Initial Coefficient Bounds for a Comprehensive Subclass of Sakaguchi Type Functions

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Abstract
In this paper, we introduce and investigate a new subclass of the function class $\Sigma$ of bi-univalent functions defined in the open unit disk. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass.

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1 Introduction and Definitions
Let $A$ denote the class of analytic functions in the unit disk

$$U = \{ z \in \mathbb{C} : |z| < 1 \}$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

Further, by $S$ we shall denote the class of all functions in $A$ which are univalent in $U$.

The Koebe one-quarter theorem [8] states that the image of $U$ under every function $f$ from $S$ contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse $f^{-1}$ which satisfies

$$f^{-1}(f(z)) = z, \quad (z \in U)$$

and

$$f(f^{-1}(w)) = w, \quad \left( |w| < r_0(f), \quad r_0(f) \geq \frac{1}{4} \right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots.$$
A function \( f(z) \in A \) is said to be bi-univalent in \( U \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( U \). Let \( \Sigma \) denote the class of bi-univalent functions defined in the unit disk \( U \).

If the functions \( f \) and \( g \) are analytic in \( U \), then \( f \) is said to be subordinate to \( g \), written as
\[
f(z) < g(z), \quad (z \in U)
\]
if there exists a Schwarz function \( w(z) \), analytic in \( U \), with
\[
w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in U)
\]
such that
\[
f(z) = g(w(z)) \quad (z \in U).
\]

Lewin [15] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient \( |a_2| \). Subsequently, Netanyahu [17] showed that \( \max |a_2| = \frac{4}{3} \) if \( f(z) \in \Sigma \). Brannan and Clunie [3] conjectured that \( |a_2| \leq \sqrt{2} \) for \( f \in \Sigma \). Brannan and Taha [4] introduced certain subclasses of the bi-univalent function class \( \Sigma \) similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and obtained estimates on the initial coefficients. Bounds for the initial coefficients of several classes of functions were also investigated in [1, 3, 7, 9, 13, 14, 16, 19, 21, 22, 23].

Not much is known about the bounds on the general coefficient \( |a_n| \) for \( n \geq 4 \). In the literature, the only a few works determining the general coefficient bounds \( |a_n| \) for the analytic bi-univalent functions [2, 6, 10, 11, 12]. The coefficient estimate problem for each of \( |a_n| \) \( n \in \mathbb{N} \setminus \{1,2\} \); \( \mathbb{N} = \{1,2,3,...\} \) is still an open problem.

Motivated by the earlier work of Sakaguchi [20] on the class of starlike functions with respect to symmetric points denoted by \( S_{\Sigma} \) consisting of functions \( f \in A \) satisfy the condition \( \text{Re} \left( \frac{z f''(z)}{f'(z) - f(-z)} \right) > 0 \), \( (z \in U) \), we introduce a new subclass of the function class \( \Sigma \) of bi-univalent functions, and find estimates on the coefficients \( |a_2| \) and \( |a_3| \) for functions in this new subclass.

**Definition 1.** Let \( h : U \rightarrow \mathbb{C} \), be a convex univalent function such that \( h(0) = 1 \) and \( h(\bar{z}) = \bar{h}(z) \), for \( z \in U \) and \( \text{Re}(h(z)) > 0 \). A function \( f \in \Sigma \) is said to be in the class \( S^{\lambda}_{\Sigma}(\beta, s, t, h) \) if the following conditions are satisfied:
\[
f(\Sigma), \quad e^{i\beta} \left[ (s-t)z \right]^{1-\lambda} f'(z) \left[ f(sz) - f(tz) \right]^{1-\lambda} < h(z) \cos \beta + i \sin \beta, \quad z \in U \tag{1.2}
\]
and
\[
e^{i\beta} \left[ (s-t)w \right]^{1-\lambda} g'(w) \left[ g(sw) - g(tw) \right]^{1-\lambda} < h(w) \cos \beta + i \sin \beta, \quad w \in U \tag{1.3}
\]
where \( g(w) = f^{-1}(w), \ s, t, \in \mathbb{C} \) with \( s \neq t, \ |t| \leq 1, \ \beta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) and \( \lambda > 0 \).

**Remark 2.** If we set \( h(z) = \frac{1 + Az}{1 + Bz}, \ -1 \leq B < A \leq 1 \), in the class \( S^{\lambda}_{\Sigma}(\beta, s, t, h) \), we have \( S^{\lambda}_{\Sigma}(\beta, s, t, \frac{1+Az}{1+Bz}) \) and defined as
\[
f(\Sigma), \quad e^{i\beta} \left[ (s-t)z \right]^{1-\lambda} f'(z) \left[ f(sz) - f(tz) \right]^{1-\lambda} < \frac{1+Az}{1+Bz} \cos \beta + i \sin \beta, \quad z \in U
\]
and
\[ e^{i\beta} \frac{(s - t)w^{1-\lambda} g'(w)}{[g(sw) - g(tw)]^{1-\lambda}} < \frac{1 + Aw}{1 + Bw} \cos \beta + i \sin \beta, \quad w \in U \]
where \( g(w) = f^{-1}(w) \), \( s, t \in \mathbb{C} \) with \( s \neq t \), \( |t| \leq 1 \), \( \beta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) and \( \lambda \geq 0 \).

**Remark 3.** If we set \( h(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \), \( 0 \leq \alpha < 1 \), in the class \( S^\lambda_{\Sigma}(\beta, s, t, h) \), we have \( S^\lambda_{\Sigma}(\beta, s, t, h) \) and defined as
\[
\begin{align*}
 f \in \Sigma, \quad \text{Re} \left\{ e^{i\beta} \frac{(s - t)z^{1-\lambda} f'(z)}{[f(sz) - f(tz)]^{1-\lambda}} \right\} > \alpha \cos \beta, \quad z \in U
\end{align*}
\]
and
\[
\begin{align*}
 \text{Re} \left\{ e^{i\beta} \frac{(s - t)w^{1-\lambda} g'(w)}{[g(sw) - g(tw)]^{1-\lambda}} \right\} > \alpha \cos \beta, \quad w \in U
\end{align*}
\]
where \( g(w) = f^{-1}(w) \), \( s, t \in \mathbb{C} \) with \( s \neq t \), \( |t| \leq 1 \), \( \beta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) and \( \lambda \geq 0 \).

**Lemma 4.** (see [13]) Let the function \( \phi(z) \) given by
\[
\phi(z) = \sum_{n=1}^{\infty} B_n z^n
\]
be convex in \( U \). Suppose also that the function \( h(z) \) given by
\[
h(z) = \sum_{n=1}^{\infty} h_n z^n
\]
is holomorphic in \( U \). If \( h(z) < \phi(z), \quad z \in U, \) then \( |h_n| \leq |B_1|, \quad n \in \mathbb{N} = \{1, 2, 3, \ldots\} \).

**2 Coefficient Estimates**

**Theorem 5.** Let \( f \) given by \([1.1]\) be in the class \( S^\lambda_{\Sigma}(\beta, s, t, h) \). Then
\[
|a_2| \leq \sqrt{\frac{2|B_1| \cos \beta}{2(\lambda - 1)(s+t)[2+(\lambda - 1)(s+t)] + 2[(\lambda - 1)(s^2+st+t^2) + 3] - \lambda(\lambda - 1)(s+t)^2}}, \quad (2.1)
\]
and
\[
|a_3| \leq \frac{2|B_1| \cos \beta}{2(\lambda - 1)(s+t)[2+(\lambda - 1)(s+t)] + 2[(\lambda - 1)(s^2+st+t^2) + 3] - \lambda(\lambda - 1)(s+t)^2}}. \quad (2.2)
\]

**Proof.** Let \( f \in S^\lambda_{\Sigma}(\beta, s, t, h) \), \( g \) be the analytic extension of \( f^{-1} \) to \( U \) and \( s, t \in \mathbb{C} \) with \( s \neq t \), \( |t| \leq 1 \) and \( \lambda \geq 0 \). It follows from \([1.2]\) and \([1.3]\) that there exists \( p, q \in P \) such that
\[
e^{i\beta} \frac{(s - t)z^{1-\lambda} f'(z)}{[f(sz) - f(tz)]^{1-\lambda}} = p(z) \cos \beta + i \sin \beta, \quad (z \in U) \quad (2.3)
\]
and
\[
e^{i\beta} \frac{(s - t)w^{1-\lambda} g'(w)}{[g(sw) - g(tw)]^{1-\lambda}} = q(w) \cos \beta + i \sin \beta, \quad (w \in U) \quad (2.4)
\]
where \( p(z) < h(z) \) and \( q(w) < h(w) \) have the forms
\[
p(z) = 1 + p_1 z + p_2 z^2 + \cdots
\]
and
\[ q(w) = 1 + q_1 w + q_2 w^2 + \cdots, \]
respectively. It follows from (2.3) and (2.4), we deduce
\[
e^{i\beta} \left[ (\lambda - 1) (s + t) + 2 \right] a_2 = p_1 \cos \beta, \tag{2.5} \]
\[
e^{i\beta} \left\{ [(\lambda - 1) (s^2 + t^2 + st) + 3] a_3 - \frac{\lambda(\lambda - 1)}{2}(s + t)^2 a_2^2 + (\lambda - 1) (s + t) [2 + (\lambda - 1)(s + t)] a_2 \right\} = p_2 \cos \beta, \tag{2.6} \]
and
\[
e^{-i\beta} \left[ (\lambda - 1) (s + t) + 2 \right] a_2 = q_1 \cos \beta, \tag{2.7} \]
\[
e^{i\beta} \left\{ 2 \left[ (\lambda - 1) (s^2 + t^2 + st) + 3 \right] - \frac{\lambda(\lambda - 1)}{2}(s + t)^2 + (\lambda - 1) (s + t) [2 + (\lambda - 1)(s + t)] \right\} a_2^2 \]
\[
- e^{i\beta} \left[ (\lambda - 1) (s^2 + t^2 + st) + 3 \right] a_3 = q_2 \cos \beta. \tag{2.8} \]
From (2.5) and (2.7) we obtain
\[ p_1 = -q_1. \]
By adding (2.6) to (2.8), we get
\[
e^{i\beta} \left\{ 2 \left[ (\lambda - 1) (s^2 + t^2 + st) + 3 \right] - \frac{\lambda(\lambda - 1)}{2}(s + t)^2 + (\lambda - 1) (s + t) [2 + (\lambda - 1)(s + t)] \right\} a_2^2 \]
\[ = (p_2 + q_2) \cos \beta. \tag{2.9} \]
Since \( p, q \in h(U) \), applying Lemma 4, we have
\[ |p_m| = \left| \frac{p^{(m)}(0)}{m!} \right| \leq |B_1|, \quad m \in \mathbb{N} \tag{2.10} \]
and
\[ |q_m| = \left| \frac{q^{(m)}(0)}{m!} \right| \leq |B_1|, \quad m \in \mathbb{N}. \tag{2.11} \]
Applying (2.10), (2.11) and Lemma 4 for the coefficients \( p_1, p_2, q_1 \) and \( q_2 \), we readily get
\[ |a_2| \leq \sqrt{\frac{2|B_1| \cos \beta}{2(\lambda - 1)(s + t)(2 + (\lambda - 1)(s + t) + 2[(\lambda - 1)(s^2 + t^2 + st) + 3] - \lambda(\lambda - 1)(s + t)^2}}. \]
Subtracting (2.8) from (2.6) we have
\[
e^{i\beta} \left\{ 2 \left[ (\lambda - 1) (s^2 + t^2 + st) + 3 \right] a_3 - 2 \left[ (\lambda - 1) (s^2 + t^2 + st) + 3 \right] a_2^2 \right\} = (p_2 - q_2) \cos \beta, \tag{2.12} \]
or, equivalently
\[
a_3 = \frac{e^{-i\beta}(p_2 + q_2) \cos \beta}{2(\lambda - 1)(s + t)(2 + (\lambda - 1)(s + t) + 2[(\lambda - 1)(s^2 + t^2 + st) + 3] - \lambda(\lambda - 1)(s + t)^2)} + \frac{e^{-i\beta}(p_2 - q_2) \cos \beta}{2(\lambda - 1)(s^2 + t^2 + st) + 3}. \]
Applying (2.10), (2.11) and Lemma 4 once again for the coefficients \( p_1, p_2, q_1 \) and \( q_2 \), we readily get
\[ |a_3| \leq \frac{2|B_1| \cos \beta}{|2(\lambda - 1)(s + t)(2 + (\lambda - 1)(s + t) + 2[(\lambda - 1)(s^2 + t^2 + st) + 3] - \lambda(\lambda - 1)(s + t)^2)|}. \]
This completes the proof of Theorem 5. \( \square \)
3 Corollaries and Consequences

Corollary 6. Let $f$ given by (1.1) be in the class $S^\lambda_{\Sigma} \left( \beta, s, t, \frac{1+Az}{1+Bz} \right)$. Then

$$|a_2| \leq \sqrt{\frac{2(A-B) \cos \beta}{2(\lambda-1)(s+t)(2+\lambda-1)(s+t)+2(\lambda-1)(s^2+st+3)\left|1-\lambda(s+t)^2\right|}}$$

and

$$|a_3| \leq \frac{2(A-B) \cos \beta}{2(\lambda-1)(s+t)(2+\lambda-1)(s+t)+2(\lambda-1)(s^2+st+3)\left|1-\lambda(s+t)^2\right|},$$

where $-1 \leq B < A \leq 1$, $s, t \in \mathbb{C}$ with $s \neq t$, $|t| \leq 1$ and $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\lambda \geq 0$.

Corollary 7. Let $f$ given by (1.1) be in the class $S^\lambda_{\Sigma} (\beta, s, t, \alpha)$. Then

$$|a_2| \leq \sqrt{\frac{4(1-\alpha) \cos \beta}{2(\lambda-1)(s+t)(2+\lambda-1)(s+t)+2(\lambda-1)(s^2+st+3)\left|1-\lambda(s+t)^2\right|}},$$

and

$$|a_3| \leq \frac{4(1-\alpha) \cos \beta}{2(\lambda-1)(s+t)(2+\lambda-1)(s+t)+2(\lambda-1)(s^2+st+3)\left|1-\lambda(s+t)^2\right|},$$

where $0 \leq \alpha < 1$, $s, t \in \mathbb{C}$ with $s \neq t$, $|t| \leq 1$ and $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

If we get $\lambda = 0$ in Theorem 5,

Corollary 8. Let $f$ given by (1.1) be in the class $S^0_{\Sigma} (\beta, s, t, \alpha)$. Then

$$|a_2| \leq \sqrt{\frac{|B_1| \cos \beta}{3-2s-2t+st}},$$

and

$$|a_3| \leq \frac{|B_1| \cos \beta}{3-2s-2t+st}.$$ 

If we get $\lambda = 0$ in Corollary 6,

Corollary 9. Let $f$ given by (1.1) be in the class $S^0_{\Sigma} (\beta, s, t, \frac{1+Az}{1+Bz})$. Then

$$|a_2| \leq \sqrt{\frac{(A-B) \cos \beta}{3-2s-2t+st}}$$

and

$$|a_3| \leq \frac{(A-B) \cos \beta}{3-2s-2t+st},$$

where $-1 \leq B < A \leq 1$, $s, t \in \mathbb{C}$ with $s \neq t$, $|t| \leq 1$ and $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

If we get $\lambda = 0$ in Corollary 7,

Corollary 10. Let $f$ given by (1.1) be in the class $S^0_{\Sigma} (\beta, s, t, \alpha)$. Then

$$|a_2| \leq \sqrt{\frac{2(1-\alpha) \cos \beta}{3-2s-2t+st}},$$

and

$$|a_3| \leq \frac{2(1-\alpha) \cos \beta}{3-2s-2t+st},$$

where $0 \leq \alpha < 1$, $s, t \in \mathbb{C}$ with $s \neq t$, $|t| \leq 1$ and $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
References


