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Probability Limit Identification Functions Revisited*

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To my colleague and friend professor Beloslav Riečan

Abstract

The concept of probability limit identification function introduced by G. Simons in 1970 is revisited and some recent advances on the topic presented. A new proof to the existence of the function under the hypothesis continuum is presented in Section 3.

Keywords Convergence in Probability, Probability Limit Identification Function

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1 Introduction

Consider a parametric family of probability distributions

$$\{P_x,\, x \in E\}$$

on a sample space $(\Omega, \mathcal{F})$, where E will be a **separable metric space**, if not stated otherwise. Its Borel σ-algebra is denoted as $\mathcal{B}(E)$. A sequence of random variables $X_n : \Omega \to E$ such that

$$X_n \to x \quad \text{in} \quad P_x \quad \text{probability for all} \quad x \in E$$

and

$$X_n \to x \quad P_x - \text{almost surely for all} \quad x \in E$$

is called a **weak and strong estimator** of the parameter x, respectively.

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Recall that a sequence of E-valued random variables X_n converge to an E-valued random variable X in P-probability if

$$\lim_n P[d(X_n, X) \geq \epsilon] = 0 \quad \forall \epsilon > 0, \tag{1.1}$$

where d is an equivalent metric for E. A useful equivalent definition is stated by

Theorem 1.1 (Riesz theorem [3, Lemma 4.2]). *The convergence* (1.1) *holds if and only if every increasing sequence of natural numbers* (n_k) *has an subsequence* (n_{k_j}) *such that*

$$\lim_{j\to\infty} X_{n_{k_j}} = X \quad P-\textit{almost surely.}$$

Hence, for arbitrary $x \in E$ there exists an increasing subsequence of natural numbers $n_k(x)$ such that

$$X_{n_k(x)} \to x \quad P_x - \text{almost surely.}$$

Provided that $S \subset E$ is **at most countable set** we are able to apply Riesz theorem and the diagonal Helly's procedure to construct a subsequence (n_k) such that

$$\lim_{k\to\infty} X_{n_k} = x \quad P_x - \text{almost surely for all} \quad x \in S.$$

Unfortunately, such an universal subsequence (n_k) is not available generally as we shall see later on. However, we might be inclined to believe that for arbitrary weak estimator X_n there are transformations $g_n(x_1, \ldots, x_n) : E^n \to E$ such that

$$g_n(X_1, X_2, \ldots, X_n) \to x \quad P_x - \text{almost surely} \quad x \in E.$$

Unfortunately, this is not a true statement, again (see Example 2.5 in Section 2).

We prefer to state our problem in a probabilistic setting. Assume for a while that E is **a Borel set in a Polish metric space**, especially a separable metric space. Given a family of probability measures $\{P_x, x \in E\}$ and a sequence of estimators $\mathbb{X} = (X_1, X_2 \ldots)$ then by Borel isomorphism theorem (see [3, Theorem 8.3.6]) there is an universal probability space $(\Omega, \mathcal{F}, P)$ and $E^{\mathbb{N}}$-valued random sequences

$$\mathbb{X}^x = (X_1^x, X_2^x, \ldots) : (\Omega, \mathcal{F}) \to (E^{\mathbb{N}}, \mathcal{B}(\mathbb{E}^{\mathbb{N}})), \quad x \in E$$

such that

$$\mathcal{L}(\mathbb{X} \,|\, P_x) \quad = \mathcal{L}(\mathbb{X}^x \,|\, P) \quad \forall x \in E, \tag{1.2}$$

where $\mathcal{L}(\mathbb{X} \,|\, P)$ denotes the Borel probability distribution of a sequence $\mathbb{X}$ on the space $E^{\mathbb{N}}$ w.r.t. the measure P. Thus, we may, in this case, operate with one probability measure P and a set of sequences of random variables that are convergent in P-probability because (1.2) implies that $X_n \to x$ in P_x-probability iff $X_n^x \to x$ in P-probability.

Remark that the Lebesgue interval $[0, 1]$ is a space rich enough to be suitable as an universal probability space to allow the above construction.

Denote by $\mathcal{E}(E)$ the space of all sequences $\mathbb{X} = (X_1, X_2, \ldots)$ of E-valued random variables X_n that are convergent in probability to a limit denoted as $p(\mathbb{X})$. Our problem is then stated as follows:

Which are the subsets $\mathcal{F}$ of $\mathcal{E}(E)$ such that it is possible to identify the probability limit $p(\mathbb{X})$ for all $\mathbb{X} \in \mathcal{F}$ almost surely?

An adapted (Borel) identification for $\mathcal{F}$ (a concept perhaps more suitable for the needs of statistics) requires the existence of (Borel) $g_n(x_1, x_2, \ldots, x_n) : E^n \to E$ transforms such that

$$g_n(X_1, X_2, \ldots, X_n) \to p(\mathbb{X}) \quad \text{almost surely} \quad \forall\, \mathbb{X} = (X_1, X_2 \ldots) \in \mathcal{F}.$$

A humble probabilist might be perhaps satisfied with the existence of a (Borel) map $f(x_1, x_2, \ldots) : E^{\mathbb{N}} \to E$ such that

$$f(X_1, X_2, \ldots) = p(\mathbb{X}) \quad \text{almost surely} \quad \forall\, \mathbb{X} = (X_1, X_2 \ldots) \in \mathcal{F},$$

the map f being called a **(Borel) probability limit identification function (PLIF)** for $\mathcal{F}$.

Observe that if $g_n(x_1, x_2, \ldots, x_n)$ define an adapted (Borel) identification, then

$$f(x_1, x_2, \ldots) = \limsup_n g_n(x_1, x_2, \ldots, x_n)$$

is a (Borel) PLIF for $\mathcal{F}$.

2 PLIFs - the present state of art

If $\mathcal{F}$ is a countable subset of $\mathcal{E}(E)$ (E a separable metric space) , then Riesz theorem yields an universal subsequence of natural numbers $n_1 < n_2 < \ldots$ such that

$$X_{n_k} \to p(\mathbb{X}) \quad \text{almost surely} \quad \forall \mathbb{X} = (X_1, X_2, \ldots) \in \mathcal{F}.$$

and therefore there is an adapted Borel identification for $\mathcal{F}$, hence also a Borel PLIF for $\mathcal{F}$.

P. Kříž in [4] proved a deeper statement

Theorem 2.1. *Let E be a separable metric space and $\mathcal{F}$ a subset of $\mathcal{E}(E)$ such that there is a Borel σ-finite measure μ on E that dominates all probability distributions in*

$$\mathcal{P}(\mathcal{F}) = \{P_{\mathbb{X}}, \mathbb{X} \in \mathcal{F}\}, \quad i.e. \quad P_{\mathbb{X}} \ll \mu \quad \forall\, \mathbb{X} \in \mathcal{F}. \tag{2.1}$$

Then there exists a Borel adapted identification, hence a Borel PLIF for $\mathcal{F}$.

We have denoted by $P_{\mathbb{X}}$ the probability distribution of the sequence $\mathbb{X}$ defined as

$$P_{\mathbb{X}}(B) = P[\mathbb{X} \in B], \quad B \subset E^{\mathbb{N}} \quad \text{a Borel set.}$$

Proof is an application of a well known theorem that states that any set $\mathcal{P}$ of probability measures on a countably generated σ-algebra ($\mathcal{B}(E^{\mathbb{N}})$ in our case) that is dominated in sense of (2.1) is separable w.r.t. the total variation metric (see [7, Theorem A.4.1]).

To construct sets $\mathcal{F} \subset \mathcal{E}(E)$ that satisfy (2.1) might not be a very easy task.

Example 2.2. Consider $E = \{0, 1\}$ that provides $E^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ as a compact Abel group with the Haar probability measure $\mu = P_{\mathbb{Y}}$, where $\mathbb{Y} = (Y_1, Y_2, \ldots)$ is a sequence of i.i.d. random variables with $P[Y_n = 0] = P[Y_n = 1] = 1/2$. Then

$$P_{\mathbb{X}} \quad \perp \quad \mu \quad \forall \mathbb{X} \in \mathcal{E}(\{0,1\}).$$

Indeed, having a sequence $\mathbb{X} = (X_1, X_2, \dots)$ of $0-1$ random variables that is convergent in probability then there is a subsequence $r = (r_1 < r_2 < \dots)$ such that $P_{\mathbb{X}}(A_r) = 1$, where

$$A_r := \{x = (x_1, x_2 \dots) \in \{0,1\}^{\mathbb{N}} : \lim_{k\to\infty} x_{r_k} \quad \text{exists}\}.$$

On the other hand, $\mu(A_r) = 0$ by Kolomogoroff 0-1 law.

A good space to start with is $\mathcal{E}(\{0,1\})$ of all sequences of 0-1 random variables that converge in probability. Its important subset is

$$\mathcal{H} = \{\mathbb{X} \in \mathcal{E}(\{0,1\}) : p(\mathbb{X}) = 1 \quad \text{or} \quad 0\}.$$

It is a good space because of

Theorem 2.3 (G. Simons (1971))**.** *If there is a (Borel) PLIF for $\mathcal{H}$ then there is a (Borel) PLIF for $\mathcal{E}([0,1])$.*

We recapitulate shortly **the Simon's proof**. Consider $\mathbb{X} \in \mathcal{E}([0,1])$ and $a \in [0,1]$. Put

$$\mathbb{X}^a = \left(I_{[X_1<a]}, I_{[X_2<a]}, \dots\right) \quad \text{and} \quad \mathbb{Y}^a = \left(I_{[X_1<a,\, p(\mathbb{X})>a]}, I_{[X_2<a,\, p(\mathbb{X})>a]}, \dots\right).$$

Note that

$$\mathbb{Y}^a \in \mathcal{H} \quad \text{with} \quad p(\mathbb{Y}) = 0 \quad \text{almost surely.}$$

So if $f(x_1, x_2, \dots)$ is a PLIF for $\mathcal{H}$, then outside a P-null set and all rational numbers $a \in [0,1]$

$$p(\mathbb{X}) > a \quad \Rightarrow \quad f(\mathbb{X}^a) = f(\mathbb{Y}^a) = p(\mathbb{Y}^a) = 0$$

and by a symmetry,

$$p(\mathbb{X}) < a \quad \Rightarrow \quad f(\mathbb{X}^a) = 1$$

hold. Define a $g : [0,1]^{\mathbb{N}} \to [0,1]$ by

$$g(x) := \sup\{a \in \mathbb{Q} \cap [0,1] \quad \text{such that} \quad f(x^a) = 0\}, \quad x = (x_1, x_1, \dots) \in [0,1]^{\mathbb{N}},$$

where $x^a = (x_1^a, x_2^a, \dots)$ and $x_n^a = I_{[x_n<a]}$. It follows that outside a P-null set and for all rational numbers $0 \le a < b \le 1$

$$a < p(\mathbb{X}) < b \quad \Rightarrow \quad a < g(\mathbb{X}) < b.$$

Hence, $p(\mathbb{X}) = g(\mathbb{X})$ almost surely. If f is a (Borel) PLIF for $\mathcal{H}$, then g is a (Borel) PLIF for $\mathcal{E}([0,1])$.

We complement slightly the above result.

Corollary 2.4. *If there is a (Borel) PLIF for $\mathcal{H}$ then there is a (Borel) PLIF for $\mathcal{E}(E)$ where E is arbitrary separable metric space.*

Proof. By Theorem 2.3 we may assume that there is a (Borel) PLIF $f(x_1, x_2, \dots)$ for $\mathcal{E}([0,1])$. It follows that

$$f^{\mathbb{N}}(x_1, x_2, \dots) := (f(x_1), f(x_2), \dots), \quad \text{where} \quad x_n \in [0,1]^{\mathbb{N}}$$

defines a (Borel) PLIF for $\mathcal{E}([0,1]^{\mathbb{N}})$. Hence, there exists a (Borel) PLIF for $\mathcal{E}(F)$ where $F \subset [0,1]^{\mathbb{N}}$ is arbitrary subset. The proof is concluded by Urysohn Theorem that states that any separable metric space E is homeomorphic to a subset of the Urysohn's cube $[0,1]^{\mathbb{N}}$. □

Obviously, the space $\mathcal{H}$ deserves our attention. We construct its subset $\mathcal{H}^*$ as follows:

Example 2.5 (G.Simons (1971))**.** Pick up a sequence

$$\mathbb{Y} = (Y_1, Y_2, \dots) \quad \text{of independent 0-1 random variables such that}$$
$$\lim_n P[Y_n = 1] = 0 \quad \text{or} \quad 1 \quad (\Rightarrow \quad \mathbb{Y} \in \mathcal{H}).$$

Also, choose a sequence of natural numbers $r = (r_1 < r_2 < \dots)$ and define $\mathbb{X} = \mathbb{X}(\mathbb{Y}, r)$ as

$$\begin{array}{cccccccccccccc} & 1 & 2 & \dots\dots & r_1 & r_1+1 & r_1+2 & \dots\dots & r_2 & r_2+1 & r_2+2 & \dots\dots \\ \mathbb{X} = & Y_1 & Y_1 & \dots\dots & Y_1 & Y_2 & Y_2 & \dots\dots & Y_2 & Y_3 & Y_3 & \dots\dots \end{array}$$

Finally, denote by $\mathcal{H}^*$ the set of all sequences constructed in the above manner. We state that **there is no adapted identification of probability limit for $\mathcal{H}^*$**. G. Simons reasons as follows. Assume that there are $g_n(x_1, x_2, \dots, x_n) : \{0,1\}^n \to \{0,1\}$ such that

$$g_n(X_1, X_2, \dots, X_n) \to p(\mathbb{X}) \quad \text{almost surely} \quad \forall\, \mathbb{X} = (X_1, X_2, \dots) \in \mathcal{H}^*.$$

For a fixed $j \in \mathbb{N}$ choose a sequence of independent r.v.'s $\mathbb{Y} = (Y_1, Y_2, \dots)$ such that

$$P[Y_k = 1] \in (0,1) \quad \text{if} \quad k \le j \quad \text{and} \quad P[Y_k = 1] = 0 \quad \text{if} \quad k > j.$$

Note that

$$g_1(Y_1), \dots, g_j(Y_1, \dots, Y_j), \dots, g_m(Y_1, \dots, Y_j, Y_{j+1}, \dots, Y_m), \dots$$

is a sequence that converges to $p(\mathbb{Y}) = 0$ almost surely and is almost surely one of the 2^j sequences

$$g_1(x_1),\ \dots,\ g_j(x_1, \dots, x_j),\ \dots,\ g_m(x_1, \dots, x_j, 0, \dots, 0), \quad (x_1, \dots, x_j) \in \{0,1\}^j.$$

Hence, there is an m_j such that for all $m \ge m_j$ and $(x_1, \dots, x_j) \in \{0,1\}^j$

$$g_m(x_1, \dots, x_j, 0, \dots, 0) = 0 \tag{2.2}$$

and by a symmetry

$$g_m(x_1, \dots, x_j, 1, \dots, 1) = 1. \tag{2.3}$$

holds.

Finally, choose a sequence of independent r.v.'s $\mathbb{Y} = (Y_1, Y_2, \dots) \in \mathcal{H}^*$ such that

$$\sum_1^\infty P[Y_n = 0] = \infty \quad \text{and} \quad \sum_1^\infty P[Y_n = 1] = \infty$$

and put $\mathbb{X} = \mathbb{X}(\mathbb{Y}, r)$, where $r = (r_1 < r_2 < \dots)$. The Borel-Cantelli 0-1 law states that the $\mathbb{X}$ owns infinitely many strings of zeros and infinitely many strings of ones. If the sequence $r = (r_1, r_2, \dots)$ grows rapidly enough (perhaps as $r_{k+1} \geq m_{r_k}$), then according to (2.2) and (2.3).

$$P[g_{r_{k+1}}(X_1, \dots, X_{r_k}, X_{r_k+1}, \dots X_{r_{k+1}}) = 0, \quad \infty \times k] = 1, \tag{2.4}$$

$$P[g_{r_{k+1}}(X_1, \dots, X_{r_k}, X_{r_k+1}, \dots X_{r_{k+1}}) = 1, \quad \infty \times k] = 1. \tag{2.5}$$

Hence, a contradiction.

Example 2.6 (G. Simons (1971))**.** An interesting subset of $\mathcal{H}^*$ is $\mathcal{H}^{**} = \{\mathbb{X}(\mathbb{Y}, r)\}$ defined by

$$\begin{aligned} p(\mathbb{Y}) = 1 \quad &\Rightarrow \quad \text{almost all} \quad r_k \quad \text{even} \\ p(\mathbb{Y}) = 0 \quad &\Rightarrow \quad \text{almost all} \quad r_k \quad \text{odd.} \end{aligned} \tag{2.6}$$

We state that **there is no adapted identification for the probability limit for $\mathcal{H}^{**}$** (to be applied the same reasoning as in Example 2.5), but **there is a Borel PLIF for $\mathcal{H}^{**}$**. For the proof of the latter statement we are indebted to P. Kříž (2012).

We denote by $\mathcal{N}_0$ the set of sequences in $\{0,1\}^{\mathbb{N}}$ that converge to zero and by $\mathcal{N}_1$ the set of those that converge to one. Note that $\mathcal{N} := \{0,1\}^{\mathbb{N}} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1)$ is the set of all sequences with an infinite number of changes. Put

$$\mathcal{N}^{even} := \{\text{the sequences in } \mathcal{N} \text{ with at most finite changes in the even coordinates}\}$$
$$\mathcal{N}^{odd} := \{\text{the sequences in } \mathcal{N} \text{ with at most finite changes in the odd coordinates}\}.$$

Note that the sets $\mathcal{N}_0$, $\mathcal{N}_1$, $\mathcal{N}^{odd}$, $\mathcal{N}^{even}$ are paire-wise disjoint.

It follows by Borel-Cantelli 0-1 law that a sequence $\mathbb{Y}$ of independent 0-1 random variables is either convergent almost surely or owns infinitely many ones and infinitely zeros. Hence, for any $\mathbb{X} = \mathbb{X}(\mathbb{Y}, r) \in \mathcal{H}^{**}$

$$\begin{aligned} p(\mathbb{X}) = 1 \quad &\Rightarrow \quad \mathbb{X} \in \mathcal{N}_1 \cup \mathcal{N}^{odd} \\ p(\mathbb{X}) = 0 \quad &\Rightarrow \quad \mathbb{X} \in \mathcal{N}_0 \cup \mathcal{N}^{even} \end{aligned}$$

holds almost surely. It follows that

$$\begin{aligned} &f(x) = 1 \quad \text{if} \quad x \in \mathcal{N}_1 \cup \mathcal{N}^{odd}, \quad f(x) = 0 \quad \text{if} \quad x \in \mathcal{N}_0 \cup \mathcal{N}^{even}, \\ &f(x) = 0 \quad \text{else} \end{aligned}$$

defines a Borel PLIF for $\mathcal{H}^{**}$.

A very deep and rather conclusive is a result of D. Blackwell (1980).

Theorem 2.7. *There is no Borel PLIF for the set*

$$\mathcal{H} := \{\mathbb{X} \in \mathcal{E}(0,1) : p(\mathbb{X}) = 1 \quad OR \quad p(\mathbb{X}) = 0\}.$$

A straightforward implication is

Corollary 2.8. *There is no Borel PLIF for $\mathcal{E}(E)$ whenever E is a separable metric space with card$(E) \geq 2$.*

The Blackwell's sophisticated **proof** goes along the following lines:

Assume that $f(x_1, x_2, \dots)$ is a Borel PLIF for $\mathcal{H}$. We may assume w.l.g. that f is the indicator of a tail Borel set $C \subset \{0,1\}^{\mathbb{N}}$ which means that it is a Borel set such that

$$\begin{aligned} &x = (x_1, x_2, \dots) \in C, \quad y = (y_1, y_2, \dots) \in \{0,1\}^{\mathbb{N}}, \\ &x_k = y_k, \quad k \geq n \quad \text{for some} \quad n \in \mathbb{N} \qquad \Rightarrow \quad y \in C \end{aligned}$$

holds. By Oxtoby's category 0-1 law [8, Theorem 21.4] either C or $\{0,1\}^{\mathbb{N}} \setminus C$ is a set of the first category which means that the other set contains a G_δ set H such that $\overline{H} = \{0,1\}^{\mathbb{N}}$. Assume that $H \subset C$. The corner stone of the proof is a lemma that says that for arbitrary dense G_δ-set H there is a sequence of 0-1 r.v.'s $\mathbb{X} = (X_1, X_2, \dots)$ such that

$$P[\mathbb{X} \in H] = 1 \quad \text{and} \quad p(\mathbb{X}) = 0 \quad \text{almost surely},$$

hence, a contradiction with $f(\mathbb{X}) = 1$.

Surprisingly, skipping the measurability requirement for PLIF's and allowing the continuum hypothesis we have all the PLIFs we might need.

Theorem 2.9. *Let E be a separable metric space. Then, under the continuum hypothesis, there exists a PLIF for $\mathcal{E}(E)$. In other words, there is a map $f : E^{\mathbb{N}} \to E$ such that*

$$f(\mathbb{X}) = p(\mathbb{X}) \quad \textit{almost surely}$$

for all sequences $\mathbb{X}$ of E-valued r.v.'s that converge in probability.

Theorem was first proved in 1973 in [10] for $E = \mathbb{R}$ and generalized to separable metric spaces in 2010 in [4]. We shall propose a new and more straightforward proof of the theorem in the coming Section.

3 Proof of Theorem 2.9

First, we employ the continuum hypothesis to prove that there is a PLIF for the set

$$\mathcal{H} = \{\mathbb{X} \in \mathcal{E}(\{0,1\}) : p(\mathbb{X}) = 1 \quad \text{or} \quad 0\}.$$

What we have to prove is that there exits a set $C \subset \{0,1\}^{\mathbb{N}}$ such that

$$\mathbb{X} \in \mathcal{H}, \quad p(\mathbb{X}) = 1 \quad \Rightarrow \quad \mathbb{X} \in C \quad \text{almost surely}$$

and

$$\mathbb{X} \in \mathcal{H}, \quad p(\mathbb{X}) = 0 \quad \Rightarrow \quad \mathbb{X} \notin C \quad \text{almost surely}.$$

This is as to say that that there exists a set $C \subset \{0,1\}^{\mathbb{N}}$ such that

$$\mu \in \mathcal{M}_1 \quad \Rightarrow \quad \mu_*(C) = 1, \qquad \mu \in \mathcal{M}_0 \quad \Rightarrow \quad \mu_*(\{0,1\}^{\mathbb{N}} \setminus C) = 1, \tag{3.1}$$

where we have denoted

$$\mathcal{M}_1 = \{P_{\mathbb{X}},\, \mathbb{X} \in \mathcal{H},\, p(\mathbb{X}) = 1\}, \quad \mathcal{M}_0 = \{P_{\mathbb{X}},\, \mathbb{X} \in \mathcal{H},\, p(\mathbb{X}) = 0\}.$$

Note that $\mathrm{card}(\mathcal{M}_1) = \mathrm{card}(\mathcal{M}_0) = c$. Hence, the continuum hypothesis allows to enumerate the set $\mathcal{M}_1$ as

$$\mathcal{M}_1 = \{\mu^\alpha,\, \alpha < \Omega\},$$

where Ω denotes the first uncountable ordinal number. Further, by the transfinite construction we get a net

$$r(\alpha) = (r_1(\alpha) < r_2(\alpha) < \dots), \quad \alpha < \Omega$$

of sequences of natural numbers such that $\mu^\alpha(A_\alpha) = 1$ holds for all $\alpha < \Omega$, where

$$A_\alpha := \{x = (x_1, x_2, \dots) \in \{0,1\}^{\mathbb{N}} : \lim_{n\to\infty} x_{r_n(\alpha)} = 1\}$$

and such that

$$\alpha < \beta \quad \Rightarrow \quad r(\beta) \quad \text{is a subsequence of} \quad r(\alpha).$$

The steps of the transfinite construction are performed as follows:

If α is an isolated ordinal, then $\alpha = \beta + 1$ for some $\beta < \alpha$ and $r(\alpha)$ is constructed as a subsequence of $r(\beta)$ by means of Riesz theorem If α is a limit ordinal number, then there are $\beta_1 < \dots < \beta_n \dots < \alpha$ such that $\alpha = \sup_n \beta_n$. Then, $r(\alpha)$ is received as a subsequence of all $r(\beta_n)$ sequences by Riesz theorem, again. Observe that $\{A_\alpha,\, \alpha < \Omega\}$ is an increasing net and put $C := \bigcup_{\alpha<\Omega} A_\alpha$. Then

$$\mu_*^\alpha(C) = 1 \quad \forall \alpha < \Omega$$

and the former implication in (3.1) is satisfied.

Define by

$$T(x_1, x_2, \dots, x_n, \dots) = (1-x_1, 1-x_2, \dots, 1-x_n, \dots), \quad (x_1, x_2, \dots, x_n, \dots) \in \{0,1\}^{\mathbb{N}}$$

a homeomorphic map $\{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ and observe that if $\mathbb{X} \in \mathcal{H}$ then

$$p(\mathbb{X}) = 1 \quad \Longleftrightarrow \quad p(T\mathbb{X}) = 0.$$

Hence,

$$\mathcal{M}_0 = \{\nu^\alpha := T \circ \mu^\alpha, \quad \alpha < \Omega\},$$

where the measure $\mu \circ T$ is defined on the Borel σ-algebra of $\{0,1\}^{\mathbb{N}}$ by $(T \circ \mu)(B) = \mu(T^{-1}B)$. Obviously, $C \cap T(C) = \emptyset$ since

$$x \in A_\alpha, \quad \mathbb{T}(x) \in A_\beta \quad (\text{say that} \quad \alpha < \beta) \quad \Rightarrow \quad x \in A_\beta \cap T(A_\beta) = \emptyset.$$

Finally, note that

$$\nu^{\alpha}\left(TA_{\alpha}\right)=\mu^{\alpha}(A_{\alpha})=1 \quad \Rightarrow \quad \nu_{*}^{\alpha}\left(\{0,1\}^{\mathbb{N}}\setminus C\right)=1, \quad \forall\alpha<\Omega$$

verifies the latter implication in (3.1) and $f=I_C$ defines a PLIF for the set $\mathcal{H}$.

To conclude the proof of Theorem 2.9 apply Corollary 2.4. □

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