ACTA UNIVERSITATIS MATTHIAE BELII, series MATHEMATICS 19 (2011), 11–20
Also available at http://actamath.savbb.sk
ISBN-978-80-557-0283-4

On a new approach towards defining intuitionistic fuzzy subtractions

Krassimir Atanassov
Institute of Biophysics and Biomedical Engineering,
Bulgarian Academy of Sciences
105 Acad. G. Bonchev Str., 1113 Sofia, Bulgaria
E-mail: krat@bas.bg

To my friend Prof. Beloslav Riečan

Abstract

A new set of operations subtraction over intuitionistic fuzzy sets are defined and some of their basic properties are studied.

Keywords Intuitionistic fuzzy set, Operation, Subtraction

MSC (2000) 03E72

Received: 23. 8. 2011; Accepted in final form: 17. 10. 2011.

Communicated with Magdaléna Renčová.

1 Introduction

In a series of papers, part of which written together with Prof. Beloslav Riečan, the concept of "substraction" operation over an Intitionistic Fuzzy Set (IFS, see [1]), was introduced for the first time (see, [2, 3, 4, 5, 6, 7, 8]).

In the first two papers [5, 6], we offered direct definitions of subtractions. Later, an approach providing a series of definitions was introduced and 67 different instances of the "substraction" operation were constructed and their properties were studied. B. Riečan participated actively in this research [7, 8].

Now, a new approach to defining different "substraction" operations is constructed and some of the basic properties of the derived new instances will be studied.

2 Some preliminary results

Up to now, different operations have been defined over IFS. Let

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where the functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ stand for the degrees of membership and non-membership of the element x from a fixed universe E to the set $A \subset E$, respectively, and every x satisfies that: $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Let for every $x \in E$:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x).$$

Therefore, function π determines the degree of uncertainty.

Below, for brevity, we write A instead of A^*. When the IFSs A and B are given, we can construct the IFS $A - B$. The currently existing forms of this operation are given below. The first two forms are taken, respectively, from [5] and [6] and we will denote them as $BR1$ and $BR2$:

$$A -_{BR1} B = \{\langle x, \mu_{A-B}(x), \nu_{A-B}(x)\rangle | x \in E\},$$

where

$$\mu_{A-B}(x) = \begin{cases} \dfrac{\mu_A(x) - \mu_B(x)}{1 - \mu_B(x)}, & \text{if } \mu_A(x) \geq \mu_B(x) \text{ and } \nu_A(x) \leq \nu_B(x) \\ & \text{and } \nu_B(x) > 0 \\ & \text{and } \nu_A(x)\pi_B(x) \leq \pi_A(x)\nu_B(x) \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_{A-B}(x) = \begin{cases} \dfrac{\nu_A(x)}{\nu_B(x)}, & \text{if } \mu_A(x) \geq \mu_B(x) \text{ and } \nu_A(x) \leq \nu_B(x) \\ & \text{and } \nu_B(x) > 0 \\ & \text{and } \nu_A(x)\pi_B(x) \leq \pi_A(x)\nu_B(x) \\ 1, & \text{otherwise} \end{cases}$$

and

$$A -_{BR2} B = \{\langle \min(\mu_A(x), \nu_B(x)), \max(\mu_B(x), \nu_A(x))\rangle | x \in E\}.$$

In some definitions below, we use functions sg and $\overline{\text{sg}}$, defined by

$$\text{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}, \qquad \overline{\text{sg}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$$

The next definitions of instances of the "substraction" operation are based on the well-known formula from set theory:

$$A - B = A \cap \neg B$$

where A and B are given sets. In the IFS case, if the IFSs A and B are given, we define the following versions of "substraction" operation:

$$A -'_i B = A \cap \neg_i B, \text{ and } A -''_i B = \neg_i \neg_i A \cap \neg_i B,$$

where $i = 1, 2, \ldots, 34$.

Of course, for every two IFSs A and B, it is valid that

$$A -'_1 B = A -''_1 B,$$

because the first negation will satisfy the Law of Excluded Middle, but in the other cases this equality is not valid.

All new subtractions are given in Table 1.

Table 1: List of intuitionistic fuzzy subtractions.

$-'_1$	$\{\langle x, \min(\mu_A(x), \nu_B(x)), \max(\nu_A(x), \mu_B(x))\rangle \| x \in E\}$
$-'_2$	$\{\langle x, \min(\mu_A(x), \overline{\text{sg}}(\mu_B(x))), \max(\nu_A(x), \text{sg}(\mu_B(x)))\rangle \| x \in E\}$
$-'_3$	$\{\langle x, \min(\mu_A(x), \nu_B(x)),$ $\max(\nu_A(x), \mu_B(x).\nu_B(x) + \mu_B(x)^2)\rangle \| x \in E\}$
$-'_4$	$\{\langle x, \min(\mu_A(x), \nu_B(x)), \max(\nu_A(x), 1 - \nu_B(x))\rangle \| x \in E\}$
$-'_5$	$\{\langle x, \min(\mu_A(x), \overline{\text{sg}}(1 - \nu_B(x))),$ $\max(\nu_A(x), \text{sg}(1 - \nu_B(x)))\rangle \| x \in E\}$
$-'_6$	$\{\langle x, \min(\mu_A(x), \overline{\text{sg}}(1 - \nu_B(x))), \max(\nu_A(x), \text{sg}(\mu_B(x)))\rangle \| x \in E\}$
$-'_7$	$\{\langle x, \min(\mu_A(x), \overline{\text{sg}}(1 - \nu_B(x))), \max(\nu_A(x), \mu_B(x))\rangle \| x \in E\}$
$-'_8$	$\{\langle x, \min(\mu_A(x), 1 - \mu_B(x)), \max(\nu_A(x), \mu_B(x))\rangle \| x \in E\}$
$-'_9$	$\{\langle x, \min(\mu_A(x), \overline{\text{sg}}(\mu_B(x))), \max(\nu_A(x), \mu_B(x))\rangle \| x \in E\}$
$-'_{10}$	$\{\langle x, \min(\mu_A(x), \overline{\text{sg}}(1 - \nu_B(x))), \max(\nu_A(x), 1 - \nu_B(x))\rangle \| x \in E\}$
$-'_{11}$	$\{\langle x, \min(\mu_A(x), \text{sg}(\nu_B(x))), \max(\nu_A(x), \overline{\text{sg}}(\nu_B(x)))\rangle \| x \in E\}$
$-'_{12}$	$\{\langle x, \min(\mu_A(x), \nu_B(x).(\mu_B(x) + \nu_B(x))),$ $\max(\nu_A(x), \mu_B(x).(\nu_B(x)^2 + \mu_B(x) + \mu_B(x).\nu_B(x)))\rangle \| x \in E\}$
$-'_{13}$	$\{\langle x, \min(\mu_A(x), \text{sg}(1 - \mu_B(x))),$ $\max(\nu_A(x), \overline{\text{sg}}(1 - \mu_B(x)))\rangle \| x \in E\}$
$-'_{14}$	$\{\langle x, \min(\mu_A(x), \text{sg}(\nu_B(x))), \max(\nu_A(x), \overline{\text{sg}}(1 - \mu_B(x)))\rangle \| x \in E\}$
$-'_{15}$	$\{\langle x, \min(\mu_A(x), \overline{\text{sg}}(1 - \nu_B(x))),$ $\max(\nu_A(x), \overline{\text{sg}}(1 - \mu_B(x)))\rangle \| x \in E\}$
$-'_{16}$	$\{\langle x, \min(\mu_A(x), \overline{\text{sg}}(\mu_B(x))), \max(\nu_A(x), \overline{\text{sg}}(1 - \mu_B(x)))\rangle \| x \in E\}$
$-'_{17}$	$\{\langle x, \min(\mu_A(x), \overline{\text{sg}}(1 - \nu_B(x))), \max(\nu_A(x), \overline{\text{sg}}(\nu_B(x)))\rangle \| x \in E\}$
$-'_{18}$	$\{\langle x, \min(\mu_A(x), \nu_B(x), \text{sg}(\mu_B(x))),$ $\max(\nu_A(x), \min(\mu_B(x), \text{sg}(\nu_B(x))))\rangle \| x \in E\}$
$-'_{19}$	$\{\langle x, \min(\mu_A(x), \nu_B(x), \text{sg}(\mu_B(x))), \nu_A(x)\rangle \| x \in E\}$
$-'_{20}$	$\{\langle x, \min(\mu_A(x), \nu_B(x)), \nu_A(x)\rangle \| x \in E\}$
$-'_{21}$	$\{\langle x, \min(\mu_A(x), 1 - \mu_B(x), \text{sg}(\mu_B(x))),$ $\max(\nu_A(x), \min(\mu_B(x), \text{sg}(1 - \mu_B(x))))\rangle \| x \in E\}$
$-'_{22}$	$\{\langle x, \min(\mu_A(x), 1 - \mu_B(x), \text{sg}(\mu_B(x))), \nu_A(x)\rangle \| x \in E\}$
$-'_{23}$	$\{\langle x, \min(\mu_A(x), 1 - \mu_B(x)), \nu_A(x)\rangle \| x \in E\}$
$-'_{24}$	$\{\langle x, \min(\mu_A(x), \nu_B(x), \text{sg}(1 - \nu_B(x))),$ $\max(\nu_A(x), \min(1 - \nu_B(x), \text{sg}(\nu_B(x))))\rangle \| x \in E\}$
$-'_{25}$	$\{\langle x, \min(\mu_A(x), \nu_B(x), \text{sg}(1 - \nu_B(x))), \nu_A(x)\rangle \| x \in E\}$
$-'_{26}$	$\{\langle x, \min(\mu_A(x), \nu_B(x)),$ $\max(\nu_A(x), \mu_B(x).\nu_B(x) + \overline{\text{sg}}(1 - \mu_B(x)))\rangle \| x \in E\}$
$-'_{27}$	$\{\langle x, \min(\mu_A(x), 1 - \mu_B(x)),$ $\max(\nu_A(x), \mu_B(x).(1 - \mu_B(x)) + \overline{\text{sg}}(1 - \mu_B(x)))\rangle \| x \in E\}$

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$-'_{28}$	$\{\langle x, \min(\mu_A(x), \nu_B(x)),$ $\max(\nu_A(x), (1-\nu_B(x)).\nu_B(x) + \overline{\rm sg}(\nu_B(x)))\rangle \mid x \in E\}$
$-'_{29}$	$\{\langle x, \min(\mu_A(x), \max(0, \mu_B(x).\nu_B(x) + \overline{\rm sg}(1-\nu_B(x)))),$ $\max(\nu_A(x), \mu_B(x).(\mu_B(x).\nu_B(x)$ $+\overline{\rm sg}(1-\nu_B(x))) + \overline{\rm sg}(1-\mu_B(x)))\rangle \mid x \in E\}$
$-'_{30}$	$\{\langle x, \min(\mu_A(x), \mu_B(x).\nu_B(x)),$ $\max(\nu_A(x), \mu_B(x).(\mu_B(x).\nu_B(x)$ $+\overline{\rm sg}(1-\nu_B(x))) + \overline{\rm sg}(1-\mu_B(x)))\rangle \mid x \in E\}$
$-'_{31}$	$\{\langle x, \min(\mu_A(x), (1-\mu_B(x)).\mu_B(x) + \overline{\rm sg}(\mu_B(x))),$ $\max(\nu_A(x), \mu_B(x).((1-\mu_B(x)).\mu_B(x)$ $+\overline{\rm sg}(\mu_B(x))) + \overline{\rm sg}(1-\mu_B(x)))\rangle \mid x \in E\}$
$-'_{32}$	$\{\langle x, \min(\mu_A(x), (1-\mu_B(x)).\mu_B(x)),$ $\max(\nu_A(x), \mu_B(x).((1-\mu_B(x)).\mu_B(x)$ $+\overline{\rm sg}(\mu_B(x))) + \overline{\rm sg}(1-\mu_B(x)))\rangle \mid x \in E\}$
$-'_{33}$	$\{\langle x, \min(\mu_A(x), \nu_B(x).(1-\nu_B(x)) + \overline{\rm sg}(1-\nu_B(x))),$ $\max(\nu_A(x), (1-\nu_B(x)).(\nu_B(x).(1-\nu_B(x))$ $+\overline{\rm sg}(1-\nu_B(x))) + \overline{\rm sg}(\nu_B(x)))\rangle \mid x \in E\}$
$-'_{34}$	$\{\langle x, \min(\mu_A(x), \nu_B(x).(1-\nu_B(x))),$ $\max(\nu_A(x), (1-\nu_B(x)).(\nu_B(x).(1-\nu_B(x))$ $+\overline{\rm sg}(1-\nu_B(x))) + \overline{\rm sg}(\nu_B(x)))\rangle \mid x \in E\}$

$-''_{1}$	$\{\langle x, \min(\mu_A(x), \nu_B(x)), \max(\nu_A(x), \mu_B(x))\rangle \mid x \in E\}$
$-''_{2}$	$\{\langle x, \min({\rm sg}(\mu_A(x)), \overline{\rm sg}(\mu_B(x))),$ $\max(\overline{\rm sg}(\mu_A(x)), {\rm sg}(\mu_B(x)))\rangle \mid x \in E\}$
$-''_{3}$	$\{\langle x, \min(\mu_A(x).\nu_A(x) + \mu_A(x)^2, \nu_B(x)),$ $\max(\nu_A(x).(\mu_A(x).\nu_A(x) + \mu_A(x)^2) + \nu_A(x)^2,$ $\mu_B(x).\nu_B(x) + \mu_B(x)^2)\rangle \mid x \in E\}$
$-''_{4}$	$\{\langle x, \min(1-\nu_A(x), \nu_B(x)), \max(\nu_A(x), 1-\nu_B(x))\rangle \mid x \in E\}$
$-''_{5}$	$\{\langle x, \min({\rm sg}(1-\nu_A(x)), \overline{\rm sg}(1-\nu_B(x))),$ $\max(\overline{\rm sg}(1-\nu_A(x)), {\rm sg}(1-\nu_B(x)))\rangle \mid x \in E\}$
$-''_{6}$	$\{\langle x, \min({\rm sg}(\mu_A(x)), \overline{\rm sg}(1-\nu_B(x))),$ $\max(\overline{\rm sg}(1-\nu_A(x)), {\rm sg}(\mu_B(x)))\rangle \mid x \in E\}$
$-''_{7}$	$\{\langle x, \min(\overline{\rm sg}(1-\mu_A(x)), \overline{\rm sg}(1-\nu_B(x))),$ $\max(\overline{\rm sg}(1-\nu_A(x)), \mu_B(x))\rangle \mid x \in E\}$
$-''_{8}$	$\{\langle x, \min(\mu_A(x), 1-\mu_B(x)), \max(1-\mu_A(x), \mu_B(x))\rangle \mid x \in E\}$
$-''_{9}$	$\{\langle x, \min({\rm sg}(\mu_A(x)), \overline{\rm sg}(\mu_B(x))), \max(\overline{\rm sg}(\mu_A(x)), \mu_B(x))\rangle \mid x \in E\}$
$-''_{10}$	$\{\langle x, \min(\overline{\rm sg}(\nu_A(x)), \overline{\rm sg}(1-\nu_B(x))),$ $\max(\nu_A(x), 1-\nu_B(x))\rangle \mid x \in E\}$
$-''_{11}$	$\{\langle x, \min(\overline{\rm sg}(\nu_A(x)), {\rm sg}(\nu_B(x))),$ $\max({\rm sg}(\nu_A(x)), \overline{\rm sg}(\nu_B(x)))\rangle \mid x \in E\}$
$-''_{12}$	$\{\langle x, \min(\mu_A(x).(\nu_A(x)^2 + \mu_A(x) + \mu_A(x).\nu_A(x)).(\mu_A(x).(\nu_A(x)^2$ $+\mu_A(x) + \mu_A(x).\nu_A(x)) + (\nu_A(x).(\mu_A(x) + \nu_A(x)))),$ $\nu_B(x).(\mu_B(x) + \nu_B(x))),$ $\max(\nu_A(x).(\mu_A(x) + \nu_A(x)).(\mu_A(x)^2.(\nu_A(x)^2 + \mu_A(x)$ $+\mu_A(x).\nu_A(x))^2 + \nu_A(x).(\mu_A(x) + \nu_A(x))) + \mu_A(x).\nu_A(x)$ $.(\nu_A(x)^2 + \mu_A(x) + \mu_A(x).\nu_A(x)).(\mu_A(x) + \nu_A(x)),$ $\mu_B(x).(\nu_B(x)^2 + \mu_B(x) + \mu_B(x).\nu_B(x)))\rangle \mid x \in E\}$

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$-''_{13}$	$\{\langle x, \min(\overline{\text{sg}}(1-\mu_A(x)), \text{sg}(1-\mu_B(x))),$ $\max(\text{sg}(1-\mu_A(x)), \overline{\text{sg}}(1-\mu_B(x)))\rangle \mid x \in E\}$
$-''_{14}$	$\{\langle x, \min(\overline{\text{sg}}(1-\mu_A(x)), \text{sg}(\nu_B(x))),$ $\max(\text{sg}(\nu_A(x)), \overline{\text{sg}}(1-\mu_B(x)))\rangle \mid x \in E\}$
$-''_{15}$	$\{\langle x, \min(\overline{\text{sg}}(1-\mu_A(x)), \overline{\text{sg}}(1-\nu_B(x))),$ $\max(\overline{\text{sg}}(1-\nu_A(x)), \overline{\text{sg}}(1-\mu_B(x)))\rangle \mid x \in E\}$
$-''_{16}$	$\{\langle x, \min(\text{sg}(\mu_A(x)), \overline{\text{sg}}(\mu_B(x))),$ $\max(\overline{\text{sg}}(\mu_A(x)), \overline{\text{sg}}(1-\mu_B(x)))\rangle \mid x \in E\}$
$-''_{17}$	$\{\langle x, \min(\overline{\text{sg}}(\nu_A(x)), \overline{\text{sg}}(1-\nu_B(x))),$ $\max(\text{sg}(\nu_A(x)), \overline{\text{sg}}(\nu_B(x)))\rangle \mid x \in E\}$
$-''_{18}$	$\{\langle x, \min(\mu_A(x), \text{sg}(\nu_A(x)), \nu_B(x), \text{sg}(\mu_B(x))),$ $\max(\min(\nu_A(x), \text{sg}(\mu_A(x))), \min(\mu_B(x), \text{sg}(\nu_B(x))))\rangle \mid x \in E\}$
$-''_{19}$	$\{\langle x, 0, 0\rangle \mid x \in E\}$
$-''_{20}$	$\{\langle x, 0, 0\rangle \mid x \in E\}$
$-''_{21}$	$\{\langle x, \mu_A(x).\text{sg}(1-\mu_A(x)),$ $\max((1-\mu_A(x)).\text{sg}(\mu_A(x)), \min(\mu_B(x),$ $\text{sg}(1-\mu_B(x))))\rangle \mid x \in E\}$
$-''_{22}$	$\{\langle x, \min(\mu_A(x).\text{sg}(\mu_A(x)), 1-\mu_B(x), \text{sg}(\mu_B(x))), 0\rangle \mid x \in E\}$
$-''_{23}$	$\{\langle x, \min(\mu_A(x), 1-\mu_B(x)), 0\rangle \mid x \in E\}$
$-''_{24}$	$\{\langle x, \min(1-\nu_A(x), \text{sg}(\nu_A(x)), \nu_B(x), \text{sg}(1-\nu_B(x))),$ $\max(\nu_A(x).\text{sg}(1-\nu_A(x))), \min(1-\nu_B(x), \text{sg}(\nu_B(x)))\rangle \mid x \in E\}$
$-''_{25}$	$\{\langle x, 0, 0\rangle \mid x \in E\}$
$-''_{26}$	$\{\langle x, \min(\mu_A(x).\nu_A(x) + \overline{\text{sg}}(1-\mu_A(x)), \nu_B(x)),$ $\max(\nu_A(x).(\mu_A(x).\nu_A(x) + \overline{\text{sg}}(1-\mu_A(x))) + \overline{\text{sg}}(1-\nu_A(x)),$ $\mu_B(x).\nu_B(x) + \overline{\text{sg}}(1-\mu_B(x)))\rangle \mid x \in E\}$
$-''_{27}$	$\{\langle x, \min(\mu_A(x), 1-\mu_B(x)),$ $\max(((1-\mu_A(x)).\mu_A(x)) + \overline{\text{sg}}(\mu_A(x)),$ $\mu_B(x).(1-\mu_B(x)) + \overline{\text{sg}}(1-\mu_B(x)))\rangle \mid x \in E\}$
$-''_{28}$	$\{\langle x, \min((1-\nu_A(x)).\nu_A(x) + \overline{\text{sg}}(\nu_A(x)), \nu_B(x)),$ $\max((1-(1-\nu_A(x)).\nu_A(x)) - \overline{\text{sg}}(\nu_A(x))).((1-\nu_A(x)).\nu_A(x)$ $+\overline{\text{sg}}((1-\nu_A(x)).\nu_A(x) + \overline{\text{sg}}(\nu_A(x))),$ $(1-\nu_B(x)).\nu_B(x) + \overline{\text{sg}}(\nu_B(x)))\rangle \mid x \in E\}$
$-''_{29}$	$\{\langle x, \min((\mu_A(x).(\mu_A(x).\nu_A(x) + \overline{\text{sg}}(1-\nu_A(x))) + \overline{\text{sg}}(1-\mu_A(x)))$ $.(\mu_A(x).\nu_A(x) + \overline{\text{sg}}(1-\nu_A(x))) + \overline{\text{sg}}(1-\mu_A(x).(\mu_A(x).\nu_A(x)$ $+\overline{\text{sg}}(1-\nu_A(x))) - \overline{\text{sg}}(1-\mu_A(x))), \mu_B(x).\nu_B(x)$ $+\overline{\text{sg}}(1-\nu_B(x))), \max((\mu_A(x).\nu_A(x) + \overline{\text{sg}}(1-\nu_A(x))).((\mu_A(x)$ $.(\mu_A(x).\nu_A(x) + \overline{\text{sg}}(1-\nu_A(x))) + \overline{\text{sg}}(1-\mu_A(x))).(\mu_A(x).\nu_A(x)$ $+\overline{\text{sg}}(1-\nu_A(x))) + \overline{\text{sg}}(1-\mu_A(x).(\mu_A(x).\nu_A(x) + \overline{\text{sg}}(1-\nu_A(x)))$ $-\overline{\text{sg}}(1-\mu_A(x)))) + \overline{\text{sg}}(1-\mu_A(x).\nu_A(x) - \overline{\text{sg}}(1-\nu_A(x))), \mu_B(x)$ $.(\mu_B(x).\nu_B(x) + \overline{\text{sg}}(1-\nu_B(x))) + \overline{\text{sg}}(1-\mu_B(x)))\rangle \mid x \in E\}$
$-''_{30}$	$\{\langle x, \min(((\mu_A(x).(\mu_A(x).\nu_A(x) + \overline{\text{sg}}(1-\nu_A(x))) + \overline{\text{sg}}(1-\mu_A(x)))$ $.\mu_A(x).\nu_A(x)), \mu_B(x).\nu_B(x)),$ $\max(\mu_A(x).\nu_A(x).((\mu_A(x).(\mu_A(x).\nu_A(x) + \overline{\text{sg}}(1-\nu_A(x)))$ $+\overline{\text{sg}}(1-\mu_A(x))).\mu_A(x).\nu_A(x) + \overline{\text{sg}}(1-\mu_A(x).(\mu_A(x).\nu_A(x)$ $+\overline{\text{sg}}(1-\nu_A(x))) - \overline{\text{sg}}(1-\mu_A(x)))) + \overline{\text{sg}}(1-(\mu_A(x).\nu_A(x))),$ $\mu_B(x).(\mu_B(x).\nu_B(x) + \overline{\text{sg}}(1-\nu_B(x))) + \overline{\text{sg}}(1-\mu_B(x)))\rangle \mid x \in E\}$

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$-''_{31}$	$\{\langle x, \min((1-(1-\mu_A(x)).\mu_A(x)-\overline{\text{sg}}(\mu_A(x))).((1-\mu_A(x)).\mu_A(x)+\overline{\text{sg}}(\mu_A(x)))+\overline{\text{sg}}(((1-\mu_A(x)).\mu_A(x)+\overline{\text{sg}}(\mu_A(x)))), (1-\mu_B(x)).\mu_B(x)+\overline{\text{sg}}(\mu_B(x))), \max(((1-\mu_A(x)).\mu_A(x)+\overline{\text{sg}}(\mu_A(x))).((1-(1-\mu_A(x)).\mu_A(x)) -\overline{\text{sg}}(\mu_A(x)).((1-\mu_A(x)).\mu_A(x)+\overline{\text{sg}}(\mu_A(x)))+\overline{\text{sg}}((1-\mu_A(x)).\mu_A(x)+\overline{\text{sg}}(\mu_A(x))))+\overline{\text{sg}}(1-(1-\mu_A(x)).\mu_A(x)-\overline{\text{sg}}(\mu_A(x))), \mu_B(x).((1-\mu_B(x)).\mu_B(x)+\overline{\text{sg}}(\mu_B(x)))+\overline{\text{sg}}(1-\mu_B(x)))\rangle \vert x \in E\}$
$-''_{32}$	$\{\langle x, \min((1-(1-\mu_A(x)).\mu_A(x)).(1-\mu_A(x)).\mu_A(x), (1-\mu_B(x)).\mu_B(x)), \max(((1-\mu_A(x)).\mu_A(x).((1-(1-\mu_A(x)).\mu_A(x)).(1-\mu_A(x)).\mu_A(x)+\overline{\text{sg}}((1-\mu_A(x)).\mu_A(x)))+\overline{\text{sg}}(1-(1-\mu_A(x)).\mu_A(x))), \mu_B(x).((1-\mu_B(x)).\mu_B(x)+\overline{\text{sg}}(\mu_B(x)))+\overline{\text{sg}}(1-\mu_B(x)))\rangle \vert x \in E\}$
$-''_{33}$	$\{\langle x, \min(((1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x))) +\overline{\text{sg}}(\nu_A(x))).(1-(1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x)) +\overline{\text{sg}}(1-\nu_A(x)))-\overline{\text{sg}}(\nu_A(x)))+\overline{\text{sg}}(1-(1-\nu_A(x)) .(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))-\overline{\text{sg}}(\nu_A(x))), \nu_B(x).(1-\nu_B(x))+\overline{\text{sg}}(1-\nu_B(x))), \max((1-(1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x))) -\overline{\text{sg}}(\nu_A(x))).(((1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x))) +\overline{\text{sg}}(\nu_A(x))).(1-(1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x))) -\overline{\text{sg}}(\nu_A(x)))+\overline{\text{sg}}(1-(1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x)) +\overline{\text{sg}}(1-\nu_A(x)))-\overline{\text{sg}}(\nu_A(x))))+\overline{\text{sg}}((1-\nu_A(x)).(\nu_A(x) .(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))+\overline{\text{sg}}(\nu_A(x))), (1-\nu_B(x)).(\nu_B(x) .(1-\nu_B(x))+\overline{\text{sg}}(1-\nu_B(x)))+\overline{\text{sg}}(\nu_B(x)))\rangle \vert x \in E\}$
$-''_{34}$	$\{\langle x, \min(((1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x))) +\overline{\text{sg}}(\nu_A(x))).(1-(1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x)) +\overline{\text{sg}}(1-\nu_A(x)))-\overline{\text{sg}}(\nu_A(x))), \nu_B(x).(1-\nu_B(x))), \max(((1-(1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x)) +\overline{\text{sg}}(1-\nu_A(x)))-\overline{\text{sg}}(\nu_A(x))).(((1-\nu_A(x)).(\nu_A(x) .(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))+\overline{\text{sg}}(\nu_A(x))).(1-(1-\nu_A(x)) .(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))-\overline{\text{sg}}(\nu_A(x)))+\overline{\text{sg}}(1-(1-\nu_A(x)) .(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))-\overline{\text{sg}}(\nu_A(x))))) +\overline{\text{sg}}((1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))+\overline{\text{sg}}(\nu_A(x))), (1-\nu_B(x)).(\nu_B(x).(1-\nu_B(x))+\overline{\text{sg}}(1-\nu_B(x))) +\overline{\text{sg}}(\nu_B(x)))\rangle \vert x \in E\}$

We immediately see that operation $-_{BR1}$ does not occur in Table 1, while operations $-_{BR2}$, $-'_1$ and $-''_2$ coincide.

3 Main results

Initially, we give the list of all intuitionistic fuzzy implications (see Table 2). They generate 34 different negations, given in Table 3. The relations between the implications and negatiions are shown in Table 4.

Table 2: List of the first 14 intuitionistic fuzzy implications.

$\to_1$	$\{\langle x, \max(\nu_A(x), \min(\mu_A(x), \mu_B(x))), \min(\mu_A(x), \nu_B(x))\rangle \vert x \in E\}$
$\to_2$	$\{\langle x, \mathrm{sg}(\mu_A(x) - \mu_B(x)), \nu_B(x).\mathrm{sg}(\mu_A(x) - \mu_B(x))\rangle \vert x \in E\}$
$\to_3$	$\{\langle x, 1 - (1 - \mu_B(x)).\mathrm{sg}(\mu_A(x) - \mu_B(x)))),$ $\nu_B(x).\mathrm{sg}(\mu_A(x) - \mu_B(x))\rangle \vert x \in E\}$
$\to_4$	$\{\langle x, \max(\nu_A(x), \mu_B(x)), \min(\mu_A(x), \nu_B(x))\rangle \vert x \in E\}$
$\to_5$	$\{\langle x, \min(1, \nu_A(x) + \mu_B(x)), \max(0, \mu_A(x) + \nu_B(x) - 1)\rangle \vert x \in E\}$
$\to_6$	$\{\langle x, \nu_A(x) + \mu_A(x)\mu_B(x), \mu_A(x)\nu_B(x)\rangle \vert x \in E\}$
$\to_7$	$\{\langle x, \min(\max(\nu_A(x), \mu_B(x)), \max(\mu_A(x), \nu_A(x)),$ $\max(\mu_B(x), \nu_B(x))),\ \max(\min(\mu_A(x), \nu_B(x)),$ $\min(\mu_A(x), \nu_A(x)), \min(\mu_B(x), \nu_B(x)))\rangle \vert x \in E\}$
$\to_8$	$\{\langle x, 1 - (1 - \min(\nu_A(x), \mu_B(x))).\mathrm{sg}(\mu_A(x) - \mu_B(x)),$ $\max(\mu_A(x), \nu_B(x)).\mathrm{sg}(\mu_A(x) - \mu_B(x)),$ $\mathrm{sg}(\nu_B(x) - \nu_A(x))\rangle \vert x \in E\}$
$\to_9$	$\{\langle x, \nu_A(x) + \mu_A(x)^2\mu_B(x), \mu_A(x)\nu_A(x) + \mu_A(x)^2\nu_B(x)\rangle \vert x \in E\}$
$\to_{10}$	$\{\langle x, \mu_B(x).\overline{\mathrm{sg}}(1 - \mu_A(x))$ $+\mathrm{sg}(1 - \mu_A(x)).(\overline{\mathrm{sg}}(1 - \mu_B(x)) + \nu_A(x).\mathrm{sg}(1 - \mu_B(x))),$ $\nu_B(x).\overline{\mathrm{sg}}(1 - \mu_A(x)) + \mu_A(x).\mathrm{sg}(1 - \mu_A(x))$ $.\mathrm{sg}(1 - \mu_B(x))\rangle \vert x \in E\}$
$\to_{11}$	$\{\langle x, 1 - (1 - \mu_B(x)).\mathrm{sg}(\mu_A(x) - \mu_B(x)),$ $\nu_B(x).\mathrm{sg}(\mu_A(x) - \mu_B(x)).\mathrm{sg}(\nu_B(x) - \nu_A(x))\rangle \vert x \in E\}$
$\to_{12}$	$\{\langle x, \max(\nu_A(x), \mu_B(x)), 1 - \max(\nu_A(x), \mu_B(x))\rangle \vert x \in E\}$
$\to_{13}$	$\{\langle x, \nu_A(x) + \mu_B(x) - \nu_A(x).\mu_B(x), \mu_A(x).\nu_B(x)\rangle \vert x \in E\}$
$\to_{14}$	$\{\langle x, 1 - (1 - \mu_B(x)).\mathrm{sg}(\mu_A(x) - \mu_B(x))$ $-\nu_B(x).\overline{\mathrm{sg}}(\mu_A(x) - \mu_B(x)).\mathrm{sg}(\nu_B(x) - \nu_A(x)),$ $\nu_B(x).\mathrm{sg}(\nu_B(x) - \nu_A(x))\rangle \vert x \in E\}$

Table 3: List of the first 5 intuitionistic fuzzy negations.

$\neg_1$	$\{\langle x, \nu_A(x), \mu_A(x)\rangle \vert x \in E\}$
$\neg_2$	$\{\langle x, \overline{\mathrm{sg}}(\mu_A(x)), \mathrm{sg}(\mu_A(x))\rangle \vert x \in E\}$
$\neg_3$	$\{\langle x, \nu_A(x), \mu_A(x).\nu_A(x) + \mu_A(x)^2\rangle \vert x \in E\}$
$\neg_4$	$\{\langle x, \nu_A(x), 1 - \nu_A(x)\rangle \vert x \in E\}$
$\neg_5$	$\{\langle x, \overline{\mathrm{sg}}(1 - \nu_A(x)), \mathrm{sg}(1 - \nu_A(x))\rangle \vert x \in E\}$

Table 4: Correspondence between intuitionistic fuzzy negations and implications.

$\neg_1$	$\to_1, \to_4, \to_5, \to_6, \to_7, \to_{10}, \to_{13}$
$\neg_2$	$\to_2, \to_3, \to_8, \to_{11}$
$\neg_3$	$\to_9$
$\neg_4$	$\to_{12}$
$\neg_5$	$\to_{14}$

Now, we introduce the definitions of the new "substraction" operations. As a basis of the new instances of this operation, we use the formula from classical set theory

$$A - B = A \cap \neg B = \neg(\neg A \cup B) = \neg(A \to B),$$

where A and B are two IFSs. Hence, for $i = 1, 2, ..., 134$ (or in the present case, for $i = 1, 2, ..., 14$)

$$A -_i B = \neg_{\delta(i)}(A \to_i B),$$

where $\delta(i)$ is the number of the negation that corresponds to i-th implication (see Table 4). Therefore, 134 new "substraction" operations can originate. This process is difficult, having in mind the very complex forms of some implications and negations from Tables 2 and 3. By this reason, here we introdice the definition of the first 14 new instances of the "substraction" operation (see Table 5) and the rest definitions will be given in future.

Table 5: List of the first 14 new intuitionistic fuzzy subtractions.

$-_1$	$\{\langle x, \min(\mu_A(x), \nu_B(x)), \max(\nu_A(x), \min(\mu_A(x), \mu_B(x)))\rangle \vert x \in E\}$
$-_2$	$\{\langle x, \text{sg}(\mu_A(x) - \mu_B(x)), \overline{\text{sg}}(\mu_A(x) - \mu_B(x))\rangle \vert x \in E\}$
$-_3$	$\{\langle x, \overline{\text{sg}}(\mu_A(x) - \mu_B(x)), \overline{\text{sg}}(\mu_A(x) - \mu_B(x))\rangle \vert x \in E\}$
$-_4$	$\{\langle x, \max(\nu_A(x), \mu_B(x)), \min(\mu_A(x), \nu_B(x))\rangle \vert x \in E\}$
$-_5$	$\{\langle x, \max(0, \mu_A(x) + \nu_B(x) - 1), \min(1, \nu_A(x) + \mu_B(x))\rangle \vert x \in E\}$
$-_6$	$\{\langle x, \mu_A(x)\nu_B(x), \nu_A(x) + \mu_A(x)\mu_B(x)\rangle \vert x \in E\}$
$-_7$	$\{\langle x, \max(\min(\mu_A(x), \nu_B(x)), \min(\mu_A(x), \nu_A(x)),$ $\min(\mu_B(x), \nu_B(x))),$ $\min(\max(\nu_A(x), \mu_B(x)), \max(\mu_A(x), \nu_A(x)),$ $\max(\mu_B(x), \nu_B(x)))\rangle \vert x \in E\}$
$-_8$	$\{\langle x, ((1 - \text{sg}(\min(\nu_A(x), \mu_B(x)))).\text{sg}(\mu_A(x) - \mu_B(x)),$ $\overline{\text{sg}}(\mu_A(x) - \mu_B(x)) + \text{sg}(\min(\nu_A(x), \mu_B(x))))$ $.\text{sg}(\mu_A(x) - \mu_B(x))\rangle \vert x \in E\}$
$-_9$	$\{\langle x, \mu_A(x).\nu_A(x) + \mu_A(x)^2\nu_B(x),$ $\mu_A(x)\nu_A(x)^2 + \mu_A(x)^2\nu_A(x)\nu_B(x) + \mu_A(x)^3\nu_A(x)\mu_B(x)$ $+\mu_A(x)^4\mu_B(x)\nu_B(x) + nu_A(x)^2 + 2 + \mu_A(x)^2\nu_A(x)\mu_B(x)$ $+\mu_A(x)^4\mu_B(x)^2\rangle \vert x \in E\}$
	Continued on next page

Table 5 – continued from previous page

$-_{10}$	$\{\langle x, \nu_B(x).\overline{\text{sg}}(1-\mu_A(x))+\mu_A(x).\text{sg}(1-\mu_A(x)).\text{sg}(1-\mu_B(x)),$ $\mu_B(x).\overline{\text{sg}}(1-\mu_A(x))+\text{sg}(1-\mu_A(x)).(\overline{\text{sg}}(1-\mu_B(x))+\nu_A(x)$ $.\text{sg}(1-\mu_B(x)))\rangle \mid x \in E\}$
$-_{11}$	$\{\langle x, \overline{\text{sg}}(\mu_A(x)-\mu_B(x)), \overline{\text{sg}}(\mu_A(x)-\mu_B(x))\rangle \mid x \in E\}$
$-_{12}$	$\{\langle x, 1-\max(\nu_A(x), \mu_B(x)), \max(\nu_A(x), \mu_B(x))\rangle \mid x \in E\}$
$-_{13}$	$\{\langle x, \mu_A(x).\nu_B(x), \nu_A(x)+\mu_B(x)-\nu_A(x).\mu_B(x)\rangle \mid x \in E\}$
$-_{14}$	$\{\langle x, \overline{\text{sg}}(1-\nu_B(x).\text{sg}(\nu_B(x)-\nu_A(x)),$ $\text{sg}(1-\nu_B(x).\text{sg}(\nu_B(x)-\nu_A(x))\rangle \mid x \in E\}$

Some of the most important properties of the subtractions are:
(a) $A - E^* = O^*$,
(b) $A - O^* = A$,
(c) $E^* - A = \neg A$,
(d) $O^* - A = O^*$,
(e) $(A - B) \cap C = (A \cap C) - B = A \cap (C - B)$,
(f) $(A \cap B) - C = (A - C) \cap (B - C)$,
(g) $(A \cup B) - C = (A - C) \cup (B - C)$,
(h) $(A - B) - C = (A - C) - B$,
(i) $(A - C) \cap B = A \cap (B - C)$,
(j) $O^* - U^* = O^*$,
(k) $O^* - E^* = O^*$,
(l) $U^* - O^* = U^*$,
(m) $U^* - E^* = O^*$,
(n) $E^* - O^* = E^*$,
(o) $E^* - U^* = O^*$.

In Table 6 are given these subtractions that satisfy these properties.

Table 6: Properties of the "subtraction" operations.

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
$\rightarrow_1$	-	+	+	+	-	-	-	-	-	+	+	+	-	+	-
$\rightarrow_2$	+	-	-	+	-	+	+	-	-	+	+	-	+	+	-
$\rightarrow_3$	+	-	+	+	-	+	+	+	-	+	+	-	+	+	-
$\rightarrow_4$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-
$\rightarrow_5$	+	+	+	+	-	+	+	+	-	+	+	+	+	+	-
$\rightarrow_6$	-	+	+	+	-	-	-	-	-	+	+	+	-	+	-
$\rightarrow_7$	-	+	+	-	-	-	-	-	-	-	+	+	-	+	-
$\rightarrow_8$	+	-	-	+	-	+	-	-	-	+	+	-	+	+	-
$\rightarrow_9$	-	-	+	+	-	-	-	-	-	+	+	+	-	+	-
$\rightarrow_{10}$	+	+	+	+	-	-	-	-	-	+	+	+	+	+	-
$\rightarrow_{11}$	+	-	+	+	-	+	+	+	-	+	+	-	+	+	-
$\rightarrow_{12}$	+	-	-	+	-	+	+	+	-	+	+	-	+	+	-
$\rightarrow_{13}$	+	+	+	+	-	+	+	+	-	+	+	+	+	+	-
$\rightarrow_{14}$	+	-	+	+	-	+	+	+	-	+	+	-	+	+	+

In a next research we will continue to study the definitions and properties of the new subtractions based on the intuitionistic fuzzy implications.

An **OPEN PROBLEM** is to find another approach to introducing variants of the "subtraction" operation over IFSs. If this is possible, the behaviour of the new operations must be studied, also.

4 Final remarks: Beloslav Riečan's group and intuitionistic fuzzy sets

In the beginning of the 21^{st} century, Prof. Beloslav Riečan established in the Matej Bel University, Banská Bystrica one of the most active research groups in the world in the area of intitionistc fuzzy set theory. After the two annual conferences on IFSs, organized in Sofia (since 1998) and Warsaw (since 2000), Banská Bystrica became the third place, where such regular meetings are being held ever since 2006.

Prof. Riečan participates actively in the organization of the Bulgarian conferences (since 2006) and in the edition of the specialized journal "Notes on Intuitionistic Fuzzy Sets". He and his PhD students and collaborators developed whole areas of IFSs theory, related to intuitionistic fuzzy integrals, probabilities, etc. To this end, Prof. Riečan has the largest number of successfully defended PhD students with theses on IFSs in the world.

On the behalf of his Bulgarian friends and colleagues, I wish him to keep up his research activity for a long years in future.

5 Acknowledgement

The author is grateful for the support provided by the projects DID-02-29 "Modelling processes with fixed development rules" and BIn-2/09 "Design and development of intuitionistic fuzzy logic tools in information technologies" funded by the National Science Fund, Bulgarian Ministry of Education, Youth.

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