} : \mathbf{2}_d\mathbf{Spat} \to \mathbf{2}_d\mathbf{Sob}$.

Lemma 69. *A bounded lattice C is $\mathbf{2}_d$-spatial iff it is distributive.*

Proof. For the necessity, notice that given a $\mathbf{2}_d$-spatial lattice C, $C \cong ED(C) = \mathcal{COU}(\mathcal{PF}(C))$, the latter lattice being a sublattice of $\mathcal{P}(\mathcal{PF}(C))$ and therefore distributive. For the sufficiency, one can use the technique applied in the proof of the Priestley representation theorem of, e.g., [19]. □

Lemma 70. *An ordered topological space X is $\mathbf{2}_d$-sober iff it is a Priestley space.*

Proof. For the necessity notice that given a $\mathbf{2}_d$-sober space X, $X \cong DE(X) = \mathcal{PF}(\mathcal{COU}(X))$. Since $\mathcal{COU}(X)$ is a sublattice of $\mathcal{P}(X)$, it is distributive and then the result follows from the technique used in the proof of Priestley duality. The same technique can be applied to obtain the sufficiency. □

All preliminaries done, we can finally state the well-known result.

Theorem 71 (Representation Theorem of H. Priestley)**.** *There exists an equivalence* $(\bar{\eta}, \bar{\varepsilon}) : \bar{E} \dashv \bar{D} : \mathbf{LoBDLat} \to \mathbf{PrSpc}$.

It is shown in [70] that the case of the representation theorems of M. Stone [75, 76] can be incorporated in the framework using the two-element frame $\mathbf{2}$ with the Sierpinski topology $\tau^s = \{\varnothing, \{\top\}, \mathbf{2}\}$ (Corollary 25).

5.2. **Representation theorem for J-distributive lattices of A. Petrovich.** Motivated by recent interest of numerous researchers in J-distributive lattices of A. Petrovich [49], in this section we incorporate his topological representation theorem for the structure into our catalg framework. As will be seen later on, the machinery for the procedure is based on the above-mentioned representation theorem of H. Priestley and the fact that the new concept has the variety **BDLat** as a reduct. For convenience of the reader, we begin with the necessary preliminaries, modifying the original notations of A. Petrovich (replacing the symbol "J" by "∇"), to fit our current framework more conveniently.

Definition 72. A ∇-*lattice* is a bounded lattice C equipped with a unary operation ∇ such that $\nabla(\bot) = \bot$ and $\nabla(c_1 \vee c_2) = \nabla(c_1) \vee \nabla(c_2)$ for every c_1, $c_2 \in C$. $\nabla\,\mathbf{BLat}$ is the variety of ∇-lattices.

Definition 73. **RPos** is the category, whose objects are triples $(X, \leqslant, R)$, where $(X, \leqslant)$ is a partially ordered set and R is a binary relation on X, and whose morphisms $(X, \leqslant, R) \xrightarrow{f} (Y, \leqslant, S)$ are order-preserving maps which also preserve the relation in question.

In the language of enriched category theory [40], **RPos** is just the category **Pos** enriched in the category $\mathbf{Rel}(2)$ (cf. Example 15). Moreover, it is easy to see that **BLat** (resp. **Pos**) is a reduct of $\nabla\,\mathbf{BLat}$ (resp. **RPos**). Slightly more sophisticated is the proof that **RPos** is an algebraic r-reduct of $\nabla\,\mathbf{BLat}$. The concrete functor in question $\nabla\,\mathbf{BLat} \xrightarrow{\|-\|} \mathbf{RPos}$ can be defined by (cf. Example 16) $\|(C_1, \nabla_1) \xrightarrow{\varphi} (C_2, \nabla_2)\| = (|C_1|, \leqslant, \langle \mathrm{Grph}\,\nabla_1 \rangle) \xrightarrow{\varphi} (|C_2|, \leqslant, \langle \mathrm{Grph}\,\nabla_2 \rangle)$, providing an r-reduct. Since ∇_i is order-preserving, the order-relation "$\leqslant$" is a subalgebra of $C_i \times C_i$, yielding algebraicity of the reduct obtained. Altogether, the considerations give the commutative diagram

$$\begin{array}{ccc} \nabla\,\mathbf{BLat} & \xrightarrow{\|-\|} & \mathbf{BLat} \\ {\scriptstyle \|-\|}\downarrow & & \downarrow{\scriptstyle \|-\|} \\ \mathbf{RPos} & \xrightarrow[\|-\|]{} & \mathbf{Pos}. \end{array}$$

The current framework fits the setting of Section 4.2.3, since the above-mentioned Priestley duality is available for the right-hand side of the diagram. Motivated by Definitions 50, 51, we introduce their particular instances for the current framework (notice that the index $(-)_P$ comes from "Petrovich").

Definition 74. $\mathbb{T}_P$ is the category, whose objects are quintuples $(X, \leqslant, R, \tau, \nabla)$ such that $(X, \leqslant, R)$ is in **RPos**, $(X, \leqslant, \tau)$ is in **PrSpc** and $(\mathcal{COU}(X), \nabla)$ is in $\nabla\,$**BLat**, and whose morphisms $(X_1, \leqslant, R_1, \tau_1, \nabla_1) \xrightarrow{f} (X_2, \leqslant, R_2, \tau_2, \nabla_2)$ are **PoTop**-morphisms $(X_1, \leqslant, \tau_1) \xrightarrow{f} (X_2, \leqslant, \tau_2)$, making the following diagram commute (notice the above-mentioned functor $\bar{E}$ from Priestley duality):

$$\begin{array}{ccc} \mathcal{COU}(X_2) & \xrightarrow{\nabla_2} & \mathcal{COU}(X_2) \\ {\scriptstyle (\bar{E}f)^{op}}\downarrow & & \downarrow{\scriptstyle (\bar{E}f)^{op}} \\ \mathcal{COU}(X_1) & \xrightarrow[\nabla_1]{} & \mathcal{COU}(X_1). \end{array}$$

Definition 75. $\mathbb{A}_P$ is the category, whose objects are triples (C, ∇, R) such that (C, ∇) is in $\nabla\,$**BDLat** and $(\mathcal{PF}(C), \subseteq, R)$ is in **RPos**, and whose morphisms $(C_1, \nabla_1, R_1) \xrightarrow{\varphi} (C_2, \nabla_2, R_2)$ are $\nabla\,$**BLat**-morphisms $(C_1, \nabla_1) \xrightarrow{\varphi} (C_2, \nabla_2)$.

By Theorem 57, there exists an equivalence $\mathbb{A}_P^{op} \sim \mathbb{T}_P$, which can be developed further using the technique of Definitions 58, 59. For the sake of convenience, we denote the enrichment of **Top** in **RPos** by **RPosTop**. Start by introducing topological axioms $\mathcal{F}_T$, suitable for the occasion. Notice that we are working with the objects of **RPosTop**, i.e., tuples $(X, \leqslant, R, \tau)$. Thus, we let $\mathcal{F}_T$ consist of the following two axioms [49]:

- $(\mathcal{A}_1)$ Given $x \in X$, $R(x) = \{y \in X \mid xRy\}$ is a closed down-set (cf. Example 31).
- $(\mathcal{A}_2)$ Given $U \in \mathcal{COU}(X)$, $R^{\uparrow}(U) = \{x \in X \mid R(x) \bigcap U \neq \varnothing\} \in \mathcal{COU}(X)$.

The set $\mathcal{F}_A$ of algebraic axioms is supposed to be empty. With these preliminaries in hand, we can introduce the two required maps:

- $(\mathcal{T}_P = \{(X, \leqslant, R, \tau) \in \mathcal{O}b(\mathbf{RPosTop}) \mid (X, \leqslant, R, \tau) \models \{(\mathcal{A}_1), (\mathcal{A}_2)\}$ and $(X, \leqslant, \tau) \in \mathcal{O}b(\mathbf{PrSpc})\}) \xrightarrow{F_T} \mathcal{O}b(\nabla\mathbf{BLat})$ defined by $F_T(X, \leqslant, R, \tau) = (\mathcal{COU}(X), R^{\uparrow})$;

- $(\mathcal{A}_P = \{(C,\nabla) \in \mathcal{O}b(\mathbf{Lo\nabla BLat}) \,|\, C \in \mathcal{O}b(\mathbf{BDLat})\}) \xrightarrow{F_A} \mathcal{O}b(\mathbf{RPos})$, $F_A(C,\nabla) = (\mathcal{PF}(C), \subseteq, R_\nabla)$, where R_∇ is defined as follows: given $F_1, F_2 \in \mathcal{PF}(C)$, $F_1 R_\nabla F_2$ iff $(C \backslash \nabla^{\leftarrow}(F_1)) \bigcap F_2 = \varnothing$.

The next categories are particular versions of Definitions 58, 59, suitable for our current framework.

Definition 76. $\bar{\mathbb{T}}_P$ is the full subcategory of $\mathbb{T}_P$ of all quintuples $(X, \leqslant, R, \tau, \nabla)$ such that $(X, \leqslant, R, \tau) \in \mathcal{T}_P$ and $\nabla = R^{\uparrow}$.

Definition 77. $\bar{\mathbb{A}}_P$ is the full subcategory of $\mathbb{A}_P$ of all triples (C, ∇, S) such that $(C, \nabla) \in \mathcal{A}_P$ and $S = R_\nabla$.

Requirements $(\mathcal{C}_T) - (\mathcal{I}_A)$ have already been checked in [49] (actually were motivated by the results of the article and the respective one of S. Celani [8]). By Corollary 61, there exists an equivalence $\bar{\mathbb{A}}_P^{op} \sim \bar{\mathbb{T}}_P$. It appears that $(\mathcal{H})$ also holds, and since the requirement is slightly off the framework of A. Petrovich, we deem it advisable to give its simple proof.

Given a continuous, order-preserving map $(X, \leqslant, R, \tau) \xrightarrow{f} (Y, \leqslant, S, \sigma)$ such that

(6)
$$\begin{array}{ccc} \mathcal{COU}(Y) & \xrightarrow{S^{\uparrow}} & \mathcal{COU}(Y) \\ {\scriptstyle f^{\leftarrow}}\downarrow & & \downarrow{\scriptstyle f^{\leftarrow}} \\ \mathcal{COU}(X) & \xrightarrow[R^{\uparrow}]{} & \mathcal{COU}(X), \end{array}$$

it will be enough to verify that $(\mathcal{PF}(\mathcal{COU}(X)), T_{R^{\uparrow}}) \xrightarrow{(f^{\leftarrow})^{\leftarrow}} (\mathcal{PF}(\mathcal{COU}(Y)), T_{S^{\uparrow}})$ is in $\mathbf{Rel}(2)$ (cf. Example 15). If $F_1, F_2 \in \mathcal{PF}(\mathcal{COU}(X))$ are such that $F_1 T_{R^{\uparrow}} F_2$, then $F_2 \subseteq (R^{\uparrow})^{\leftarrow}(F_1)$ and therefore $U \in F_2$ implies $R^{\uparrow}(U) \in F_1$. On the other hand, $(f^{\leftarrow})^{\leftarrow}(F_i) = \{U \in \mathcal{COU}(Y) \,|\, f^{\leftarrow}(U) \in F_i\}$ for $i \in \{1,2\}$. It follows that for $U \in (f^{\leftarrow})^{\leftarrow}(F_2)$, $f^{\leftarrow}(U) \in F_2$ and therefore $f^{\leftarrow} \circ S^{\uparrow}(U) = R^{\uparrow} \circ f^{\leftarrow}(U) \in F_1$, yielding $S^{\uparrow}(U) \in (f^{\leftarrow})^{\leftarrow}(F_1)$. Altogether, $(f^{\leftarrow})^{\leftarrow}(F_2) \subseteq (S^{\uparrow})^{\leftarrow}((f^{\leftarrow})^{\leftarrow}(F_1))$ and thus, $((f^{\leftarrow})^{\leftarrow}(F_1)) T_{S^{\uparrow}} ((f^{\leftarrow})^{\leftarrow}(F_2))$.

The final touch is now made by Definition 66 and Theorem 68.

Definition 78. $\nabla\,\mathbf{PrSpc}$ is the subcategory of $\mathbf{RPosTop}$, whose objects are the elements of $\mathcal{T}_P$, and whose morphisms $(X, \leqslant, R, \tau) \xrightarrow{f} (Y, \leqslant, S, \sigma)$ make Diagram (6) commute.

Theorem 79 (Representation Theorem of A. Petrovich)**.** *There exists an equivalence* $(\bar{\eta}, \bar{\varepsilon}) : \bar{E} \dashv \bar{D} : \mathbf{Lo\nabla\,BDLat} \to \nabla\,\mathbf{PrSpc}$.

It is easy to show a simple (but extremely useful) characterization of $\nabla\,\mathbf{PrSpc}$-morphisms [49].

Lemma 80. *Given two* ∇ **PrSpc**-*objects* $(X, \leqslant, R, \tau)$, $(Y, \leqslant, S, \sigma)$, *a map* $X \xrightarrow{f} Y$ *is a* ∇ **PrSpc**-*morphism iff the following hold:*

(1) f *is an* **RPosTop**-*morphism;*
(2) if $x \in X$, $y \in Y$ *and* $f(x)Sy$, *then there exists* $x' \in X$ *such that* $x' \in R(x)$ *and* $y \leqslant f(x')$.

Using the machinery of Section 4.1 and the equivalence of Theorem 79, one can obtain the topological representation theorem for *Q-distributive lattices* of R. Cignoli [12], which are ∇ **BDLat**-lattices (C, ∇) satisfying for every $c, c' \in C$ two additional conditions:

($\mathcal{D}_1$) $c \wedge \nabla(c) = c$;
($\mathcal{D}_2$) $\nabla(c \wedge \nabla(c')) = \nabla(c) \wedge \nabla(c')$.

The respective result has already been considered by A. Petrovich [49], the corresponding relations being *quasiequivalences*, i.e., relations $R \subseteq X \times X$ which are reflexive, transitive (the so-called *preorders*) and satisfy the following condition:

($\mathcal{S}$) For every $x, y \in X$ with xRy, there exists $z \in X$ such that $y \leqslant z$, xRz and zRx.

Notice that if the partial order "$\leqslant$" is given by equality, quasiequivalences reduce to equivalence relations.

5.3. **Representation theorem for ¬-lattices of S. Celani.** The results of this section were motivated by our research on *quasi-Stone algebras* introduced by N. H. Sankappanavar and H. P. Sankappanavar [65] and studied later on by various researchers [8, 9, 10, 26, 27]. In particular, there exists a topological representation theorem for the structure proved by H. Gaitán [26] and induced by Priestley duality. From the applicational point of view, however, a much more transparent result of S. Celani [8] seems to be advisable. The representation in question is based on the concept of *¬-lattice* [8, 9, 10], which is similar to the notion of ∇-lattice considered in the previous section. It is our current purpose to incorporate the duality in the catalg framework. We begin again with the necessary algebraic preliminaries.

Definition 81. A *¬-lattice* is a bounded lattice C equipped with a unary operation $\neg$ such that $\neg(\bot) = \top$ and $\neg(c_1 \vee c_2) = \neg(c_1) \wedge \neg(c_2)$ for every $c_1, c_2 \in C$. ¬**BLat** is the variety of ¬-lattices.

It is easy to see that **BLat** is a reduct of ¬**BLat**. Moreover, using the underlying functor from the previous section, one can show that **RPos** is an r-reduct of ¬**BLat**. Unlike the results for ∇-lattices, the reduct in question is not algebraic, since given a ¬-lattice $(C, \neg)$, $\neg$ is order-reversing and therefore "$\leqslant$" is not a subalgebra of $C \times C$. Thus, we have a commutative diagram

$$\begin{array}{ccc} \neg\mathbf{BLat} & \xrightarrow{\|-\|} & \mathbf{BLat} \\ {\scriptstyle \|-\|}\downarrow & & \downarrow{\scriptstyle \|-\|} \\ \mathbf{RPos} & \xrightarrow[\|-\|]{} & \mathbf{Pos}, \end{array}$$

where the left-hand side never satisfies requirement $(\mathcal{R})$ and therefore the procedure of catalg duality of Section 3 is not applicable. On the other hand, the machinery of Section 4.2.3 is fruitful even in this setting. By analogy with Definitions 74, 75, one can introduce the categories $\mathbb{T}_C$ and $\mathbb{A}_C$ (notice that the index $(-)_C$ comes from "Celani") and obtain an equivalence $\mathbb{A}_C^{op} \sim \mathbb{T}_C$. The set of topological (resp. algebraic) axioms is similar to the already considered one, with $(\mathcal{A}_2)$ changed as follows:

$(\mathcal{A}'_2)$ Given $U \in \mathcal{COU}(X)$, $R^{\downarrow}(U) = \{x \in X \mid R(x) \bigcap U = \varnothing\} \in \mathcal{COU}(X)$.

The map $\mathcal{T}_C \xrightarrow{F_T} \mathcal{O}b(\neg\mathbf{BLat})$ has the respective modification of $(-)^{\uparrow}$ to $(-)^{\downarrow}$, whereas in the map $\mathcal{A}_C \xrightarrow{F_A} \mathcal{O}b(\mathbf{RPos})$, the relation $R_\neg$ is defined by $F_1 R_\neg F_2$ iff $\neg^{\leftarrow}(F_1) \bigcap F_2 = \varnothing$. All the other proceedings are similar to the already considered ones, resulting in the following theorem.

Theorem 82 (Representation Theorem of S. Celani)**.** *There exists an equivalence* $(\bar{\eta}, \bar{\varepsilon}) : \bar{E} \dashv \bar{D} : \mathbf{Lo}\neg\mathbf{BDLat} \to \neg\mathbf{PrSpc}$.

Moreover, characterization Lemma 80 is applicable in the new setting. Using the machinery of Section 4.1 and the equivalence of Theorem 82, one can obtain the topological representation theorem for *quasi-Stone algebras* [65], which are $\neg\mathbf{BDLat}$-lattices $(C, \neg)$ satisfying for every $c, c' \in C$ four additional conditions:

$(\mathcal{Q}_1)$ $\neg(\top) = \bot$;
$(\mathcal{Q}_2)$ $\neg(c \wedge \neg(c')) = \neg(c) \vee \neg(\neg(c'))$;
$(\mathcal{Q}_3)$ $c \wedge \neg(\neg(c)) = c$;
$(\mathcal{Q}_4)$ $\neg c \vee \neg(\neg(c)) = \top$.

The respective result has already been considered by S. Celani in [8], with the corresponding binary relations appearing to be equivalences. An important point, however, should be noticed at once. In [8, 10] S. Celani shows that some axioms in the definition of quasi-Stone algebras are dependant on the others. In particular, axioms $(\mathcal{Q}_1)$ and $(\mathcal{Q}_2)$ are supposed to superfluous. While the latter statement is true, the former one contains a flaw. A possible counterexample is quite simple, i.e., consider the lattice $\mathbf{2} = \{\bot, \top\}$ with $\neg(\bot) = \top = \neg\top$. It follows that the structure satisfies all the required axioms with the exception of $(\mathcal{Q}_1)$.

In [10] S. Celani went even further, introducing a particular generalization of equivalence relations, i.e., considering a relation $R \subseteq X \times X$ which is *serial*

($R(x) \neq \varnothing$ for every $x \in X$), *euclidean* ($R^{-1} \circ R \subseteq R$) and transitive. Every equivalence relation satisfies the properties, but not vice versa. Using the duality of Theorem 82 and the reversed technique of Section 4.1 (from topology to algebra), one obtains the subcategory $\mathfrak{A}$ of $\neg$**BDLat** equivalent to the subcategory of $\neg$**PrSpc**, where the relations on objects satisfy the above-mentioned three properties. It appears [10] that the objects C of $\mathfrak{A}$ are characterized by two axioms:

($\mathcal{W}_1$) $\neg c \wedge \neg(\neg(c)) = \bot$;
($\mathcal{W}_2$) $\neg c \vee \neg(\neg(c)) = \top$.

The new structure was coined by S. Celani *weak-quasi-Stone algebra* and studied extensively in [10].

6. Conclusion and open problems

In the paper we have presented a *categorically-algebraic* (*catalg*) approach to the natural dualities of D. Clark and B. Davey [15], motivated by the outlook on the Stone representation theorems of P. T. Johnstone [39] and our recent attempt [70] on a generalization of the topological representation theorem for bounded distributive lattices of H. Priestley [53]. The new setting underlines catalg properties of the dualities in question, on one hand, and serves as a tool for generating new topological representation theorems for algebraic structures, on the other. In particular, we have presented several procedures for obtaining new dualities from the already existing ones, based on relations between a given variety and its reduct, and motivated by the multitude of techniques encountered in the literature. The results obtained were illustrated by two examples relying on Priestley duality, the first one employing a modal operator of *possibility* and the second one providing analogous results for a *negation* operator. The examples extend their influence from the realm of *algebraic logic* (Q-distributive lattices of R.Cignoli [12]) to the setting of *pseudocomplemented lattices* ((weak-) quasi-Stone algebras of N. H. Sankappanavar and H. P. Sankappanavar [65] and S Celani [10]). The most important property of the machinery proposed is its applicability to both crisp and fuzzy developments, making another step towards our ultimate goal of erasing the border between traditional and fuzzy approaches in mathematics. Moreover, the results of this paper show once again the advantage of our *catalg* approach over the *poslat* one of S. E. Rodabaugh [56], where one is tied to varieties of *lattices*, being unable to shift to those of *arbitrary algebras*. In conclusion of the paper, we would to draw the attention of the reader to several open problems related to the topic.

Lemma 80 shows a characterization of morphisms of both the category ∇**PrSpc** and $\neg$**PrSpc**. Since the characterization appears to be extremely useful in applications, the first problem is immediate.

Problem 83. Is it possible to generalize Lemma 80 to the general setting of the category $Q_{\mathbf{B}}$-$\mathbf{RTop}_{\mathcal{T}}$?

In [13] R. Cignoli uses Priestley duality to construct free Q-distributive lattices from bounded distributive lattices, whereas H. Gaitán [26] does the same job for quasi-Stone algebras. In our catalg framework, the result is equivalent to the functor $\mathbf{D} \xrightarrow{\|-\|} (Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Spat})^{op}$ from Diagram (5) having a left adjoint in those particular two cases, where $\mathbf{D}$ is the respective full subcategory of $\mathbf{LoC}_{\mathcal{A}}$, suitable for the occasion. The second problem can be thus stated as follows.

Problem 84. Under what conditions the functor $\mathbf{LoC}_{\mathcal{A}} \xrightarrow{\|-\|} (Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Spat})^{op}$ from Diagram (5) has a left adjoint?

The next problem stems from [70], being still actual. The current manuscript presented several examples illustrating the new approach, all of which are *crisp* (based on the standard vbp-theory $\mathcal{P}$). Since the fruitfulness of every new theory is measured by the amount of useful applications arising of it, the next problem springs into mind immediately.

Problem 85. Find other examples of catalg dualities based on both crisp and fuzzy topological spaces.

Since our approach incorporates natural dualities of [15], possible candidates can be found in [15, Chapter 4]. It will be the topic of our further research to translate them into catalg language as well as to find new ones.

The last problem is a reiteration of the one touched in Section 3.3, when trying to switch the developed theory to composite topological spaces.

Problem 86. Under what conditions the functor $(Q_{i\mathbf{B}_i})_I$-$\mathbf{CRTop} \xrightarrow{E_I} \prod_{i\in I}\mathbf{LoC}_i$ of Lemma 40 has a right adjoint?

Notice that by Theorem 27, the sufficient condition is the set I being a singleton. It is (probably) a nice challenge to answer the question on whether the condition is also a necessary one.

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