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$S(2,1)$-labeling of graphs with cyclic structure

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Abstract

We present the values of the σ-number of two infinite classes of graphs with cyclic structure, namely prisms and the Isaacs graphs, depending on their order.

Keywords Channel assignment, $S(2,1)$-labeling, σ-number, action of cyclic group, prism, Isaacs snark.

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1 Introduction

In this paper, we give a brief review of our work concerning the σ-number of two infinite classes of graphs with cyclic structure – prisms and Isaacs graphs. The paper also contains several research announcements of results to be published in the extended version of the paper containing detailed proofs.

The $S(2,1)$-labeling problem is a variation of the $L(2,1)$-labeling problem or, more general, of the $L(d_1,d_2)$-labeling problem – a survey on the $L(d_1,d_2)$-labeling problem is given by Calamoneri in [1]. An r-$S(2,1)$-*labeling* of a graph G is a mapping from the vertex-set of G to the cyclic group $\mathbb{Z}_r$ such that every pair of vertices adjacent in G has labels at least 2 apart in $\mathbb{Z}_r$ and simultaneously every pair of vertices at distance 2 in G has distinct labels in $\mathbb{Z}_r$. The σ-*number* of a graph G is the smallest r such that G admits an r-$S(2,1)$-labeling. A survey on the σ-number is presented by Yeh in [5].

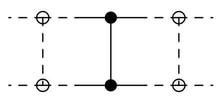

Figure 1: Prism

Although a prism can be regarded as the Cartesian product of a cycle and of the complete graph of order 2, a different equivalent description is more suitable for us. A *prism* Y_m, for $m \geq 3$, is a graph consisting of m segments isomorphic to the complete graph K_2 arranged into a cycle, where both vertices of a given segment are connected to the corresponding vertices of the preceding and of the succeeding segments – a plain given segment and its incidence with the dashed preceding and the dashed succeeding segments are shown in Figure 1. The Isaacs graphs form a superclass of the Isaacs snarks constructed by Isaacs in [4], as only odd members of the class are snarks. *The Isaacs graph* J_m, for $m \geq 3$, is a graph consisting of m segments isomorphic to the claw $K_{1,3}$ arranged into a cycle, where the leaves of a given segment are connected to the leaves of the preceding and of the succeeding segments in the manner indicated in Figure 2 – the given segment is plain while the preceding and the succeeding segments are dashed.

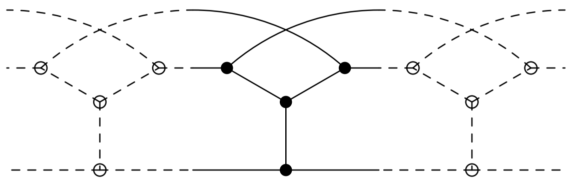

Figure 2: The Isaacs graph

2 Strategy

In order to determine the σ-number of a graph, we investigate whether or not the graph admits an r-$S(2,1)$-labeling. It follows from the definition of the σ-number that if a graph does admit an r-$S(2,1)$-labeling, then its σ-number is at most r and, conversely, if a graph does not admit an r-$S(2,1)$-labeling, then its σ-number is at least $r+1$.

To find out whether a graph G admits an r-$S(2,1)$-labeling, we proceed in two steps. First, we cover G with overlapping subgraphs in such a way that for every pair of vertices adjacent in G, as well as for every pair of vertices at distance 2 in G, there is at least one of the covering subgraphs in which the vertices have the same distances as in G. Second, we take such r-$S(2,1)$-labelings of the subgraphs that the labels of vertices in the overlapping parts match, so that they naturally form an r-$S(2,1)$-labeling of G. In the second step, we obtain an r-$S(2,1)$-labeling of G because of the choice of the covering subgraphs in the first step. Conversely, if no such r-$S(2,1)$-labelings of the subgraphs exist that the labels of vertices in the overlapping parts match, then no r-$S(2,1)$-labeling of G exists.

In the following, let G_m stand for Y_m or J_m. Due to the cyclic structure of G_m, we can choose m isomorphic subgraphs to cover G_m. To fulfil the conditions for the cover, we take m subgraphs isomorphic to the graph Y_C shown in Figure 3 to cover Y_m and m subgraphs isomorphic to the graph J_C shown in Figure 4 to cover J_m. Now, let G_C stand for Y_C whenever $G_m = Y_m$ and let G_C stand for J_C whenever $G_m = J_m$. We cover the i-th, the $(i+1)$-st and the $(i+2)$-nd segment of G_m with the i-th copy of G_C; throughout, indices are taken modulo m. Now, we have to determine all r-$S(2,1)$-labelings of G_C and to find out whether there are m-tuples of these r-$S(2,1)$-labelings which can be assigned to the copies of G_C in such a way that

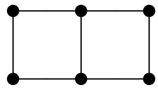

Figure 3: The graph Y_C

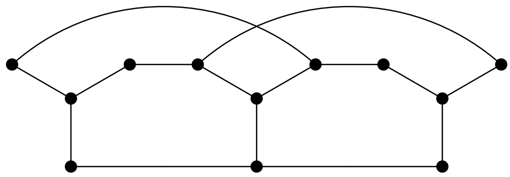

Figure 4: The graph J_C

the labels of vertices in the overlapping parts match. We call an r-$S(2,1)$-labeling l_i of G_C *concatenable* with the r-$S(2,1)$-labeling l_{i+1} of G_C if l_{i+1} labels the vertices of the left and of the central segments of G_C with the same labels as l_i labels the corresponding vertices of the central and of the right segments of G_C. Furthermore, we call a cyclic m-tuple of r-$S(2,1)$-labelings of G_C *concatenable* if the i-th r-$S(2,1)$-labeling of G_C is concatenable with the $(i+1)$-st r-$S(2,1)$-labeling of G_C for every i. Since there is a one-to-one correspondence between the r-$S(2,1)$-labelings of G_m and the concatenable cyclic m-tuples of r-$S(2,1)$-labelings of G_C, the existence of a concatenable cyclic m-tuple of r-$S(2,1)$-labelings of G_C is a sufficient and a necessary condition for the existence of an r-$S(2,1)$-labeling of G_m. We define a directed graph $D_r(G_C)$ whose vertex-set is formed by the r-$S(2,1)$-labelings of G_C and which has an arc from one r-$S(2,1)$-labeling of G_C to an another precisely when the former r-$S(2,1)$-labeling of G_C is concatenable with the latter. It follows from the definition of $D_r(G_C)$ that the existence of a concatenable cyclic m-tuple of r-$S(2,1)$-labelings of G_C is equivalent to the existence of a closed walk of length m in $D_r(G_C)$.

Since the number of r-$S(2,1)$-labelings of G_C might be very large for both $G_C = Y_C$ and $G_C = J_C$, we use the action of translations and reflections of $\mathbb{Z}_r$ and the action of the automorphism group of G_C to partition the r-$S(2,1)$-labelings of G_C into orbits. Subsequently, we only consider the representatives of these orbits. Having determined all representatives, we can reconstruct all r-$S(2,1)$-labelings of G_C by applying automorphisms of G_C and translations and reflections of $\mathbb{Z}_r$.

Since G_m is a cubic graph, it contains the claw $K_{1,3}$ as a subgraph. Observe that the σ-number of $K_{1,3}$ is equal to 6. Since the σ-number of a subgraph does not exceed the σ-number of the supergraph, we conclude that the σ-number of G_m is at least 6. Thus, we only have to investigate the r-$S(2,1)$-labelings of G_m and G_C for r at least 6.

Due to the large number of distinct r-$S(2,1)$-labelings of G_C the proofs are rather involved, although straightforward. We omit them by referring to the author's PhD.

Thesis [2].

3 Prisms

For $r = 6$, we have twelve distinct r-$S(2,1)$-labelings of Y_C. It is easy to see that in the directed graph $D_6(Y_C)$, there are closed walks of length m if and only if $m \equiv 0 \pmod 3$. Therefore $\sigma(Y_m) = 6$ for $m \equiv 0 \pmod 3$ and $\sigma(Y_m) \geq 7$ for $m \not\equiv 0 \pmod 3$.

For $r = 7$, we have 196 distinct r-$S(2,1)$-labelings of Y_C. From the directed graph $D_7(Y_C)$, we construct a directed voltage graph D' of order 28 with voltages in $\mathbb{Z}_7$ such that there exists a closed walk of length m in $D_7(Y_C)$ if and only if there exists a closed walk of length m with net voltage of 0 in D'. More details on voltage graphs can be found in [3] by Gross and Tucker. It can be shown that in D', there are closed walks of length m with net voltage of 0 if and only if $m \notin \{4,5,8,11\}$. Therefore $\sigma(Y_m) = 7$ for $m \not\equiv 0 \pmod 3$ and $m \notin \{4,5,8,11\}$ and $\sigma(Y_m) \geq 8$ for $m \in \{4,5,8,11\}$.

For $r = 8$, we can find r-$S(2,1)$-labelings of Y_C that can form a concatenable cyclic m-tuple of r-$S(2,1)$-labelings of Y_C for every $m \geq 3$. Therefore $\sigma(Y_m) = 8$ for $m \in \{4,5,8,11\}$.

Summarizing previous results, we obtain the following theorem.

Theorem 3.1. *[2, Theorem 3.1] Let Y_m be a prism of order $2m$, for $m \geq 3$. Then*

$$\sigma(Y_m) = \begin{cases} 6 & \text{for } m \equiv 0 \pmod 3 \\ 7 & \text{for } m \not\equiv 0 \pmod 3 \text{ and } m \notin \{4,5,8,11\} \\ 8 & \text{for } m \in \{4,5,8,11\}. \end{cases}$$

4 The Isaacs graphs

For $r = 6$, we have no r-$S(2,1)$-labelings of J_C. Therefore $\sigma(J_m) \geq 7$ for every $m \geq 3$.

For $r = 7$, we have 1176 distinct r-$S(2,1)$-labelings of J_C. From the directed graph $D_7(J_C)$, we construct its adjacency matrix A. Observe that a closed walk of length m in $D_7(J_C)$ exists if and only if there exists a non-zero diagonal element in A^m. By calculating the powers of A, we can see that there are non-zero diagonal elements in A^m if and only if $m \notin \{3,4,5,7,8,9,11\}$. Therefore $\sigma(J_m) = 7$ for $m \notin \{3,4,5,7,8,9,11\}$ and $\sigma(J_m) \geq 8$ for $m \in \{3,4,5,7,8,9,11\}$.

For $r = 8$, we can find r-$S(2,1)$-labelings of J_C that can form a concatenable cyclic m-tuple of r-$S(2,1)$-labelings of J_C for every $m \geq 3$. Therefore $\sigma(J_m) = 8$ for $m \in \{3,4,5,7,8,9,11\}$.

Summarizing previous results, we obtain the following theorem.

Theorem 4.1. *[2, Theorem 4.1] Let J_m be an Isaacs graph of order $4m$, for $m \geq 3$. Then*

$$\sigma(J_m) = \begin{cases} 7 & \text{for } m \notin \{3,4,5,7,8,9,11\} \\ 8 & \text{for } m \in \{3,4,5,7,8,9,11\}. \end{cases}$$

5 Remarks

The presented strategy can be used to calculate the σ-number of other graphs with cyclic structure. Besides this, after minor modifications, it can be used to determine also the σ-number of graphs with nearly cyclic structure – for instance the σ-number of the generalized Blanuša snarks investigated in a subsequent paper.

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