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The partially ordered metric semigroup valued lower integral

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Abstract

The lower integral defined on the lattice ordered group with values in a partially ordered metric semigroup and the integrable elements are defined in this paper.

Keywords lower integral, ℓ-group, partially ordered metric semigroup, integrability

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1 Introduction

The lower integral on the real-valued functions was defined by Topsoe in [7]. An axiomatic definition of the real-valued upper or lower integral and the fuzzy number valued lower integral defined on the ℓ-group and the notion of integrability were introduced in [6], [8], [9] and [10]. We generalize this theory for the partially ordered metric semigroup valued lower integral. The metric semigroup was used in the case of Kurzweil–Henstock integral in [2], [3], [4], [11] and also as the range space of BV mappings of two real variables in [1]. The aim of this paper is to define the lower integral and the integrable elements, and to prove that the upper limits of the countable set of integrable elements are integrable.

2 Preliminaries

Definition 2.1. A partially ordered metric semigroup is a structure $(X, \varrho, +, \leq)$, where $\varrho : X \times X \to \mathbb{R}$, $+ : X \times X \to X$ satisfy the following conditions:

(i) (X, ϱ) is a metric space

(ii) $(X, +)$ is a commutative semigroup endowed with a neutral element 0

(iii) $(X, \leq)$ is a partially ordered set

(iv) if $u, v, z \in X$ and $u \le v$ then $u + z \le v + z$.

(v) ϱ is translation invariant: $\varrho(u, v) = \varrho(u + w, v + w)$ for all $u, v, w \in X$

(vi) $\varrho(u + y, v + z) \le \varrho(u, v) + \varrho(y, z)$ for all $u, v, y, z \in X$

Examples 2.2. 1. A simple example of the partially ordered metric semigroup is the set $\mathbb{R}$ with metric $\varrho(u, v) = |u - v|$ and the usual order.

2. Let X be a set of all pointwise ordered bounded real-valued functions defined on a compact metric space M with the metric $\varrho(f, g) = \sup\{|f(u) - g(u)|;\ u \in M\}$. Then $(X, \varrho, +, \le)$ is a partially ordered metric semigroup.

3. An example of the partially ordered metric semigroup which is not a group is the set of fuzzy numbers $E = (E, D, +, \le)$. The sum of fuzzy numbers u, v is a fuzzy number z such that

$$z = u + v \Leftrightarrow (z)^\alpha = (u)^\alpha + (v)^\alpha \text{ for every } \alpha \in (0, 1],$$

where $(u)^\alpha = \{x \in \mathbb{R},\ u(x) \ge \alpha\}$ and the sum of intervals $[a, b] + [c, d] = [a + c, b + d]$. The partial ordering on the set E is defined in the following way:

$$u \le v \ \Leftrightarrow\ (u)^\alpha \le (v)^\alpha \text{ for every } \alpha \in (0, 1],$$

where $[a, b] \le [c, d] \ \Leftrightarrow\ (a \le c \wedge b \le d)$. The Hausdorff distance d of closed possibly degenerate intervals is defined by equation:

$$d([a, b], [c, d]) = \max\{|c - a|, |d - b|\}.$$

Then (E, D), where $D : E \times E \to [0, \infty)$,

$$D(u, v) = \sup\left\{d\left((u)^\alpha, (v)^\alpha\right);\ \alpha \in (0, 1]\right\}$$

is a complete metric space. The properties (v) and (vi) of the metric D can be found in [12].

3 The lower integral

Definition 3.1. Let G be an ℓ-group. The lower integral is a mapping $I : G^+ \to X$ which fulfills the following conditions:

1) $I(0) = 0$,

2) if $x \le y$ then $I(x) \le I(y)$ for all $x, y \in G^+$,

3) $I(x) + I(y) \le I(x + y)$ for all $x, y \in G^+$,

4) if $x_n \downarrow x,\ x, x_n \in G^+$ $(n = 1, 2, \dots)$ then $\lim_{n\to\infty} \varrho(I(x_n), I(x)) = 0$.

Definition 3.2. Let G be an ℓ-group and I be a lower integral on G^+. An element $x \in G^+$ is called I-integrable iff

$$I(a) = I(a \wedge x) + I(a - (a \wedge x))$$

for any $a \in G^+$. We denote $G_I^+ = \{x \in G^+;\ x \text{ is } I\text{-integrable}\}$.

Theorem 3.3. *(i) If $a \in G^+$ and $x \in G_I^+$ then $I(a + x) = I(a) + I(x)$.*
(ii) If $x, y \in G_I^+$ then $x + y, x \wedge y \in G_I^+$. Furthermore, if $x, y \in G_I^+, y \le x$ then $x - y \in G_I^+$.
(iii) If $x, y \in G_I^+$ then $x \vee y \in G_I^+$.

Proof. (i) Let $a \in G^+$ and $x \in G_I^+$. Then from properties of ℓ-group we get

$$I(a+x) = I((a+x) \wedge x) + I(a+x-(a+x)\wedge x) = I(x) + I(a).$$

(ii) If $a, x, y \in G, y \geq 0$ then

$$a \wedge x + ((a - a \wedge x) \wedge y) = a \wedge ((a \wedge x) + y) = a \wedge (a+y) \wedge (x+y) = a \wedge (x+y).$$

Let $x, y \in G_I^+,\ a \in G^+$. From the property 3) of the lower integral we have

$$\begin{aligned} I(a) &= I(a \wedge x) + I(a - a \wedge x) \\ &= I(a \wedge x) + I((a - a \wedge x) \wedge y) + I(a - a \wedge x - (a - a \wedge x) \wedge y) \\ &\leq I(a \wedge (x+y)) + I(a - a \wedge (x+y)) \leq I(a). \end{aligned}$$

Hence, $x + y \in G_I^+$. Similarly

$$\begin{aligned} I(a) &= I(a \wedge x) + I(a - a \wedge x) \\ &= I(a \wedge x \wedge y) + I(a \wedge x - a \wedge x \wedge y) + I(a - a \wedge x) \\ &\leq I(a \wedge (x \wedge y)) + I(a - a \wedge (x \wedge y)) \leq I(a). \end{aligned}$$

It follows $x \wedge y \in G_I^+$. It holds $(a+y) \wedge x = a \wedge (x-y) + y$ in every ℓ-group. Let $x, y \in G_I^+$, $y \leq x$, $a \in G^+$. By the proof of the assertion (i),

$$\begin{aligned} I(a) + I(y) &= I(a+y) = I((a+y) \wedge x) + I(a+y-(a+y)\wedge x) \\ &= I(a \wedge (x-y) + y) + I(a - a \wedge (x-y)) \\ &= I(y) + I(a \wedge (x-y)) + I(a - a \wedge (x-y)). \end{aligned}$$

Using the property (v) of the metric ϱ we can write

$$\begin{aligned} & \varrho(I(a), I(a \wedge (x-y)) + I(a - a \wedge (x-y))) \\ = \ & \varrho(I(a) + I(y), I(a \wedge (x-y)) + I(a - a \wedge (x-y)) + I(y)) = 0. \end{aligned}$$

Because $\varrho(x, y) = 0$ iff $x = y$ we get

$$I(a) = I(a \wedge (x-y)) + I(a - a \wedge (x-y)).$$

Hence, $x - y \in G_I^+$.
(iii) The assertion follows from the part (ii) and the equation $x \vee y = (x+y) - x \wedge y$. □

Theorem 3.4. *Let $x_n \uparrow x, x_n \in G_I^+, n = 1, 2, \ldots, x \in G^+$. Then $x \in G_I^+$ and $I(x_n) \to I(x)$ on the metric ϱ.*

Proof. Let $x_n \uparrow x, x_n \in G_I^+, n = 1, 2, \ldots, x \in X^+$. Using the integrability of x_n and the property (vi) of the metric ϱ we get

$$\begin{aligned} & \varrho(I(a \wedge x) + I(a - a \wedge x), I(a)) \\ = \ & \varrho(I(a \wedge x) + I(a - a \wedge x), I(a \wedge x_n) + I(a - a \wedge x_n)) \\ \leq \ & \varrho(I(a \wedge x), I(a \wedge x_n)) + \varrho(I(a - a \wedge x), I(a - a \wedge x_n)) \end{aligned}$$

for every $n \in \mathbb{N}$, hence

$$\begin{aligned} & \varrho(I(a \wedge x) + I(a - a \wedge x), I(a)) \\ \leq \ & \lim_{n \to \infty} \varrho(I(a \wedge x), I(a \wedge x_n)) + \lim_{n \to \infty} \varrho(I(a - a \wedge x), I(a - a \wedge x_n)). \end{aligned}$$

From the property 4) of the lower integral the second limit equals zero. The first limit equals zero too, because

$$\begin{aligned}
&\lim_{n\to\infty} \varrho\left(I\left(a\wedge x\right), I\left(a\wedge x_n\right)\right)\\
&= \lim_{n\to\infty} \varrho\left(I\left(a\wedge x\wedge x_n\right) + I\left(a\wedge x - a\wedge x\wedge x_n\right), I\left(a\wedge x_n\right)\right)\\
&= \lim_{n\to\infty} \varrho\left(I\left(a\wedge x_n\right) + I\left(a\wedge x - a\wedge x_n\right), I\left(a\wedge x_n\right)\right)\\
&\le \lim_{n\to\infty} \varrho\left(I\left(a\wedge x_n\right), I\left(a\wedge x_n\right)\right) + \lim_{n\to\infty} \varrho\left(I\left(a\wedge x - a\wedge x_n\right), 0\right)\\
&= 0 + \lim_{n\to\infty} \varrho\left(I\left(a\wedge x - a\wedge x_n\right), I\left(0\right)\right) = 0
\end{aligned}$$

by the property (vi) of ϱ and 4) of I. So

$$\varrho\left(I\left(a\wedge x\right) + I\left(a - a\wedge x\right), I\left(a\right)\right) = 0,$$

that is $I\left(a\wedge x\right) + I\left(a - a\wedge x\right) = I\left(a\right)$ and $x \in G_I^+$. Now we prove $I\left(x_n\right) \to I\left(x\right)$ in the metric ϱ. It holds

$$\begin{aligned}
\lim_{n\to\infty} \varrho\left(I\left(x\right), I\left(x_n\right)\right) &= \lim_{n\to\infty} \varrho\left(I\left(x\wedge x_n\right) + I\left(x - x\wedge x_n\right), I\left(x_n\right)\right)\\
&= \lim_{n\to\infty} \varrho\left(I\left(x_n\right) + I\left(x - x_n\right), I\left(x_n\right)\right)\\
&\le \lim_{n\to\infty} \varrho\left(I\left(x_n\right), I\left(x_n\right)\right) + \lim_{n\to\infty} \varrho\left(I\left(x - x_n\right), 0\right)\\
&= 0.
\end{aligned}$$

□

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