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On number of interior periodic points of a Lotka-Volterra map

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Dedicated to the 75th birthday of Beloslav Riečan

Abstract
Given the plane triangle $\Delta = \{ [x, y] : 0 \le x,\ 0 \le y,\ x + y \le 4 \}$ and the transformation $F : \Delta \to \Delta$, $[x, y] \mapsto [x(4 - x - y),\ xy]$ we give a lower estimate of the number of interior periodic orbits with period $n \le 36$.

Keywords Periodic point, Jacobi matrix, saddle fixed point, itinerary, Brouwer theorem

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1 Introduction

We study periodic points of the map $F : [x, y] \mapsto [x(4 - x - y), xy]$ lying inside the triangle

$$\Delta = \{ [x, y] : 0 \le x,\ 0 \le y,\ x + y \le 4 \}.$$

The map F maps the triangle Δ onto itself. This map has been studied in the papers [8], [4],[5],[6] and is sometimes called Lotka–Volterra. Y. Avishai and D. Berend in [1] (see also [2] and [3]) studied a discrete system related with the dynamics of the map $F : \Delta \to \Delta$. The basic transformation considered in [1] is $H[x, y] = [y,\ x^2y - 2x^2 + 2]$ defined on $\mathbb{R}^2$. The system (Δ, F) was obtained from $(\mathbb{R}^2, H)$ employing some conjugacy reductions. A. N. Sharkovskiĭ in [7] stated some open problems on the dynamics of the map F. It is easy to find three fixed points of the map F, namely $[0, 0]$, $[3, 0]$ and $[1, 2]$. (Periodic points on the lower side of Δ are well known, because the restriction of F to the lower side is the logistic map $f : x \mapsto x(4 - x)$ which is conjugate with the tent map.)

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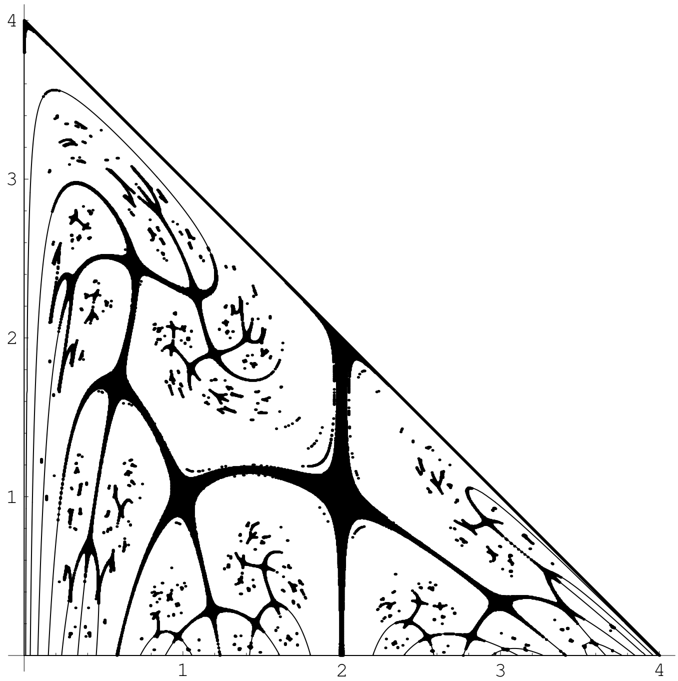

Figure 1: Interior periodic points of the map F whose periods are ≤ 36. Such interior periodic points are white. The black part of the triangle does not contain those points.

Until recently nothing has been known on the existence of interior periodic points different from $[1, 2]$. Only in 2006, in [4], the interior point $[1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}]$ with period 4 was found. Trying to find other interior periodic points, we started to study periodic points by numerical experiments and soon we found the point $[1, \frac{3+\sqrt{5}}{2}]$ with period 6 and numerically also many other periodic points. We omit these numerical experiments because they are not necessary for reading the present paper. In fact, after a careful analysis of them we were able to prove an *exact result*, Theorem 4.3, which was proved in [6]. It implies the existence of interior periodic points of all periods $n \geq 4$ inside Δ. The results of our numerical experiments are illustrated on Fig. 1. It contains about $5.4 \cdot 10^{10}$ periodic points with period $n \leq 36$.

The present paper is a continuation of [6]. Our main result is Theorem 3.3 and Table 1.

2 Notations and preliminary results

We denote by $[x, y]$ a point in the plane, while (α, β) and $\langle \alpha, \beta \rangle$ are open and closed intervals on the real line. Throughout the paper we denote by F the map of the plane $\mathbb{R}^2$ given by $F[x, y] = [x(4 - x - y), xy]$. Let $\Delta = \{ [x, y] : 0 \leq x, 0 \leq y, x + y \leq 4 \}$. The sides of the triangle Δ are denoted by a, b and c as it is shown in Fig. 2. It is easy to see that $F(\Delta) = \Delta$. Note that $F[x, 0] = [f(x), 0]$, where

$$f : \langle 0, 4 \rangle \to \langle 0, 4 \rangle, \; f(x) = x(4 - x)$$

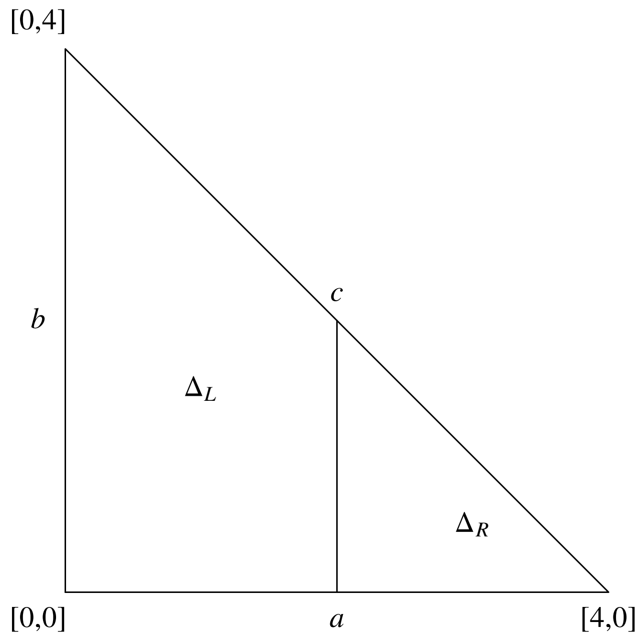

Figure 2: Notations concerning the triangle Δ.

is the logistic map. Note that any point $x \in \langle 0,4\rangle$ may be written in the form $x = 4\sin^2 t$ with $t \in \langle 0, \frac{\pi}{2}\rangle$ and in this case

$$\begin{aligned} f(x) &= f(4\sin^2 t) = 4\sin^2 t(4-4\sin^2 t) = 16\sin^2 t\cos^2 t \\ &= 4\sin^2 2t = 4\sin^2(\pi - 2t)\ . \end{aligned} \tag{2.1}$$

The logistic map f is conjugate with the tent map $g : \langle 0,1\rangle \to \langle 0,1\rangle,\ g(t) = 1-|1-2t|$ via the conjugation $h : \langle 0,1\rangle \to \langle 0,4\rangle,\ h(t) = 4\sin^2(\pi t/2)$. Since any fixed point of the map g^n is of the form $\frac{2k}{2^n\pm1}$, any lower fixed point of the map F^n is of the form $\left[4\sin^2\frac{k\pi}{2^n\pm1}, 0\right]$ where n and k are integers such that $0 < n$ and $0 \le 2k < 2^n \pm 1$. It is easy to see that the Jacobi matrix of the map F at the point $[x,y]$ has the form

$$\begin{pmatrix} 4-2x-y & -x \\ y & x \end{pmatrix}.$$

Therefore the Jacobi matrix of the map F at the point $[x,0]$ has the form

$$\begin{pmatrix} 4-2x & -x \\ 0 & x \end{pmatrix}.$$

It means that the Jacobi matrix of the map F^n at the point $[x_0,0]$ has the form

$$\begin{pmatrix} \prod_{i=0}^{n-1}(4-2x_i) & \mu \\ 0 & \prod_{i=0}^{n-1} x_i \end{pmatrix},$$

where $x_i = f^i(x_0)$. As we shall see, the value of μ is unimportant. Clearly, the Jacobi matrix of the map F^n at the point $[0,0]$ has the form

$$\begin{pmatrix} 4^n & 0 \\ 0 & 0 \end{pmatrix}.$$

(As we shall see it is an exception. For the other lower fixed points of the map F^n we have the eigenvalue 2^n instead of 4^n). Let $x_0 > 0$ and $P = [x_0, 0] \in \Delta$ be a fixed point of the map F^n. So $x_0 = 4\sin^2 \frac{k\pi}{2^n \pm 1}$ where $k \geq 1$ and

$$\begin{aligned}
x_i &= 4\sin^2 \frac{2^i k\pi}{2^n \pm 1}, \\
4 - 2x_i &= 4\cos \frac{2^{i+1} k\pi}{2^n \pm 1}, \\
\sin \frac{2^n k\pi}{2^n \pm 1} &= \mp(-1)^k \sin \frac{k\pi}{2^n \pm 1}, \\
\cos \frac{2^n k\pi}{2^n \pm 1} &= (-1)^k \cos \frac{k\pi}{2^n \pm 1}, \\
\sin \frac{2^n k\pi}{2^n \pm 1} &= 2^n \sin \frac{k\pi}{2^n \pm 1} \prod_{i=0}^{n-1} \cos \frac{2^i k\pi}{2^n \pm 1}, \\
\prod_{i=0}^{n-1} \cos \frac{2^i k\pi}{2^n \pm 1} &= \frac{\mp(-1)^k}{2^n}, \\
\prod_{i=0}^{n-1} (4 - 2x_i) &= 4^n \prod_{i=0}^{n-1} \cos \frac{2^{i+1} k\pi}{2^n \pm 1} = (-1)^k 4^n \prod_{i=0}^{n-1} \cos \frac{2^i k\pi}{2^n \pm 1} = \mp 2^n.
\end{aligned}$$

Hence the Jacobi matrix of the map F^n at the point P has the form

$$\begin{pmatrix} \lambda_1 & \mu \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \mp 2^n & \mu \\ 0 & \prod_{i=0}^{n-1} x_i \end{pmatrix} = \begin{pmatrix} \mp 2^n & \mu \\ 0 & \prod_{i=0}^{n-1} 4\sin^2 \frac{2^i k\pi}{2^n \pm 1} \end{pmatrix}. \tag{2.2}$$

So,

$$\lambda_2 = \prod_{i=0}^{n-1} 4\sin^2 \frac{2^i k\pi}{2^n \pm 1}.$$

For λ_2 we have the possibilities

(i) $0 \leq \lambda_2 < 1$, i.e. $[x_0, 0]$ is a saddle point, e.g. $x_0 = 4\sin^2 \frac{\pi}{17}$,

(ii) $\lambda_2 = 1$, i.e. $[x_0, 0]$ is a non-hyperbolic point, e.g. $x_0 = 4\sin^2 \frac{\pi}{15}$,

(iii) $1 < \lambda_2$, i.e. $[x_0, 0]$ is a repulsive point, e.g. $x_0 = 4\sin^2 \frac{3\pi}{17}$.

Remark 2.1. All the chosen points $[x_0, 0]$ in (i)-(iii) have period 4. Lower periodic points with period n and $0 < \lambda_2 < 1$ appear for all $n \geq 4$. Lower periodic points with period n and $\lambda_2 = 1$ appear for infinitely many n, e.g. $n = 4 \cdot 3^i \cdot 5^j$, where $i \geq 0$, $j \geq 0$. Lower periodic points with period n and $1 < \lambda_2$ appear for all $n \geq 1$.

3 Estimates of the number of lower saddle periodic points.

In connection with saddle points and the main result, Theorem 4.3, it is necessary to have at least a sufficient condition for a fixed point of F^n to be saddle. Therefore we include the following theorem.

Theorem 3.1 ([6]). *Let* $P = \left[4\sin^2 \frac{k\pi}{2^n \pm 1}, 0\right]$ *where* $n > 0$ *and* k *are integers such that*

$$1 \le k \le \frac{\sqrt{2}(2^n \pm 1)}{\pi \cdot 2^{\sqrt{2n+1/4}}} . \tag{3.1}$$

Then P *is a saddle fixed point of* F^n.

Remark 3.2. Note that for $4 \le n \le 13$ all points $P = \left[4\sin^2 \frac{k\pi}{2^n \pm 1}, 0\right]$, where k satisfies (3.1), have period n (and not less). If $n = 14$ and $k = 127$ or 129 then (3.1) is satisfied (with the sign $-$) and the point $P = \left[4\sin^2 \frac{k\pi}{2^n - 1}, 0\right]$ has period 7.

Unfortunately, the previous theorem gives only sufficient condition for a saddle point. So fix an integer n, the choice of signs $\pm$ and an integer k such that $1 \le k < \frac{2^n \pm 1}{2}$. We want to decide whether the point $P = \left[4\sin^2 \frac{k\pi}{2^n \pm 1}, 0\right]$ is a saddle point of the map F^n and whether its period is n (because it is a divisor of n in general). So we need to decide whether $\lambda_2 < 1$, $\lambda_2 = 1$ or $\lambda_2 > 1$, where

$$\lambda_2 = \prod_{i=0}^{n-1} 4\sin^2 \frac{2^i k\pi}{2^n \pm 1} .$$

Put $k_0 = k$ and

$$k_{i+1} = \begin{cases} 2k_i & \text{if } 2k_i < \frac{2^n \pm 1}{2} , \\ 2^n \pm 1 - 2k_i & \text{otherwise.} \end{cases}$$

Then $\sqrt{\lambda_2} = \prod_{i=0}^{n-1} 2\sin \frac{k_i\pi}{2^n \pm 1}$. If $k_i = k$ for $0 < i < n-1$ then the period of the point $P = \left[4\sin^2 \frac{k\pi}{2^n \pm 1}, 0\right]$ is less than n. If $k_i < k$ for $0 < i < n-1$ than the point P belongs to the orbit of the point $\left[4\sin^2 \frac{k_i\pi}{2^n \pm 1}, 0\right]$ and this point has been already considered (we assume that we consider k from 1 to $\frac{2^n \pm 1}{2} - \frac{1}{2}$ with the step 1). So, if $k_i \le k$ the evaluation of $\sqrt{\lambda_2} = \prod_{i=0}^{n-1} 2\sin \frac{k_i\pi}{2^n \pm 1}$ is not necessary and this evaluation may be interrupted. To find the number of saddle periodic points of the map F with period n it is sufficient to find the number of saddle periodic orbits and multiply this number by n. For any lower saddle periodic orbit it is sufficient to find that point which has the smallest x–coordinate.

Theorem 3.3. *Consider integers* $n \ge 1$ *and* $1 \le k < \frac{2^n \pm 1}{2}$ *for a fixed choice of* $\pm$. *Let*

$$\lambda_2 = \prod_{i=0}^{n-1} 4\sin^2 \frac{2^i k\pi}{2^n \pm 1} < 1$$

and

$$4\sin^2 \frac{k\pi}{2^n \pm 1} < 4\sin^2 \frac{2^i k\pi}{2^n \pm 1} \ \textit{for } 1 \le i \le n-1 .$$

Then k *is odd and*

$$\frac{k}{2^n \pm 1} < \frac{1}{12} . \tag{3.2}$$

Proof. If $k = 2j$, then $4\sin^2 \frac{2^{n-1}k\pi}{2^n \pm 1} = 4\sin^2 \frac{j\pi}{2^n \pm 1} < 4\sin^2 \frac{k\pi}{2^n \pm 1}$. Put $x_i = 4\sin^2 \frac{2^i k\pi}{2^n \pm 1}$. Clearly, $x_{i+1} = f(x_i)$ and $f^n(x_0) = x_0$. Assume that $\lambda_2 = \prod_{i=0}^{n-1} x_i < 1$, and

$x_i > x_0$ for $0 < i < n$. If $\frac{k}{2^n \pm 1} > \frac{1}{6}$ then $x_0 > 1$, $x_i > x_0 > 1$ for $1 \le i \le n-1$ and $\lambda_2 > 1$. So we obtain a contradiction. If $\frac{k}{2^n \pm 1} = \frac{1}{6}$ then $x_0 = 1$ and $x_i = 3$ for $i > 0$ and we have again a contradiction. We shall show that the assumption $\frac{1}{12} < \frac{k}{2^n \pm 1} < \frac{1}{6}$ leads to a contradiction. Let I be the set of all integers i such that $0 \le i < n$ and $x_i < 1$. Let $i_0 < i_1 \cdots < i_j$ be all elements of I. Put also $i_{j+1} = n$. Then $\lambda_2 = \prod_{s=0}^{j} \prod_{i=i_s}^{i_{s+1}-1} x_i$. Since $\lambda_2 < 1$ then $\prod_{i=i_s}^{i_{s+1}-1} x_i < 1$ at least for one $s = 0, \cdots, j$. Since $2 - \sqrt{3} = 4\sin^2 \frac{\pi}{12} < x_0 < 4\sin^2 \frac{\pi}{6} = 1$ and $x_0 \le x_{i_s}$ we have

$$
\begin{aligned}
2 - \sqrt{3} = 4\sin^2 \frac{\pi}{12} &< x_{i_s} < 4\sin^2 \frac{\pi}{6} = 1\,, \\
1 < f(x_{i_s}) &= x_{i_s+1} < 3\,, \\
f(x_{i_s+1}) &= x_{i_s+2} > 3\,.
\end{aligned}
$$

If $i_{s+1} = i_s + 3$ then $2 - \sqrt{3} = 4\sin^2 \frac{\pi}{12} < x_{i_s+3} < 4\sin^2 \frac{\pi}{6} = 1$. It is possible only for $x_{i_s+2} > 2 + \sqrt{3}$, because f is decreasing on $\langle 2, 4 \rangle$, $x_{i_s+3} = f(x_{i_s+2})$ and $f(2+\sqrt{3}) = 1$. We obtain $x_{i_s} \cdot x_{i_s+1} \cdot x_{i_s+2} > (2 - \sqrt{3}) \cdot 1 \cdot (2 - \sqrt{3}) = 1$ what is a contradiction. If $i_{s+1} - i_s > 3$ then the difference $i_{s+1} - i_s$ is odd, $x_{i_s+2j} > 3$ and $1 < x_{i_s+2j+1} < 3$ for $2j < i_{s+1} - i_s - 1$. Therefore $\prod_{i=i_s}^{i_{s+1}-1} x_i > (2 - \sqrt{3}) \cdot 9 > 1$ what is a contradiction. So $\frac{k}{2^n \pm 1} \le \frac{1}{12}$. If $\frac{k}{2^n \pm 1} = \frac{1}{12}$ then $x_0 = 2 - \sqrt{3}$, $x_1 = 1$ and $x_i = 3$ for $i \ge 2$ which is impossible. We have $\frac{k}{2^n \pm 1} < \frac{1}{12}$. □

Remark 3.4. The previous theorem shows that it is not necessary to consider all possible k but only odd k which satisfy (3.2). It shortens the computation of saddle periodic orbits and points 12 times. In fact, with a little care but essentially in the same way, for $n \ge 5$ the inequality

$$\frac{k}{2^n \pm 1} < \frac{1}{17}$$

can be proved. (For $n = 4$ we have 3 periodic orbits. Only one of them is a saddle orbit, see Remark 2.1.) Thus for $n \ge 5$ the computation can be shortened 17 times.

We denote by s_n the number of lower saddle periodic orbits and by $p_n = n \cdot s_n$ the number of lower saddle periodic points of the map F with period n. Table 1 contains values s_n and p_n for $1 \le n \le 36$.

4 Relationship between lower and interior periodic points

Let $P = [x, y] \in \Delta$ be a periodic point of the map F and $F^i[x_0, y_0] = [x_i, y_i]$. Then $x_i \neq 2$, because otherwise we would have $F[x_i, y_i] = [4 - 2y_i, 2y_i]$, $F^2[x_i, y_i] = [0, 8y_i - 4y_i^2]$, $F^3[x_i, y_i] = [0, 0]$, $F^j[x_i, y_i] = [0, 0]$ for $j \ge 3$ and $F^m[x_0, y_0] = [0, 0]$ for all $m \ge i + 3$ which is a contradiction. For any fixed point P of the map F^n we define its *itinerary* as a sequence $W = (w_i)_{i=0}^{n-1}$, where

$$w_i = \begin{cases} L \text{ if } x_i < 2 \\ R \text{ if } x_i > 2\,. \end{cases}$$

More generally, any sequence $W = (w_i)_{i=0}^{n-1}$ of letters L and R will also be called an *itinerary*. Such an itinerary is said to be *aperiodic* if for any proper divisor k of n there is $j < n - k$ such that $w_j \neq w_{j+k}$.

n	s_n	$p_n = n \cdot s_n$	$\frac{n \cdot s_n}{2^n}$
1	1	1	0.5
2	0	0	0
3	0	0	0
4	1	4	0.250000
5	2	10	0.312500
6	3	18	0.281250
7	5	35	0.273438
8	11	88	0.343750
9	18	162	0.316406
10	37	370	0.361328
11	72	792	0.386719
12	122	1464	0.357422
13	223	2899	0.353882
14	418	5852	0.357178
15	793	11895	0.363007
16	1500	24000	0.366211
17	2903	49351	0.376518
18	5477	98586	0.376076
19	10412	197828	0.377327
20	19890	397800	0.379372
21	38090	799890	0.381417
22	72892	1603624	0.382334
23	140345	3227935	0.384800
24	270239	6485736	0.386580
25	520870	13021750	0.388078
26	1005368	26139568	0.389510
27	1945782	52536114	0.391425
28	3766954	105474712	0.392924
29	7298398	211653542	0.394235
30	14159124	424773720	0.395601
31	27492108	852255348	0.396862
32	53415336	1709290752	0.397975
33	103871727	3427766991	0.399045
34	202193966	6874594844	0.400154
35	393867993	13785379755	0.401207
36	767755134	27639184824	0.402203

Table 1: Number of saddle orbits and saddle periodic points with period n for $1 \leq n \leq 36$.

Remark 4.1. Itineraries are usually defined as infinite sequences. In this paper we consider only itineraries of fixed points of the iterates F^n and so finite sequences are sufficient.

Proposition 4.2 ([6])**.** *For any itinerary $W = (w_i)_{i=0}^{n-1}$ there is a unique lower fixed point P of the map F^n with itinerary W. The period of P is n if and only if W is aperiodic.*

Now we are ready to formulate the main result on periodic point of the map F.

Theorem 4.3 ([6])**.** *Let P be a lower saddle periodic point of the map F. Then there is an interior periodic point Q of F with the same itinerary and period.*

Let $\mathrm{Fix}_{\mathrm{Int}}(F^n)$ be the set of all interior fixed points of the map F^n and $\mathrm{Per}_{\mathrm{Int}}(F, n)$ be the set of all interior n-periodic points of the map F.

Theorem 4.4 ([6])**.** *For cardinalities of $\mathrm{Fix}_{\mathrm{Int}}(F^n)$ and $\mathrm{Per}_{\mathrm{Int}}(F, n)$ we have the estimates*

(i) $\#\,\mathrm{Fix}_{\mathrm{Int}}(F^n) \geq \frac{2\sqrt{2}}{\pi} \cdot 2^{n-\sqrt{2n+1/4}} - 2$

(ii) $\#\,\mathrm{Per}_{\mathrm{Int}}(F, n) \geq \frac{2\sqrt{2}}{\pi} 2^{n-\sqrt{2n+1/4}} - 2^{1+\frac{n}{2}} + 1$

(iii) $\#\,\mathrm{Per}_{\mathrm{Int}}(F, n) \geq (2 - \varepsilon)^n$ *for* $0 < \varepsilon < 1$ *and sufficiently large* n.

Remark 4.5. The estimate given in (ii) is useless for $n \leq 12$. In such a case it may be used that $\#\,\mathrm{Per}_{\mathrm{Int}}(F, n) \geq \frac{2\sqrt{2}}{\pi} 2^{n-\sqrt{2n+1/4}} - 2$. Moreover, for small n the number of lower saddle n-periodic points of F may be easily found.

The points $[4 \sin^2 \frac{\pi}{2^n \pm 1}, 0]$ have period n. It follows from Theorem 3.1 that they are saddle fixed points of F^n for $n \geq 4$ and $n \geq 5$ provided we choose the sign $+$ and $-$, respectively. So we obtain the following theorem.

Theorem 4.6 ([6])**.** *For any $n \geq 4$ there is an interior point Q of the map F with period n.*

The following theorem is also a consequence of Theorem 4.3.

Theorem 4.7. *For $1 \leq n \leq 36$ the third column of Table 1 gives a lower estimate of $\#\,\mathrm{Per}_{\mathrm{Int}}(F, n)$.*

Note that these estimates, for $1 \leq n \leq 36$, are much better than those from Theorem 4.4.

5 Existence and nonexistence of periodic points with prescribed itineraries

Proposition 4.2 says that the lower periodic points may be described by their itineraries. In this section we prove that for some itineraries interior periodic points need not exist. It is sufficient to consider itineraries $W = (w_i)_{i=0}^{n-1}$ with $w_0 = L$ and $w_{n-1} = R$, see the proof of Theorem 4.3. We shall write such itineraries in the form $W = L^{j_1} R^{k_1} \ldots L^{j_m} R^{k_m}$, where all j_i and k_i are positive integers and $n = j_1 + k_1 + \ldots + j_m + k_m$.

Now we show that interior fixed points of the map F^n with itineraries containing too many R's do not exist.

Theorem 5.1 ([6]). *Let $W = L^{j_1}R^{k_1}\cdots L^{j_m}R^{k_m}$ be an itinerary such that $j_i > 0$, $k_i > 0$ and $\sum_{i=1}^{m}(j_i + k_i) = n$. If*

$$\sum_{i=1}^{m} k_i \geq \frac{\ln 2}{\ln 3}\sum_{i=1}^{m} j_i^2 - \frac{\ln(4-2\sqrt{2})}{\ln 3}\sum_{i=1}^{m} j_i + m\,, \tag{5.1}$$

then there is no interior fixed point of the map F^n with the itinerary W.

The following theorem can be sometimes more useful than the previous one.

Theorem 5.2 ([6]). *Let $W = L^{j_1}R^{k_1}\cdots L^{j_m}R^{k_m}$ be an itinerary such that $j_i > 0$, $k_i > 0$ and $\sum_{i=1}^{m}(j_i + k_i) = n$. If*

$$\sum_{i=1}^{m} k_i \geq \sum_{i=1}^{m} j_i^2 - \frac{\ln(4-2\sqrt{2})}{\ln 2}\sum_{i=1}^{m} j_i\,, \tag{5.2}$$

then there is no interior fixed point of the map F^n with the itinerary W.

On the other hand, the following theorem shows that if an itinerary W of length n contains sufficiently many L's then the map F^n has an interior fixed point with this itinerary.

Theorem 5.3 ([6]). *Let $W = L^{j_1}R^{k_1}\cdots L^{j_m}R^{k_m}$ be an itinerary such that $j_i > 0$, $k_i > 0$ and $\sum_{i=1}^{m}(j_i + k_i) = n$. If*

$$\sum_{i=1}^{m} k_i \leq \frac{\ln 2}{\ln 3}\sum_{i=1}^{m} j_i^2 - \frac{\ln\frac{\pi^2}{2}}{\ln 3}\sum_{i=1}^{m} j_i - \frac{\ln\frac{32}{3\pi^2}}{\ln 3}m\,, \tag{5.3}$$

then there exists an interior fixed point of the map F^n with itinerary W.

6 Conclusion and future directions

Many problems concerning periodic points of the Lotka–Volterra map remain open. On the base of our numerical experiments and Table 1 we formulate the following conjectures.

Conjecture 6.1. *If $P \in \Delta$ is a lower repulsive (non-hyperbolic) fixed point of the map F^n, then there is no interior fixed point of F^n with the same itinerary.*

Conjecture 6.2. *If $P \in \Delta$ is a lower saddle fixed point of F^n, then there is a unique interior fixed point of F^n with the same itinerary.*

Conjecture 6.3.

$$\liminf_{n\to\infty} \frac{\#\,\mathrm{Fix}_{\mathrm{Int}}(F^n)}{2^n} > 0\,.$$

It turns out that the eigenvalue λ_2 is related to some open problems in number theory. In the near future we plan to publish corresponding results.

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