Acta Universitatis Matthiae Belii, series Mathematics
Volume 20 (2012), 11–17, ISSN 1338-7111
Online version, `http://actamath.savbb.sk`

Classification of regular maps whose automorphism groups are 2-groups of maximal class

Kan Hu*
Faculty of Natural Sciences, Matej Bel University Tajovského 40, 974 01, Banská Bystrica, Slovakia
`kanhu@savbb.sk`

Naer Wang
Faculty of Natural Sciences, Matej Bel University Tajovského 40, 974 01, Banská Bystrica, Slovakia
`naerwang@savbb.sk`

Abstract

It is proved by Malnič, Nedela and Škoviera [Regular maps with nilpotent automorphism groups, *European J. Combin.* **33** (2012), no. 8, 1974–1986] that regular maps with nilpotent automorphism groups can be decomposed into a direct product of two regular maps, a regular map whose automorphism group is a 2-group and a semistar of odd valency. This reduction theorem motivates the classification of regular maps whose automorphism groups are 2-groups. In this paper, we classify regular maps whose automorphism groups are 2−groups of maximal nilpotency class.

Received October 1, 2012
Revised October 26, 2012
Accepted in final form October 29, 2012
Communicated with Ján Karabáš.

Keywords regular map, nilpotent group, maximal class.
MSC(2010) 20B25, 05C10.

1 Introduction

A map is a 2-cell embedding of a connected graph into a closed surface. A map on a orientable surface is regular if its group of orientation preserving automorphisms acts regularly on its darts. The best known examples of regular maps are the Platonic solids, which together with the dihedra and hosohedra give a full list of regular maps on the sphere. Thorough investigation was not carried out until in the 19th century the connection between modern topology, group theory, graph theory and the theory of complex functions was developed [13, 6]. Modern foundations of the theory of maps on orientable surfaces were built by Jones and Singerman [10], Gross and Tucker [7] and others. There are three different approaches to the classification of regular maps on orientable surfaces:

(1) classification of regular maps with prescribed supporting surfaces;

(2) classification of regular maps with prescribed automorphism groups;

(3) classification of regular maps with prescribed underlying graphs.

*Supported partially by APVV-0223-10

In the second direction, the classification of regular maps with automorphism group isomorphic to the group $\mathrm{PSL}(2,q)$, where q is a power of some prime p, was in principle done by Macbeath [14]. A finite group G is called a *Hurwitz group* if it is a finite nontrivial quotient of the infinite group

$$\langle x, y | x^3 = y^2 = (xy)^7 = 1\rangle.$$

It is well-known that Hurwitz groups are corresponding to the automorphism groups of some Riemann surface with highest symmetry [9]. It is also obvious that every Hurwitz group gives rise to a regular map. Determination of the simple groups which are Hurwitz groups is an important problem. In this context, the symmetric and alternating groups [4], Suzuki groups [11], Ree groups [12], and various sporadic simple groups [21] have been investigated. A survey can be found in [1, 3]. In present paper, we consider the following problem:

Problem 1. *Determine and classify regular maps whose automorphism groups are nilpotent.*

This problem was studied by Malnič, Nedela and Škoviera in [16], where the authors have proved that each such map can be decomposed into a direct product of two regular maps, a regular map whose automorphism group is a 2-group and a semistar of odd valency. Therefore, the above problem is reduced into the following

Problem 2. *Determine and classify regular maps whose automorphism groups are* 2*-groups.*

If the nilpotency class is small enough, Problem 2 was resolved [16, 5]. For convenience, we define a *regular* 2*-map* to be a regular map whose automorphism group is a 2-group. If the automorphism group of a regular 2-map has class c, we will refer to the map as a *regular* 2*-map of class* c. Regular 2-maps of class 1 and 2 have been classified in the aforementioned paper [16]. Regular 2-maps of class 3 have been recently classified by Du et al [5]. In present paper, we will classify regular 2-maps of maximal class.

2 Preliminaries

As mentioned before, a *topological map* $\mathcal{M}$ is a 2-cell embedding $i : X \hookrightarrow S$ of a connected graph X into a closed surface S such that each component of $S - i(X)$ is homeomorphic to an open disc in $\mathbb{C}$. The vertices, edges of $\mathcal{M}$ are inherited from the embedded graph X, whereas the faces are the components of $S - i(X)$. If the supporting surface S is orientable, the map $\mathcal{M}$ is also called *orientable*, otherwise, $\mathcal{M}$ is called *non-orientable*. As usual, we define a *(combinatorial oriented) map* to be a triple $(D; R, L)$, where D is a nonempty finite set whose elements are called *darts*, and R and L are two permutations on D such that $L^2 = 1$ and the group generated by R and L acts transitively on D. Here the permutation R describes the local orientation of darts around the vertices of $\mathcal{M}$ and is called a *rotation*, whereas L inverts each pair of darts with the same underlying edge of $\mathcal{M}$. The group generated by R and L is called the *monodromy group* of $\mathcal{M}$ and is denoted by $\mathrm{Mon}(\mathcal{M})$.

A *homomorphism* from a map $\mathcal{M}_1 = (D_1; R_1, L_1)$ to a map $\mathcal{M}_2 = (D_2; R_2, L_2)$ is a mapping $\phi : D_1 \to D_2$ such that

$$\phi R_1 = R_2\phi \quad \text{and} \quad \phi L_1 = L_2\phi.$$

The mapping ϕ is necessarily surjective due to the transitivity of monodromy groups. If ϕ is a bijection, then we say $\mathcal{M}_1$ is *isomorphic* to $\mathcal{M}_2$ and denote it by $\mathcal{M}_1 \cong \mathcal{M}_2$.

An isomorphism of a map $\mathcal{M}$ to itself is called an *automorphism* of $\mathcal{M}$. The set of all automorphisms of $\mathcal{M}$ forms the *automorphism group* of $\mathcal{M}$ under the composition operation and is denoted by $\mathrm{Aut}\,(\mathcal{M})$.

It follows from a classical result in the theory of maps that the action of $\mathrm{Aut}\,(\mathcal{M})$ on D is semi-regular (namely, the stabiliser of $\mathrm{Aut}\,(\mathcal{M})$ is trivial) [10]. If this action is transitive and hence regular, then $\mathcal{M}$ is called a *regular* map as well. Therefore, regular maps exhibit highest possible symmetry.

In a regular map $\mathcal{M} = (D; R, L)$ with $G = \mathrm{Aut}\,(\mathcal{M})$, we can identify D with G and regard the actions of $\mathrm{Aut}\,(\mathcal{M})$ and $\mathrm{Mon}\,(\mathcal{M})$ on D as the right and left multiplication by the elements of G, respectively. More precisely, if $G = \langle x, y\rangle$, $y^2 = 1$, we denote by (G, x, y) the regular map $(D; R, L)$ defined by setting $D = G$, $Rg = xg$, $Lg = yg$ for any $g \in G$. For a map $\mathcal{M} = (D; R, L)$, there are two associated maps: the *dual* $\mathcal{M}^* = (D; RL, L)$ and the *mirror map* $\mathcal{M}^{-1} = (D; R^{-1}, L)$ of $\mathcal{M}$. A map $\mathcal{M}$ is called *self-dual* if $\mathcal{M} \cong \mathcal{M}^*$; a map $\mathcal{M}$ is called *reflexible* if $\mathcal{M} \cong \mathcal{M}^{-1}$. Those maps which are not reflexible are called *chiral*.

To answer Problem 2 for a given finite 2-group G, we have first to see whether G is the automorphism group of a regular map $\mathcal{M}$, which is equivalent to decide whether G is generated by two elements, say x and y, and $y^2 = 1$. If this is the case, then such a group G will be called *admissible* and the associated generating pair (x, y) will be called *admissible* as well. Secondly, according to [18], for an admissible group G, two admissible generating pairs (x_i, y_i) $(i = 1, 2)$ of G are called *equivalent* if there is an automorphism ϕ of G such that

$$x_1^{\phi} = x_2 \quad \text{and} \quad y_1^{\phi} = y_2. \tag{2.1}$$

The isomorphism classes of regular maps $\mathcal{M}$ with $\mathrm{Aut}\,(\mathcal{M}) = \mathrm{G}$ are therefore in a one-to-one correspondence with the orbits of admissible generating pairs of G under the action of $\mathrm{Aut}\,(\mathrm{G})$.

By definition, the automorphism groups of regular 2-maps of maximal class are 2-groups of maximal class. The classification of 2-groups of maximal class is known as Taussky's Theorem in group theory. We rephrase it as follows.

Proposition 3. *[20][Taussky's Theorem] Let G be a finite $2-$group of maximal nilpotency class with $|G| = 2^n$, $n \geq 3$. Then, up to isomorphism, G belongs to one and only one of the following three classes of groups:*

1. *generalized quaternion groups defined by*

$$Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, b^{-1}ab = a^{-1}\rangle, n \geq 3; \tag{2.2}$$

2. *dihedral groups defined by*

$$D_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, b^{-1}ab = a^{-1}\rangle, n \geq 3; \tag{2.3}$$

3. *semidihedral groups defined by*

$$SD_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, b^{-1}ab = a^{-1+2^{n-2}}\rangle, n \geq 4. \tag{2.4}$$

3 Regular 2-maps of maximal class

In this section, we apply Taussky's Theorem to give a classification of regular 2-maps of maximal class. First, we define several families of regular maps. The *semistar* is defined to be a map $\mathcal{S}_n = (C_n, x, y)$ with

$$C_n = \langle x, y | x^n = y = 1 \rangle. \tag{3.1}$$

The *cycle of length* n is a map $\mathcal{C}_n = (D_{2n}, x, y)$ with

$$D_{2n} = \langle x, y | x^2 = y^2 = (xy)^n = 1 \rangle. \tag{3.2}$$

Moreover, the regular embeddings of n-*dipoles* are the maps $\mathcal{D}(n, e) = (G, x, y)$ with

$$G = \langle x, y | x^n = y^2 = 1, y^{-1}xy = x^e \rangle, \tag{3.3}$$

where $e^2 \equiv 1 \pmod n$ (Theorem 9.1 [17]). Note that $\mathcal{D}^*(n, -1) = \mathcal{C}_n$.

Moreover, by replacing each edge of a cycle C_n by m parallel edges incident to the same vertices, we get a *multicycle* $C_n^{(m)}$. The regular embeddings of multicycles $C_{2n}^{(2m)}$ are maps $\mathcal{C}(2n, 2m, e, f) = (G, x, y)$ with

$$G = \langle x, y | x^{4m} = y^2 = 1, y^{-1}x^2y = x^{2e}, (xy)^{2n} = x^{2f} \rangle, \tag{3.4}$$

where $e^2 \equiv 1 \pmod{2m}$ and $f \equiv (e+1)n/4 \pmod{2m}$ or $f \equiv ((e+1)n + 4m)/4 \pmod{2m}$ (Theorem 2 [8]).

To present our classification of regular 2-maps of maximal class, we need to identify admissible 2-groups from the groups given in Proposition 3.

Lemma 4. *In Proposition 3, only the dihedral groups and semidihedral groups are admissible. Moreover, the admissible generating pairs of the dihedral groups have form (a^rb^k, a^jb), where either $k = 0$, r is odd, or $k = 1$, $r - j$ is odd; the admissible generating pairs of the semidihedral groups have form (a^rb^k, a^jb), where $k \in \{0, 1\}$, r is odd and j is even. Two admissible pairs (a^rb^k, a^jb) and $(a^{r'}b^{k'}, a^{j'}b)$ of D_{2^n} are equivalent if and only if $k = k'$ and two admissible pairs (a^rb^k, a^jb) and $(a^{r'}b^{k'}, a^{j'}b)$ of SD_{2^n} are equivalent if and only if $k = k'$.*

Proof. It is known that the generalised quaternion group Q_{2^n} has a unique involution $y = a^{2^{n-2}}$. Obviously, y is a central element. Since Q_{2^n} is non-abelian, Q_{2^n} has no admissible generating pairs. Furthermore, for both D_{2^n} and SD_{2^n}, it is clear the generators (a, b) given in Proposition 3 are admissible. To determine all admissible generating pairs (x, y) of D_{2^n}, we see that the involutions of D_{2^n} are either $a^{2^{n-2}}$ or of the form a^jb. But $a^{2^{n-2}} \in Z(D_{2^n})$ and D_{2^n} is non-abelian, which implies that $a^{2^{n-2}}$ can not be a generator in any admissible generating pair. On the other hand, let $y = a^jb$ and $x = a^rb^k$, where $0 \le j, r \le 2^{n-1} - 1$ and $0 \le k \le 1$. If $k = 0$, we show that $\langle x, y \rangle = D_{2^n}$ if and only if $(r, 2) = 1$. In fact, if $(r, 2) = 1$, then $\langle a^r \rangle = \langle a \rangle$. Hence,

$$\langle x, y \rangle = \langle a^r, a^jb \rangle = \langle a, a^jb \rangle = \langle a, b \rangle.$$

Consequently, $D_{2^n} = \langle x, y \rangle$. Conversely, assume $D_{2^n} = \langle x, y \rangle$, if r was an even number, let $m = o(a^r)$, then $m < o(a)$, we have

$$\langle x, y \rangle = \langle x, y \mid x^m = y^2 = 1, yxy = x^{-1} \rangle \cong D_{2m},$$

which implies that $\langle x, y\rangle$ is a proper subgroup of D_{2^n}, a contradiction. Similarly, if $k = 1$, we show that $\langle x, y\rangle = D_{2^n}$ if and only if $(r - j, 2) = 1$. In fact, since

$$\langle x, y\rangle = \langle a^r b, a^j b\rangle = \langle a^{r-j}, a^j b\rangle,$$

by the preceding case we have $\langle x, y\rangle = D_{2^n}$ if and only if $(r - j, 2) = 1$.

As concerns the admissible pairs of the semidihedral groups SD_{2^n}, we see the involutions of SD_{2^n} are either $a^{2^{n-2}}$ or of the form $a^j b$, where j is even. The former is a central element and hence can not be a generating involution in any admissible pair. Now let $y = a^j b$ and $x = a^r b^k$, where $0 \leq r, j \leq 2^{n-1} - 1$, j is even and $k \in \{0, 1\}$. If $k = 0$, then we show that $\langle x, y\rangle = SD_{2^n}$ if and only if r is odd. First assume r odd, then $\langle a\rangle = \langle a^r\rangle$. We have

$$\langle x, y\rangle = \langle a^r, a^j b\rangle = \langle a, a^j b\rangle = \langle a, b\rangle.$$

It follows that $\langle x, y\rangle = SD_{2^n}$. Conversely, assume $\langle x, y\rangle = SD_{2^n}$, if r was an even number, let $m = o(a^r)$, then $m < o(a)$. We have

$$\langle x, y | x^m = y^2 = 1, yxy = x^{-1}\rangle \cong D_{2m}.$$

Therefore, $|\langle x, y\rangle| < |SD_{2^n}|$, which implies that $\langle x, y\rangle$ is a proper subgroup of SD_{2^n}, a contradiction. Similarly, if $k = 1$, we have

$$\langle x, y\rangle = \langle a^r b, a^j b\rangle = \langle a^r a^{j(-1+2^{n-2})}, a^j b\rangle \overset{j \text{ even}}{=} \langle a^{r-j}, a^j b\rangle.$$

It follows the first case that $\langle x, y\rangle = SD_{2^n}$ if and only if $r - j$ is odd.

Finally, we decide the equivalence relationship of aforementioned admissible pairs. The admissible pair (x, y) of D_{2^n} has a defining relation

$$\langle x, y | x^{2^{n-1}} = y^2 = (xy)^2 = 1\rangle,$$

if $x = a^r$, $y = a^j b$, where r is odd, or

$$\langle x, y | x^2 = y^2 = (xy)^{2^{n-1}} = 1\rangle,$$

if $x = a^r b$, $y = a^j b$, where $r - j$ is odd. Therefore, two admissible pairs $(a^r b^k, a^j b)$ and $(a^{r'} b^{k'}, a^{j'} b)$ of D_{2^n} are equivalent if and only if $k = k'$. Similarly, the admissible pair (x, y) of SD_{2^n} has a defining relation

$$\langle x, y | x^{2^{n-1}} = y^2 = (xy)^4 = 1, [x, y] = x^{-2+2^{n-2}}\rangle,$$

if $x = a^r$, $y = a^j b$, where r is odd, j is even, or

$$\langle x, y | x^4 = y^2 = (xy)^{2^{n-1}} = 1, [x, y] = (xy)^{2-2^{n-2}}\rangle,$$

if $x = a^r b$, $y = a^j b$, where r is odd, j is even. Therefore, two admissible pairs $(a^r b^k, a^j b)$ and $(a^{r'} b^{k'}, a^{j'} b)$ of SD_{2^n} are equivalent if and only if $k = k'$, as claimed. □

Now we are ready to formulate our classification theorem of regular 2-maps of maximal class.

Theorem 5. *Up to isomorphism, regular maps whose automorphism groups are non-abelian 2-groups of maximal nilpotency class are one of the following maps:*

(1) a cycle $\mathcal{C}_{2^{n-1}}$ or its dual, a dipole $\mathcal{D}(2^{n-1}, -1)$, where $n \geq 3$, both of genus 0;

(2) a dipole $\mathcal{D}(2^{n-1}, -1 + 2^{n-2})$ or its dual $\mathcal{C}(2^{n-2}, 2, 1, 1)$, the regular embedding of a multicycle, where $n \geq 4$, both of genus 2^{n-3}.

Moreover, the above maps are all reflexible.

Proof. Let $\mathcal{M}$ be a regular 2-map of maximal class and $G = \mathrm{Aut}(\mathcal{M})$, let $|G| = 2^n$. Since G is non-abelian, $n \geq 3$. By Taussky's Theorem, G is isomorphic to one of the groups in Proposition 3. By Lemma 4, the generalised quaternion group Q_{2^n} is not admissible, and every admissible pair (x, y) of D_{2^n} either has a form $x = a^r, y = a^j b$, where r is odd, or has a form $x = a^r b, y = a^j b$, where $r - j$ is odd. The former defines the map $\mathcal{D}(2^{n-1}, -1)$ and the latter defines the map $\mathcal{C}_{2^{n-1}}$, which is the dual of $\mathcal{D}(2^{n-1}, -1)$. Similarly, every admissible pair (x, y) of SD_{2^n} is of a form $x = a^r b^k, y = a^j b$, where r is odd, j is even, $k \in \{0, 1\}$. If $k = 0$, then (SD_{2^n}, x, y) defines the map $\mathcal{D}(2^{n-1}, -1+2^{n-2})$; if $k = 1$, we have its dual map $\mathcal{C}(2^{n-2}, 2, 1, 1)$, the regular embedding of a multicycle $C^{(2)}_{2^{n-2}}$. It is clear that the maps are all reflexible, as claimed. □

Acknowledgements

The authors thank Professor Shao-Fei Du for suggesting this topic. Without many useful discussions with Professor Roman Nedela, we would never try to publish this work. We are also thankful to the anonymous referee for pointing out a flaw in our manuscript and particularly, whose detailed remarks and comments have improved the presentation of this paper.

References

[1] M. D. E. Conder, *Hurwitz groups: a brief survey*, Bull. Amer. Math. Soc. **23** (1990), 359–370.

[2] M. D. E. Conder, *Hurwitz groups with given centre*, Bull. London. Math. Soc. **34** (2002), 725–728.

[3] M. D. E. Conder, *An update on Hurwitz groups*, Groups, Complexity and Cryptology **34** (2010), 35–49.

[4] M. D. E. Conder, *Generators for alternating and symmetric groups*, J. London Math. Soc. (2) **22** (1980), 75–86.

[5] Y. F. Ban, S. F. Du, Y. Liu, R. Nedela and M. Škoviera, Classification of regular maps whose automorphism groups are 2-groups of class three, in preparation.

[6] W. Dyck, *Uber Aufstellung und Untersuchung von Gruppe und Irrationalität regularer Riemannscher Flächen*, Math. Ann. **17** (1880), 473–508.

[7] J. L. Gross and T.W. Tucker, "Topological Graph Theory", Dover, 2001.

[8] K. Hu, R. Nedela, M. Škoviera and Naer Wang, Regular embeddings of cycles with multiple edges revisited, in preparation.

[9] A. Hurwitz, *Über algebraische Gebilde mit eindeutigen Transformationen in sich*, Math. Ann. **41** (1893), 403–442.

[10] G. A. Jones and D. Singerman, *Theory of maps on orientable surfaces*, Proc. London. Math. Soc. **37** (1978), 273–307.

[11] G. A. Jones and S. A. Silver, *Suzuki groups and surfaces*, J. London Math. Soc. (2) **48** (1993), 117–125.

[12] G. A. Jones, *Ree groups and Riemann surfaces*, J. Algebra **165** (1994), 41–62.

[13] F. Klein, *Über die Transformation siebenter Ordnnung der elliptischen Functionen*, Math. Ann. **14** (1879), 428–471.

[14] A. M. Macbeath, *Generators of the linear fractional groups,* in: "1969 Number Theory Institute, Proc. Sympos. Pure Math.," XII, Amer. Math. Soc., Providence, R.I, 1969, 14–32.

[15] G. Malle, *Hurwitz groups and* $G_2(q)$, Canad. Math. Bull. **33** (1990), 349–357.

[16] A. Malnič, R.Nedela and M. Škoviera, *Regular maps with nilpotent automorphism groups*, European J. Combin. **33** (2012), no. 8, 1974–1986.

[17] R. Nedela and M. Škoviera, *Regular maps of canonical double coverings of graphs*, J. Combin. Theory Ser. B **67** (1996), 249–277.

[18] R. Nedela and M. Škoviera, *Exponents of orientable maps*, Proc. London Math. Soc. **75** (1997), 1–31.

[19] K. B. Tchakerian, *An explicit* $(2,3,7)$*-generation of the simple Ree groups* ${}^2G_2(3^n)$, C. R. Acad. Bulgare Sci. **58** (2005), 749–752.

[20] O. Taussky, *Remark on the class field tower*, J. Lond. Math. Soc. **12** (1937), 82–85.

[21] A. J. Woldar, *Sporadic simple groups which are Hurwitz*, J. Algebra **144** (1991), 443–450.