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Central limit theorem on MV-algebras

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Abstract

The aim is to approve the Central limit theorem on MV-algebras by the new approach, presented by Riečan in [1]. We use the observable as a distribution function instead of the σ-homomorphism. The main idea is in local representation of σ-algebras.
The following theorem is proved: Let M be a σ-complete MV-algebra with product, $m : M \longrightarrow \langle 0, 1\rangle$ be a σ- additive state, $(x_n)_n$ be a sequence of independent, equally distributed, square integrable strong observables. Hence $E\left[x_1\right] = E\left[x_2\right] = ... = a$, $\sigma(x_1) = \sigma(x_2) = ... = \sigma$. Then for any t $\in R$

$$\lim_{n\to\infty} m\left(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i - a}{\frac{\sigma}{\sqrt{n}}}\left((-\infty, t)\right)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}}du$$

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Seeing that the paper [1] haven't been published yet, let us summarize the main ideas of Riečan's new approach to probability on MV-algebras. We will do it in the first two paragraphs. In Paragraph 3 will we present our main result - a proof of the central limit theorem.

1 MV-algebras

We shall use an excellent characterization of MV-algebras given by D . Mundici ([3]) by the help of l-groups. An l-group is an algebraic system

$$(G, +, \leq)$$

such that

$(G, +)$ is an Abelian group,

$(G, \leq)$ is a partially ordered set being a lattice,

$a \leq b \Longrightarrow a + c \leq b + c$ for and a, b, c in G.

A typical example of an MV-algebra is the unit interval $\langle 0, 1\rangle$ of real numbers with two binary operations $\oplus, \odot$,

$$a \oplus b = (a + b) \wedge 1$$
$$a \odot b = (a + b - 1) \vee 0$$

with the partial ordering $\leq$, and two fixed elements $0, 1$ (0 is the least element of $\leq$, 1 is the greatest element of $\leq$).

Generally the situation is analogous.

Definition 1.1. An MV-algebra is an algebraic system

$$(M, \oplus, \odot, \leq, 0, u),$$

where

$$M = \langle 0, u\rangle$$

is an interval in an l-group $G = (G, +, \leq), 0$ is the neutral element of G (i.e. $a + 0 = a$ for any $a \in G$), u is the strong unit of G (i.e. to any $a \in G$ there exists $n \in N$ such that $a \leq u + u + ... + u(n\text{-times})$),

$$\begin{aligned} a \oplus b &= (a + b) \wedge u, \\ a \odot b &= (a + b - u) \vee 0. \end{aligned}$$

In two-valued logic Boolean algebras can be represented e.g. by charateristic functions $\chi_A : \Omega \to \{0, 1\}$ where,

$$\chi_A(v) = \begin{cases} 1, & \text{if } v \in A, \\ 0, & \text{if } v \notin A. \end{cases}$$

In multi-valued logic MV-algebras instead of two-valued functions

$$\chi_A : \Omega \to \{0, 1\},$$

multivalued functions

$$\mu_A : \Omega \to \langle 0, 1\rangle$$

are considered. Evidently

$$\chi_A \oplus \chi_B = \chi_{A \cup B}, \chi_A \odot \chi_B = \chi_{A \cap B}.$$

Hence $\mu_A \odot \mu_B$ can be considered as the conjunction, $\mu_A \oplus \mu_B$ as the disjunction, $1 - \mu_A$ as the negation.

Unlike in [4], Riečan uses more general definition of a state and an observable in [1].

Definition 1.2. A state on an MV-algebra M is a mapping $m : M \to \langle 0, 1\rangle$ satisfying the following conditions:

(i) $m(u) = 1, m(0) = 0$;

(ii) $a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a)$;

(iii) (iii) $a_n \searrow a \Longrightarrow m(a_n) \searrow m(a)$.

Definition 1.3. Let $\mathcal{J} = \{(-\infty, t); t \in R\}$. An observable on M is any mapping $x : \mathcal{J} \to M$ satisfying the conditions:

(i) $t_n \nearrow \infty \Longrightarrow x((-\infty, t_n)) \nearrow u$;

(ii) $t_n \searrow -\infty \Longrightarrow x((-\infty, t_n)) \searrow 0$;

(iii) $t_n \nearrow t \Longrightarrow x((-\infty, t_n)) \nearrow x((-\infty, t))$.

Thus, a distribution function can be created as a composition of the observable and the state (see also [1]).

Theorem 1.4. *Let $m : M \to \langle 0, 1\rangle$ be a state, $x : \mathcal{J} \to M$ be an observable. Define $F : R \to \langle 0, 1\rangle$ by the formula*

$$F(t) = m(x((-\infty, t))), t \in R$$

Then F has the following properties:

(i) F is non-decreasing;

(ii) $\lim_{t\to\infty} F(t) = 1$;

(iii) $\lim_{t\to-\infty} F(t) = 0$;

(iv) F is left continuous in any point $t \in R$.

Proof is straightforward.

Recall that in the Kolmogorov theory the mean value $E(\xi)$ of a random variable $\xi : (\Omega, \mathcal{S}, P) \to R$ is defined as an integral

$$E(\xi) = \int_\Omega \xi dP.$$

Let $g : R \to R$ be a Borel measurable function. The transformation formula states

$$E(g \circ \xi) = \int_R g(t)dF(t),$$

where F is the distribution function of ξ. Therefore

$$\begin{aligned} E(\xi) &= \int_R tdF(t), \\ E(\xi^2) &= \int_R t^2 dF(t), \\ \sigma(\xi)^2 &= E(\xi^2) - E(\xi)^2 = \int_R t^2 dF(t) - (\int_R tdF(t))^2. \end{aligned}$$

Definition 1.5. An observable $x : \mathcal{J} \to M$ is called to be integrable if there exists

$$E(x) = \int_R t dF(t),$$

where $F : R \to \langle 0, 1\rangle$ is the distribution function of the observable x. The observable x is square integrable, if there exists

$$\int_R t^2 dF(t).$$

2 MV algebras with product

Similarly as in [2] we shall work with a further binary operation called product (see also [4] and [5]). Recall that an MV algebra in an interval is an l-group $(G, +, \leq)$ and denote by $-$ the inverse group operation, i.e. $a - a = 0$ for any $a \in G$.

Definition 2.1. An MV-algebra with product is a pair$(M, \cdot)$, where M is an MV-algebra and $\cdot$ is a commutative and associative binary operation on M satisfying the following conditions:

(i) $u \cdot a = a$ for any $a \in M$;

(ii) $a \cdot ((b - c) \vee 0) = (a \cdot b - a \cdot c) \vee 0$ for any $a, b, c \in M$.

The first problem we shall solve is the construction of sums of observables. In the classical case

$$(\xi + \eta) = g \circ T, \text{ where } T = (\xi, \eta)\,;\ \Omega \longrightarrow R^2,\ g : R^2 \longrightarrow R,\ g(u, v) = u + v.$$

Therefore

$$(\xi + \eta)^{-1} = T^{-1} \circ g^{-1}.$$

Of course,

$$g^{-1}\left((-\infty, t)\right) = \{(u, v)\,; u + v < t\} = D_t.$$

Hence instead of T^{-1} we shall construct a mapping

$$h : \{D_t; t \in R\} \longrightarrow M.$$

We will use the notation

$$x\left(\langle a, b)\right) = x\left((-\infty, b)\right) - x\left((-\infty, a)\right)$$

Definition 2.2. For $t \in R$ put

$$h\left(D_t\right) = \bigvee_{n=1}^{\infty} \bigoplus_{i=-\infty}^{\infty} \left[x\left(\langle \frac{i-1}{2^n}, \frac{i}{2^n})\right) \cdot y\left(\left(-\infty, t - \frac{i}{2^n}\right)\right) \right]$$

Theorem 2.3. *Let M be a σ-complete MV-algebra with product, $x, y : \mathcal{J} \longrightarrow M$ be observables, then the mapping $z : \mathcal{J} \longrightarrow M$ defined by*

$$z\left((-\infty, t)\right) = h\left(D_t\right),$$

is an observable.

Proof. The properties (i) and (ii) of Definition 1.3 follows by the inequalities

$$x\left((-\infty,k)\right).y\left((-\infty,k)\right)\leq z\left((-\infty,2k)\right)\leq x\left((-\infty,2k)\right)\vee y\left((-\infty,k)\right)$$

holding for any $k\in Z$. By [4, Proposition 3.2] we have

$$c.\ (a\vee b)=(c\,.a)\vee(c.\,b)$$

for any $a,b,c\in M$. Therefore, if $t_k\nearrow t$, then

$$\bigoplus_{k=1}^{\infty}z\left((-\infty,t_k)\right)=\bigoplus_{k=1}^{\infty}\bigvee_{n=1}^{\infty}\bigoplus_{i=-\infty}^{\infty}x\left(\langle\frac{i-1}{2^n},\frac{i}{2^n})\right).y\left(\left(-\infty,t_k-\frac{i}{2^n}\right)\right)=$$

$$=\bigvee_{n=1}^{\infty}\bigoplus_{i=-\infty}^{\infty}x\left(\langle\frac{i-1}{2^n},\frac{i}{2^n})\right).\left(\bigoplus_{k=1}^{\infty}y\left(\left(-\infty,t_k-\frac{i}{2^n}\right)\right)\right)=z\left((-\infty,t)\right).$$

□

Definition 2.4. Let M be a σ-complete MV-algebra with product, $x,y:\mathcal{J}\longrightarrow M$ be observables. Then its sum is defined by the formula

$$(x+y)\left((-\infty,t)\right)=h\left(D_t\right)=h\left(g^{-1}\left((-\infty,t)\right)\right).$$

Remark 2.5. Evidently Theorem 2.3 and Definition 2.4 can be generalized for n summs and $x_1,...,x_n:\mathcal{J}\longrightarrow M$:

$$D_t^n=\left\{(m_1,...,m_n)\,;m_1+...+m_n<t\right\},M_n=\left\{D_t^n;\,t\in R\right\},$$

$$g_n\left(m_1,...,m_n\right)=m_1+...+m_n$$

$$h_n:M_n\longrightarrow M,x_1+...+x_n\left((-\infty,t)\right)=h_n\left(D_t^n\right)=h_n\left(g_n^{-1}\left((-\infty,t)\right)\right),$$

hence

$$\left(\sum_{i=1}^{n}x_i\right)\left((-\infty,t)\right)=h_n\left(g_n^{-1}\left((-\infty,t)\right)\right),$$

$$\sum_{i=1}^{n}x_i=h_n\circ g_n^{-1}.$$

The second problem solved in the section is a characterization of independence of observables in MV-algebras.

Definition 2.6. Observables $x_1,...,x_n$ are independent, if for any $t_1,...,t_n\in R$

$$m\left(h_n\left((-\infty,t_1)\times(-\infty,t_2)\times...\times(-\infty,t_n)\right)\right)=$$

$$=m\left(x_1\left((-\infty,t_1)\right)\right).m\left(x_2\left((-\infty,t_2)\right)\right).\,...\,.m\left(x_n\left((-\infty,t_n)\right)\right).$$

Remark 2.7. In the Kolmogorov theory also the following two assertions hold:

1. If $F : R \longrightarrow \langle 0,1\rangle$ is a distribution function, then there exists exactly one probability measure $\lambda_F : \mathcal{B}(R) \longrightarrow \langle 0,1\rangle$ such that

$$\lambda_F(\langle a,b)) = F(b) - F(a)$$

for any $a, b \in R,\ a \leq b$

2. If $F_1, ..., F_n : R \longrightarrow \langle 0,1\rangle$ are distribution functions, then there exists exactly one probability measure $\lambda_F : \mathcal{B}(R^n) \longrightarrow \langle 0,1\rangle$ such that

$$\lambda_F(A_1 \times A_1 \times ... \times A_n) = \lambda_{F_1}(A_1) . \lambda_{F_2}(A_2) . ... . \lambda_{F_n}(A_n)$$

for any $A_1, A_2, ..., A_n \in \mathcal{B}(R)$. Notation $\lambda_F = \lambda_{F_1} \times \lambda_{F_2} \times ... \times \lambda_{F_n}$.

Definition 2.8. An observable $x : \mathcal{J} \longrightarrow M$ is called strong, if

$$\langle a,b) \cap \langle c,d) = \emptyset \Longrightarrow (x(\langle a,b)) .\alpha) . (x(\langle c,d)) .\beta) = 0$$

for any $\alpha, \beta \in M$.

Definition 2.9. A state $m : M \longrightarrow \langle 0,1\rangle$ is called σ-additive, if

$$m\left(\bigoplus_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} m(a_n)$$

whenever $a_n \wedge a_m = 0\ (n \neq m), a_n \in M$.

Theorem 2.10. *Let M be σ- complete MV-algebra with product, m be a σ-additive state, $x_1, \ldots, x_n$ independent strong observables. Then*

$$\lambda_{F_1} \times ... \times \lambda_{F_n}(D_t^n) = m(h(D_t^n))$$

for any $t \in R$.

Proof. We have

$$\begin{aligned}
D_t &= \bigcup_{n=1}^{\infty} \bigcup_{i=-\infty}^{\infty} \langle \frac{i-1}{2^n}, \frac{i}{2^n}\rangle \times \left(-\infty, t - \frac{i}{2^n}\right) \\
h(D_t) &= \bigvee_n \bigoplus_i x_1\left(\langle \frac{i-1}{2^n}, \frac{i}{2^n}\right)\right) \cdot x_2\left(\left(-\infty, t - \frac{i}{2^n}\right)\right) \\
\lambda_{F_1}(\langle a,b)) &= F_1(b) - F_1(a) \\
F_1(b) &= m(x_1((-\infty, b)))
\end{aligned}$$

We shall present it for $n = 2$. Of course, since x_1 is strong,

$$x_1\left(\langle \frac{i-1}{2^n}, \frac{i}{2^n}\right)\right) \cdot x_2\left(\left(-\infty, t - \frac{i}{2^n}\right)\right) . x_1\left(\langle \frac{j-1}{2^n}, \frac{j}{2^n}\right)\right) \cdot x_2\left(\left(-\infty, t - \frac{j}{2^n}\right)\right) =$$

$$= x_1(\langle a,b)) .\alpha . x_1(\langle c,d)) \cdot \beta = 0$$

for $i \neq j$. Therefore

$$m\left(\bigoplus_{i=-\infty}^{\infty} x_1\left(\left\langle\frac{i-1}{2^n},\frac{i}{2^n}\right)\right)\cdot x_2\left(\left(-\infty,t-\frac{i}{2^n}\right)\right)\right)=$$

$$=\sum_{i=-\infty}^{\infty} m\left(x_1\left(\langle\frac{i-1}{2^n},\frac{i}{2^n}\right)\right)\cdot x_2\left(\left(-\infty,t-\frac{i}{2^n}\right)\right)$$

hence

$$\lambda_{F_1}\times\lambda_{F_2}\left(D_t\right)=\lambda_{F_1}\times\lambda_{F_2}\left(\bigcup_{n=1}^{\infty}\bigcup_{i=-\infty}^{\infty}\left(\langle\frac{i-1}{2^n},\frac{i}{2^n}\right)\times\left(-\infty,t-\frac{i}{2^n}\right)\right)=$$

$$=\lim_{n\to\infty}\sum_{i=-\infty}^{\infty}\lambda_{F_1}\times\lambda_{F_2}\left(\langle\frac{i-1}{2^n},\frac{i}{2^n}\right)\times\left(-\infty,t-\frac{i}{2^n}\right)\right)=$$

$$=\lim_{n\to\infty}\sum_{i=-\infty}^{\infty}\lambda_{F_1}\left(\langle\frac{i-1}{2^n},\frac{i}{2^n}\right)\right).\lambda_{F_2}\left(\left(-\infty,t-\frac{i}{2^n}\right)\right)=$$

$$=\lim_{n\to\infty}\sum_{i=-\infty}^{\infty} m\left(x_1\left(\langle\frac{i-1}{2^n},\frac{i}{2^n}\right)\right)\right).m\left(x_2\left(\left(-\infty,t-\frac{i}{2^n}\right)\right)\right)=$$

$$=\lim_{n\to\infty}\sum_{i=-\infty}^{\infty} m\left(x_1\left(\langle\frac{i-1}{2^n},\frac{i}{2^n}\right)\right).x_2\left(\left(-\infty,t-\frac{i}{2^n}\right)\right)\right).$$

$$=m\left(\bigvee_{n=1}^{\infty}\bigoplus_{i=-\infty}^{\infty} x\left(\langle\frac{i-1}{2^n},\frac{i}{2^n}\right)\right)\cdot\left(\bigoplus_{k=1}^{\infty} y\left(\left(-\infty,t_k-\frac{i}{2^n}\right)\right)\right)\right)=z\left((-\infty,t)\right)$$

$$=m\left(h_2\left(D_t^n\right)\right).$$

□

3 Central limit theorem

We have founded our reasoning on the Kolmogorov theory.

Theorem 3.1. *Let $(\Omega,\mathcal{S},P)$ be a probability space, $(\xi_n)_n$ be a sequence of independent, square integrable, equally distributed random variables. Let $E(\xi_1)=E(\xi_2)=a,\sigma(\xi_1)=\sigma(\xi_2)=...=\sigma$. Then for any $t\in R$*

$$\lim_{n\to\infty} P\left(\frac{\left(\frac{1}{n}\sum_{i=1}^n\xi_i\right)-a}{\frac{\sigma}{\sqrt{n}}}<t\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}e^{-\frac{u^2}{2}}du.$$

Let $(x_n)_n$ be a sequence of independent observables. We have already defined

$$\sum_{i=1}^{n}x_i((-\infty,t))=h_n\circ g_n^{-1}((-\infty,t)),$$

where

$$g_n(u_1, ..., u_n) = u_1 + ... + u_n.$$

And we have to define

$$\frac{\left(\frac{1}{n}\sum_{i=1}^{n}\xi_i - a\right)}{\frac{\sigma}{\sqrt{n}}}\left((-\infty, t)\right).$$

Where

$$\frac{\frac{1}{n}\sum_{i=1}^{n}\xi_i - a}{\frac{\sigma}{\sqrt{n}}} < t$$

if and only if

$$\sum_{i=1}^{n}\xi_i < \left(t\frac{\sigma}{\sqrt{n}} + a\right)n.$$

Then we can define

Definition 3.2. Let M be a σ-complete MV-algebra with product, $x_1, ..., x_n$ be independent observables. Then we define

$$\frac{\left(\frac{1}{n}\sum_{i=1}^{n}x_i - a\right)}{\frac{\sigma}{\sqrt{n}}}\left((-\infty, t)\right) = \left(\sum_{i=1}^{n}x_i\right)\left(\left(-\infty, (t\frac{\sigma}{\sqrt{n}} + a)n\right)\right)$$

$$= h_n\left(g_n^{-1}\left(\left(-\infty, (t\frac{\sigma}{\sqrt{n}} + a)n\right)\right)\right) = h_n \circ g_n^{-1}\left(\left(-\infty, (t\frac{\sigma}{\sqrt{n}} + a)n\right)\right).$$

Theorem 3.3. *Let M be a σ-complete MV-algebra with product, $m : M \to \langle 0, 1\rangle$ be a σ-additive state, $(x_n)_n$ be a sequence of independent, equally distributed, square integrable strong observables. Denote $E[x_1] = E[x_2] = ... = a$, $\sigma(x_1) = \sigma(x_2) = ... = \sigma$. Then for any $t \in R$*

$$\lim_{n\to\infty} m\left(\frac{\frac{1}{n}\sum_{i=1}^{n}x_i - a}{\frac{\sigma}{\sqrt{n}}}\left((-\infty, t)\right)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}}du$$

Proof. Now we have to construct Kolmogorov probability space (depending on the sequence $(x_n)_n$). For any $n \in N$ define

$$\mu_n = \lambda_{F_1} \times \lambda_{F_2} \times ... \times \lambda_{F_n} : \mathcal{B}(R^n) \to \langle 0, 1\rangle.$$

Then $(\mu_n)_n$ presents a consistent system of probability measures, i.e.

$$\mu_{n+1}|\mathcal{B}(R^n) = \mu_n.$$

Let $\mathcal{C}$ be the family of all cylinders in R^N, i.e. all sets of the form

$$\{(u_n)_{n=1}^{\infty} \in R^N; u_{i_1} \in A_1, u_{i_2} \in A_2, ..., u_{i_k} \in A_k\}.$$

Then, using the Kolmogorov consistence theorem, there exists exactly one probability measure

$$P : \sigma(\mathcal{C}) \to \langle 0, 1\rangle$$

such that

$$P(\pi_n^{-1}(A)) = \mu_n(A)$$

for any $n \in N, A \in \mathcal{B}(R^n)$, where $\pi_n : R^N \to R^n$ is the projection

$$\pi_n((u_i)_{i=1}^{\infty}) = (u_1, ..., u_n).$$

Now we have obtained a probability space

$$(R^N, \sigma(\mathcal{C}), P).$$

Next, define

$$\xi_k : R^N \to R$$

by the formula

$$\xi_k((u_i)_{i=1}^{\infty}) = u_k.$$

Compute

$$P_{\xi_k}((-\infty, t)) = P(\{u_n \in R^n; u_k < t\}) =$$

$$= \lambda_{F_1} \times ... \times \lambda_{F_k}(R \times R \times ... \times R \times (-\infty, t)) =$$

$$= \lambda_{F_k}((-\infty, t)).$$

hence $\xi_k : R^N \to R$ and $x_k : \mathcal{J} \to M$ have the same distribution function. Especially

$$E(\xi_k) = \int_R t dF_k(t) = E(x_k).$$

Moreover, put

$$\eta_n = \frac{\frac{1}{n}\sum_{i=1}^{n} \xi_i - a}{\frac{\sigma}{\sqrt{n}}}$$

and

$$y_n = \frac{\frac{1}{n}\sum_{i=1}^{n} x_i - a}{\frac{\sigma}{\sqrt{n}}}$$

i.e.

$$\eta_n^{-1}((-\infty, t)) = \pi_n^{-1}\left(g_n^{-1}\left(\left(-\infty, (t\frac{\sigma}{\sqrt{n}} + a)n\right)\right)\right)$$

and

$$y_n\left((-\infty, t)\right) = \left(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i - a}{\frac{\sigma}{\sqrt{n}}}\right)((-\infty, t)) =$$

$$= h_n\left(g_n^{-1}\left(\left(-\infty,(t\frac{\sigma}{\sqrt{n}}+a)n\right)\right)\right)$$

Then

$$P\left(\eta_n^{-1}\left((-\infty,t)\right)\right) = P\left(\pi_n^{-1}\left(g_n^{-1}\left(\left(-\infty,(t\frac{\sigma}{\sqrt{n}}+a)n\right)\right)\right)\right) =$$

$$= m\left(h_n\left(g_n^{-1}\left(\left(-\infty,(t\frac{\sigma}{\sqrt{n}}+a)n\right)\right)\right)\right) = m\left(y_n\left((-\infty,t)\right)\right).$$

Now we shall prove that $(\xi_n)_n$ are independent:

$$P\left(\xi_1^{-1}((-\infty,t_1)) \cap \xi_2^{-1}((-\infty,t_2)) \cap ... \cap \xi_n^{-1}((-\infty,t_n))\right) =$$

$$= P \circ \pi_n^{-1}\left((-\infty,t_1) \times (-\infty,t_2) \times ... \times (-\infty,t_n)\right) =$$

$$= m \circ h_n\left((-\infty,t_1) \times (-\infty,t_2) \times ... \times (-\infty,t_n)\right) \;) =$$

$$= m\left(h_n\left((-\infty,t_1) \times (-\infty,t_2) \times ... \times (-\infty,t_n)\right)\right) =$$

$$= m\left(x_1((-\infty,t_1))\right).m\left(x_2((-\infty,t_2))\right). ... .m\left(x_n((-\infty,t_n))\right) =$$

$$= P\left(\xi_1^{-1}((-\infty,t_1))\right).P\left(\xi_2^{-1}((-\infty,t_2))\right). ... .P\left(\xi_n^{-1}((-\infty,t_n))\right).$$

Therefore by Theorem 3.1, Theorem 2.10 and the statement

$$P(\eta_n^{-1}((-\infty,t))) = m\left(y_n((-\infty,t))\right),$$

for $a = E(\xi_1) = E(\xi_2) = ... = E(\xi_n)$ we have

$$\lim_{n\to\infty} P\left(\frac{\frac{1}{n}\sum_{i=1}^{n}\xi_i - a}{\frac{\sigma}{\sqrt{n}}} < t\right) = \lim_{n\to\infty} P\left(\eta_n^{-1}\left((-\infty,t)\right)\right) =$$

$$= \lim_{n\to\infty} P\left(\pi_n^{-1}\left(g_n^{-1}\left(\left(-\infty,(t\frac{\sigma}{\sqrt{n}}+a)n\right)\right)\right)\right) =$$

$$= \lim_{n\to\infty} P \circ \pi_n^{-1}\left(g_n^{-1}\left(\left(-\infty,(t\frac{\sigma}{\sqrt{n}}+a)n\right)\right)\right) =$$

$$= \lim_{n\to\infty} m\left(h_n\left(g_n^{-1}\left(\left(-\infty,(t\frac{\sigma}{\sqrt{n}}+a)n\right)\right)\right)\right) =$$

$$= \lim_{n\to\infty} m\left(y_n((-\infty,t))\right) = \lim_{n\to\infty} m\left(\frac{\frac{1}{n}\sum_{i=1}^{n}x_i - a}{\frac{\sigma}{\sqrt{n}}}\left((-\infty,t)\right)\right) =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}}\,du.$$

□

Remark 3.4. We have shown, that on σ-complete MV-algebra with product, each observable corresponds to a random variable with the same distribution, and a couple of independent observables corresponds to a couple of independent random variables. This way all version of central limit theorems can be translated into the languange of σ-complete MV-algebras with product. However, that was not the aim of this paper.

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