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Probability measures on interval–valued fuzzy events*

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Dedicated to Prof. B. Riečan

Abstract

Probability measures on intuitionistic fuzzy events were axiomatically characterized by B. Riečan in 2004 and subsequently studied in several papers. All these results can be straightforwardly transformed for the case of probability measures on interval–valued fuzzy sets. We give an alternative representation of all such probability measures. Comparison with the previous results of Riečan with co–authors is also included.

Keywords IFS event, IFS–probability, T_L–tribe, interval–valued event

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1 Introduction

Fuzzy sets were introduced by Zadeh in 1965 [9]. Recall that each fuzzy set in the universe X is characterized by its membership function $A : X \to [0,1]$ (we will not distinguish in notation fuzzy sets and their respective membership functions). In 1968, Zadeh has introduced probability measures on fuzzy events. Note that, for a measurable universe $(X, \mathcal{A})$, $\mathcal{A}$ being a σ–algebra of subsets of X, fuzzy events are just measurable fuzzy sets. For any classical probability measure P on $(X, \mathcal{A})$, the induced fuzzy probability measure $\mathcal{P}_P(A) = E_P(A)$, where E_P is the classical P–based expected value. An axiomatic approach to fuzzy probability measures was proposed by Butnariu [4], where the additivity was modelled by means of the Lukasiewicz t–norm $\odot : [0,1]^2 \to [0,1]$, $a \odot b = max\,\{a + b - 1, 0\}$ and of the Lukasiewicz t–conorm $\oplus : [0,1]^2 \to [0,1]$, $a \oplus b = min\,\{a + b, 1\}$. The main result of [4] shows

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that Zadeh's fuzzy probability measures coincide with axiomatically defined fuzzy probability measures.

Atanassov in [2], see also [3], has introduced intuitionistic fuzzy set $A : X \to [0,1]^2$ as a couple of fuzzy sets, $A = (B, C)$, such that $B \odot C = 1$ (i.e., $B(x) \odot C(x) \leq 1$ for all $x \in X$). Observe that $B \odot C = 0$ if and only if $B \leq \mathbf{1} - C$, and thus the intuitionistic fuzzy set A can be isomorphically seen as an interval valued fuzzy set $\tilde{A} = [B, \mathbf{1} - C] = \left[\underline{A}, \overline{A}\right]$, where $\underline{A}$, $\overline{A}$ are fuzzy sets satisfying $\underline{A} \leq \overline{A}$. Clearly, fuzzy sets can be embedded into interval fuzzy sets, supposing $\underline{A} = \overline{A}$. Grzegorzewski and Mrówka in 2002 [6] have proposed probability measures on intuitionistic fuzzy sets generalizing the original Zadeh's approach from 1968 [10]. Based on a probability measure P on $(X, \mathcal{A})$, intuitionistic fuzzy probability $\mathcal{P}_P$ was given as an interval–valued mapping by

$$\mathcal{P}_P\left([B, C]\right) = \left[E_P(B), 1 - E_P(C)\right]. \tag{1.1}$$

Transforming formula (1.1) for interval–valued fuzzy events, we get

$$\mathcal{P}_P\left(A\right) = \mathcal{P}_P\left(\left[\underline{A}, \overline{A}\right]\right) = \left[E_P(\underline{A}), E_P(\overline{A})\right]. \tag{1.2}$$

Riečan in 2004 [7] has proposed an axiomatic characterization of intuitionistic fuzzy probability measures, and later in [8, 5] has studied the structure of these mappings.

All these results can be easily reformulated for interval–valued fuzzy sets, yielding more transparent look (event values and probability values are then in both cases intervals, and thus one can look on these probabilities as a kind of expected values).

The aim of this contribution is an alternative characterization of interval–valued probability measures of interval–valued fuzzy events on a general measurable space $(X, \mathcal{A})$. The paper is organized as follows. In the next section, Riečan's results are transformed for the interval–valued case. Section 3 brings the main result – complete characterization of interval–valued probability measures of interval–valued fuzzy events. In Section 4 results of Riečan are compared with our results and the convex structure of discussed probabilities is completely determined. Finally, some concluding remarks are added.

2 Riečan's results on probability measures on interval–valued fuzzy events

For a measurable space $(X, \mathcal{A})$, denote by $\mathcal{J}$ the class of all interval–valued fuzzy events, $\mathcal{J} = \left\{\left[\underline{A}, \overline{A}\right] \mid \underline{A}, \overline{A} : X \to [0,1] \text{ are } \mathcal{A} - \text{measurable}, \underline{A} \leq \overline{A}\right\}$. Let $\mathcal{J} = \{[a,b] \mid 0 \leq a \leq b \leq 1\}$.

The next definition is a version of the original Riečan's definition from [7].

Definition 2.1. A mapping $\mathcal{P} : \mathcal{J} \to I$ is called an interval probability measure if the next axioms are satisfied:

(i) $\mathcal{P}\left([\mathbf{1}, \mathbf{1}]\right) = [1, 1]$, $\quad \mathcal{P}\left([\mathbf{0}, \mathbf{0}]\right) = [0, 0]$;

(ii) for any $A = \left[\underline{A}, \overline{A}\right]$, $\quad B = \left[\underline{B}, \overline{B}\right]$, if $\overline{A} \odot \overline{B} = 0$ then $\mathcal{P}(A \oplus B) = \mathcal{P}(A) + \mathcal{P}(B)$, where $A \oplus B(x) = \left[\underline{A}(x) \oplus \underline{B}(x), \overline{A}(x) \oplus \overline{B}(x)\right]$, and $+$ is the standard addition of intervals;

(iii) if $A_n \nearrow A$ then $\mathcal{P}(A_n) \nearrow \mathcal{P}(A)$.

In [8], Riečan has shown the existence of interval probability measures differing from those given by (1.2), i.e., he has shown that the probabilistic environment for interval–valued events is much more richer than that for fuzzy events.

Theorem 2.2. *Let $P : \mathcal{A} \to [0,1]$ be a probability measure and let $f, g : [0,1]^2 \to [0,1]$ be functions. Then the mapping $\mathcal{P} : \mathcal{J} \to I$ given by*

$$\mathcal{P}\left(\left[\underline{A}, \overline{A}\right]\right) = \left[f\left(E_P(\underline{A}), E_P(\overline{A})\right), g\left(E_P(\underline{A}), E_P(\overline{A})\right]\right] \tag{2.1}$$

is an interval probability measure if and only if $f(u,v) = (1-\alpha)\, u + \alpha\, v$ and $g(u,v) = (1-\beta)\, u + \beta\, v$ for some $\alpha, \beta \in [0,1]$, $\alpha \le \beta$, i.e., if

$$\mathcal{P}\left(\left[\underline{A}, \overline{A}\right]\right) = \left[(1-\alpha)\, E_P(\underline{A}) + \alpha\, E_P(\overline{A}), (1-\beta)\, E_P(\underline{A}) + \beta\, E_P(\overline{A})\right]. \tag{2.2}$$

Evidently, Grzegorzewski and Mrówka's proposal (1.2) corresponds to the case $\alpha = 0$ and $\beta = 1$, i.e., it is the largest solution of the problem (2.1).

A complete characterization of IFS–probabilities was shown by Theorem 3.1 in Ciungu and Riečan, 2010 [5]. In interval–valued approach this result can be reformulated as follows.

Theorem 2.3. *A mapping $\mathcal{P} : \mathcal{J} \to I$ is an interval probability measure if and only if there are probability measures $P_1, R_1, R_2 : \mathcal{A} \to [0,1]$ and constants β, γ such that $0 \le \beta \le \gamma \le 1$, $\beta\, R_1 \le \gamma\, R_2$, so that*

$$\mathcal{P}\left(\left[\underline{A}, \overline{A}\right]\right) = \left[E_{P_1}(\underline{A}) + \beta\, E_{R_1}(\overline{A} - \underline{A}), E_{P_1}(\underline{A}) + \gamma\, E_{R_2}(\overline{A} - \underline{A})\right]. \tag{2.3}$$

3 An alternative approach to probability measures on interval–valued fuzzy events

We introduce now an alternative approach how to characterize interval–valued probability measures. Due to their continuity (axiom (iii) in Definition 2.1) it is enough to consider finite spaces $X = \{1, \dots, n\}$ and $\mathcal{A} = 2^X$ only.

Theorem 3.1. *Let $X = \{1, \dots, n\}$ for some $n \in N$ and $\mathcal{A} = 2^X$. Then a mapping $\mathcal{P} : \mathcal{J} \to I$ is an interval–valued probability measure if and only if there are probability measures $P, R, Q : \mathcal{A} \to [0,1]$ and constants $u, v \in [0,1]$ such that $v\, Q \le u\, R$, so that*

$$\begin{aligned}\mathcal{P}\left(\left[\underline{A}, \overline{A}\right]\right) = \big[&(1-u)\, E_P(\underline{A}) + u\, E_R(\overline{A}) - v\, E_Q(\overline{A} - \underline{A}),\\ &(1-u)\, E_P(\underline{A}) + u\, E_R(\overline{A})\big].\end{aligned} \tag{3.1}$$

Proof. Observe first that each interval–valued fuzzy event $A = \left[\underline{A}, \overline{A}\right] \in \mathcal{J}$ can be seen as an $n \times 2$ matrix $A = (a_{ij})$ such that $0 \le a_{i1} \le a_{i2} \le 1$, $i = 1, \dots, n$.

Suppose that $\mathcal{P} : \mathcal{J} \to I$ is an interval–valued probability measure on X. Due to the additivity (axiom ii in Definition 1) and due to the classical Cauchy equation [1] it holds

$$\mathcal{P}(A) = \left[\sum_{i,j} \lambda_{ij}\, a_{ij}, \sum_{i,j} \mu_{ij}\, a_{ij}\right]$$

for some non–negative constants λ_{ij}, μ_{ij} independently of $A \in \mathcal{J}$. The boundary condition $\mathcal{P}([\mathbf{1}, \mathbf{1}]) = [1,1]$ forces $\sum_{i,j} \lambda_{ij} = 1 = \sum_{i,j} \mu_{ij}$. On the other hand, denoting $B_k = \left(b_{ij}^{(k)}\right)$ the matrix given by $b_{ij}^{(k)} = \delta_i(k)$ (Dirac function), we have $[\mathbf{1}, \mathbf{1}] = B_1 \oplus \dots \oplus B_n$ and $[1,1] = \mathcal{P}([\mathbf{1}, \mathbf{1}]) = \mathcal{P}(B_1) \oplus \dots \oplus \mathcal{P}(B_n)$. Then necessarily $\mathcal{P}(B_k)$ is a trivial singleton interval and hence $\lambda_{k1} + \lambda_{k2} = \mu_{k1} + \mu_{k2}$, $k = 1, \dots, , n$. Further,

$$\mathcal{P}\left(\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{array}\right]\right) = [\lambda_{12}, \mu_{12}]\,,$$

i.e., $\lambda_{12} \leq \mu_{12}$ (and hence $\lambda_{11} \geq \mu_{11}$). Similarly, $\lambda_{k2} \leq \mu_{k2}$ for $k = 2, \ldots, , n$. Denote $u = \sum_{i=1}^{n} \mu_{i2}$. Then $\sum_{i=1}^{n} \mu_{i1} = 1 - u$.

If $u = 0$, $(\lambda_{11}, \ldots, \lambda_{n1}) = (\mu_{11}, \ldots, \mu_{n1})$ is a probability vector linked to a probability measure P, $v = 0$ and $\mathcal{P}(A) = [E_P(\underline{A}), E_P(\underline{A})]$, i.e., (3.1) holds.

If $u = 1$ then the probability vector $(\mu_{12}, \ldots, \mu_{n2})$ is linked to a probability measure R, and $v = \sum_{i=1}^{n} \lambda_{i1}$. If $v = 0$, $\mathcal{P}(A) = \left[E_R(\overline{A}), E_R(\overline{A})\right]$ and (3.1) holds. If $v > 0$, the probability vector $\left(\frac{\lambda_{i1}}{v}, \ldots, \frac{\lambda_{in}}{v}\right)$ is linked to a probability measure Q, and clearly $v\,Q \leq R$, i.e., (3.1) holds.

Finally, let $0 < u < 1$. Then the probability vectors $\left(\frac{\mu_{11}}{1-u}, \ldots, \frac{\mu_{n1}}{1-u}\right)$ and $\left(\frac{\mu_{12}}{u}, \ldots, \frac{\mu_{n2}}{u}\right)$ are linked to probability measures P and R, respectively. If we denote $v = \sum_{i=1}^{n}(\mu_{i2} - \lambda_{i2})$, evidently $v \in [0, u]$. If $v = 0$, $\mathcal{P}(A) = \left[(1-u)\,E_P(\underline{A}) + u\,E_R(\overline{A}), (1-u)\,E_P(\underline{A}) + u\,E_R(\overline{A})\right]$. Finally, if $v > 0$, we define a probability measure Q by means of a probability vector $\left(\frac{\mu_{12}-\lambda_{12}}{v}, \ldots, \frac{\mu_{n2}-\lambda_{n2}}{v}\right)$, so that evidently $v\,Q \leq R$, and (3.1) holds.

On the other hand, if (3.1) holds, it is an easy verification to see that $\mathcal{P}$ is an interval–valued probability measure on X. □

As already mentioned, formula (3.1) applies also in the case of a general measurable space $(X, \mathcal{A})$.

4 Comparison of two different representations of probability measures and their convex structure on interval–valued fuzzy events

To see the coincidence of formulas (2.3) and (3.1), one should verify the validity (for all $A \in \mathcal{J}$) of the equality

$$\begin{aligned} &\left[E_{P_1}(\underline{A}) + \beta E_{R_1}(\overline{A} - \underline{A}), E_{P_1}(\underline{A}) + \gamma E_{R_2}(\overline{A} - \underline{A})\right] = \\ &= \left[(1-u)E_P(\underline{A}) + uE_R(\overline{A}) - vE_Q(\overline{A} - \underline{A}), (1-u)E_P(\underline{A}) + uE_R(\overline{A})\right]. \end{aligned}$$

Evidently, $u = \gamma$ and $v = \gamma - \beta$. Moreover, $R_2 = R$ (it is enough to consider $\underline{A} = \mathbf{0}$ if $u > 0$). Putting $\underline{A} = \frac{1}{2}\,1_S$, $\overline{A} = 1_S$, $S \subseteq X$, we see that $P_1 = (1-u)\,P + u\,R$. If $v = u$, i.e., $\beta = 0$, R_1 can be chosen arbitrarily. Otherwise, again applying $\underline{A} = \frac{1}{2}\,1_S$, $\overline{A} = 1_S$, $S \subseteq X$, we see that $R_1 = \frac{u\,R - v\,Q}{u-v}$.

Summarizing, we can conclude that the formulas (2.3) and (3.1) fully describe the same class of all probability measures on interval–valued fuzzy events.

Due to Definition 2.1, it is evident that the class $\mathcal{I}_P$ of all probability measures of interval–valued events on a measurable space $(X, \mathcal{A})$ is convex.

Using formula (3.1), each $\mathcal{P} \in \mathcal{I}_P$ can be characterized by a pentuple (P, Q, R, u, v), such that P, Q, R are classical probability measures on $(X, \mathcal{A})$, $u, v \in [0, 1]$, $v\,Q \leq u\,R$. Let $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{I}_P$, $\mathcal{P}_i \sim (P_i, Q_i, R_i, u_i, v_i)$, $i = 1, 2$. Then $\mathcal{P} = \lambda\,\mathcal{P}_1 + (1-\lambda)\,\mathcal{P}_2$ is characterized by (P, Q, R, u, v), where

$$P = \frac{\lambda\,(1-u_1)\,\mathcal{P}_1 + (1-\lambda)\,(1-u_2)\,\mathcal{P}_2}{1 - \lambda u_1 - (1-\lambda)u_2},$$

$$R = \frac{\lambda u_1 R_1 + (1-\lambda)u_2 R_2}{\lambda u_1 + (1-\lambda)u_2},$$

$$Q = \frac{\lambda v_1 Q_1 + (1-\lambda)v_2 Q_2}{\lambda v_1 + (1-\lambda)v_2},$$

$$u = \lambda u_1 + (1-\lambda)u_2,$$

$$v = \lambda v_1 + (1-\lambda)v_2,$$

with convention that if $u = 0$ then $R = \lambda R_1 + (1-\lambda)R_2$, and if $v = 0$ then $Q = \lambda Q_1 + (1-\lambda)Q_2$. Obviously, P, Q, R are probability measures on $(X, \mathcal{A})$, and $v\,Q \leq u\,R$, and thus $\mathcal{P} \in \mathcal{I}_P$.

In the case of a finite space $X = \{1, \dots, n\}$, the convex class $\mathcal{I}_P$ has the next vertices (here D_i is the Dirac measure concentrated in point $\{i\}$):

$$\underline{\mathcal{P}}_i \sim (D_i, D_j, D_k, 0, 0),\ j,k \text{ can be chosen arbitrarily, } \underline{\mathcal{P}}_i(A) = [a_{i1}, a_{i1}];$$

$$\underline{\mathcal{P}}_{ik} \sim (D_i, D_j, D_k, 1, 0),\ j \text{ can be chosen arbitrarily, } \underline{\mathcal{P}}_{ik}(A) = [a_{i1}, a_{i1} + a_{k2} - a_{k1}];$$

$$\overline{\mathcal{P}}_{ij} \sim (D_i, D_j, D_k, 1, 1),\ k \text{ can be chosen arbitrarily, } \overline{\mathcal{P}}_{ij}(A) = [a_{i1}+a_{j2}-a_{j1}, a_{i1}+a_{j2}-a_{j1}],$$

where $i, j, k \in X$. Hence $\mathcal{I}_P$ has exactly $2n^2 + n$ vertices.

Note that the convex closure of vertices $\underline{\mathcal{P}}_{11}, \dots, \underline{\mathcal{P}}_{nn}$yields just the class of Grzegorzewski and Mrówka's probabilities given by formula (1.2). Riečan's probabilities given by formula (2.2) are a convex closure of the vertices $\underline{\mathcal{P}}_{11}, \dots, \underline{\mathcal{P}}_{nn}, \underline{\mathcal{P}}_1, \dots, \underline{\mathcal{P}}_n, \overline{\mathcal{P}}_{11}, \dots, \overline{\mathcal{P}}_{nn}$.

5 Concluding remarks

We have given an alternative look on representation of probability measures on interval–valued fuzzy events by means of 3 classical probability measures, confirming the original results of Riečan with co-author. Note that there are several classical set functions closely related to probability measures, such as belief and plausibility measures, k–additive capacities, etc. In the further investigation of the measure theory on interval–valued (or intuitionistic–valued) fuzzy events we propose to consider the above mentioned generalizations of probability measures.

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