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Generalized sequence spaces over n-normed spaces

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Abstract

In the present paper we introduce generalized sequence spaces over a n-normed space defined by Musielak-Orlicz function $\mathcal{M} = (M_k)$.

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1 Introduction and Preliminaries

The concept of 2-normed spaces was initially developed by Gähler [4] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [11]. Since then, many others have studied this concept and obtained various results, see Gunawan [5, 6] and Gunawan and Mashadi [7]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and
4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.
For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_n||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k \to \infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,p \to \infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.
An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.
Lindenstrauss and Tzafriri [9] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [9] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$. A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function see [10, 13]. A sequence $\mathcal{N} = (N_k)$ is defined by

$$N_k(v) = \sup\{|v|u - (M_k) : u \geq 0\}, \ k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

By w we denote the space of all real or complex valued sequences. If $x \in w$, then we simply write $x = (x_k)$ instead of $x = (x_k)_{k=1}^{\infty}$. Also, we will use the conventions that $e = (1, 1, \cdots)$. Any vector subspace of w is called a sequence space. We will write l_∞, c and c_0 for the sequence spaces of all bounded, convergent, and null sequences, respectively. Further, by $l_p (1 \leq p < \infty)$, we denote the sequence space of all p-absolutely convergent series, that is, $l_p = \Big\{x = (x_k) \in w : \sum_{k=0}^{\infty} |x_k|^p < \infty\Big\}$ for $1 \leq p < \infty$. Throughout the article, $w(X)$, $l_\infty(X)$, and $l_p(X)$ denote, respectively, the spaces of all, bounded, and $p-$absolutely summable sequences with the elements in X, where (X, q) is a seminormed space. By $\theta = (0, 0, \cdots)$, we denote the zero element in X. P_s denotes the set of all subsets of $\mathbb{N}$, that do not contain more than s elements. With (φ_s), we will denote a non-decreasing sequence of positive real numbers such that $(s-1)\varphi_{s-1} \leq (s-1)\varphi_s$ and $\varphi_s \to \infty$, as $s \to \infty$. The class of all the sequences (φ_s) satisfying this property is denoted by φ.

In paper [12], the notion of λ-convergent and bounded sequences is introduced as follows: let $\lambda = (\lambda_k)$ be a strictly increasing sequence of positive reals tending to infinity, that is

$$0 < \lambda_0 < \lambda_1 < \cdots, \quad \lambda_k \to \infty \text{ as } k \to \infty.$$

We say that a sequence $x = (x_k) \in w$ is λ-convergent to the number $l \in \mathbb{C}$, called as the λ-limit of x, if $\Lambda_n(x) \to l$ as $n \to \infty$, where

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k, \quad n \in \mathbb{N}.$$

In particular, we say that x is a λ-null sequence if $\Lambda_n(x) \to 0$ as $n \to \infty$. Further, we say that x is λ-bounded if $\sup |\Lambda_n(x)| < \infty$. Here and in the sequel, we will use the convention that any term with a negative subscript is equal to naught, for example, $\lambda_{-1} = 0$ and $x_{-1} = 0$. Now, it is well known in [12] that if $\lim_n x_n = a$ in the ordinary sense of convergence, then

$$\lim_n \Big(\frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) |x_k - a|\Big) = 0.$$

This implies that

$$\lim_n |\Lambda_n(x) - a| = \lim_n \Big|\frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(x_k - a)\Big| = 0,$$

which yields that $\lim_n \Lambda_n(x) = a$ and hence x is λ-convergent to a. We therefore deduce that the ordinary convergence implies the λ-convergence to the same limit. The notion

of difference sequence spaces was introduced by Kızmaz [8], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_o(\Delta)$. The notion was further generalized by Et and Çolak [3] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_o(\Delta^n)$.
Let m, n be non-negative integers, then for Z a given sequence space, we have

$$Z(\Delta_m^n) = \{x = (x_k) \in w : (\Delta_m^n x_k) \in Z\}$$

for $Z = c, c_0$ and l_∞ where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} x_{k+mv}.$$

Taking $m = 1$, we get the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_o(\Delta^n)$ studied by Et and Çolak [3]. Taking $m = n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_o(\Delta)$ introduced and studied by Kızmaz [8]. For more details about sequence spaces see [2, 14, 15, 16] and references therein.

The space $m(\phi)$ introduced and studied by Sargent [17] is defined as follows:

$$m(\phi) = \Big\{x = (x_k) \in s : \|x\|_{m(\phi)} = \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} |x_k| < \infty\Big\}.$$

Let M be an Orlicz function, then Tripathy and Mahanta [18] defined and studied the following sequence space:

$$m(M, \varphi) = \Big\{x = (x_k) \in s : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M\big(\frac{|x_k|}{\rho}\big) < \infty, \text{ for some } \rho > 0\Big\}.$$

Recently, Altun and Bilgin [1] defined and studied the following sequence spaces:

$$m(M, A, \varphi, p) = \Big\{x = (x_k) \in s : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M\big(\frac{|A_i x|}{\rho}\big)^{p_k} < \infty, \text{ for some } \rho > 0\Big\}.$$

We define the following sequence spaces which we shall discuss in the second section of the present paper:

$$m\Big(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, ||., \cdots, .||\Big) =$$

$$\Big\{x = (x_k) \in w : \lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k\Big(q\Big(||\frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} = 0, \text{ for some } \rho > 0\Big\}.$$

If we take $M_k(x) = x$, we get

$$m\Big(\varphi, q, \Lambda, \Delta_m^n, p, ||., \cdots, .||\Big)$$

$$= \Big\{x = (x_k) \in w : \lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} \Big(q\Big(||\frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} = 0, \text{ for some } \rho > 0\Big\}.$$

If we take $p = (p_k) = 1$, we get

$$m\Big(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, ||., \cdots, .||\Big)$$

$$= \Big\{x = (x_k) \in w : \lim_k \sum_{k \in \sigma, \sigma \in P_s} M_k\Big(q\Big(||\frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big) = 0, \text{ for some } \rho > 0\Big\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

In this paper we study some topological properties and inclusion relation between spaces $m\Big(\mathcal{M}, \varphi, q, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big)$ which we have defined above and the spaces $m(\mathcal{M}, A, \varphi, \Delta^n_m, p, ||., \cdots, .||)$ which we shall discuss in the third section of the paper.

2 Sequence spaces defined by Musielak-Orlicz function

In this section we study some toplogical properties and inclusion relation between the spaces $m\Big(\mathcal{M}, \varphi, q, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big)$.

Theorem 1. *Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function and $p = (p_k)$ be a sequence of strictly positive real number, the sequence space $m\Big(\mathcal{M}, \varphi, q, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big)$ is a linear space over the set of complex number $\mathbb{C}$.*

Proof. Let $x = (x_k)$ and $y = (y_k) \in m\Big(\mathcal{M}, \varphi, q, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$\lim_k \frac{1}{\varphi_s} \sum_{k\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k \Delta^n_m x_k}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} = 0$$

and

$$\lim_k \frac{1}{\varphi_s} \sum_{k\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k \Delta^n_m y_k}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} = 0.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\|., ..., .\|$ is a n-norm on X and M_k are non-decreasing and convex function so by using inequality (1.1), we have

$$\begin{aligned}
&\lim_k \tfrac{1}{\varphi_s} \sum_{k\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k \Delta^n_m (\alpha x_k + \beta y_k)}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} \\
\leq\ & \lim_k \frac{1}{\varphi_s} \sum_{k\in\sigma,\sigma\in P_s} M_k\Big[\Big(q\Big(||\frac{\Lambda_k \Delta^n_m (\alpha x_k)}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big) \\
+\ & \Big(q\Big(||\frac{\Lambda_k \Delta^n_m (\beta y_k)}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \\
\leq\ & K \lim_k \frac{1}{\varphi_s} \sum_{k\in\sigma,\sigma\in P_s} \frac{1}{2^{p_k}} M_k\Big(q\Big(||\frac{\Lambda_k \Delta^n_m x_k}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} \\
+\ & K \lim_k \frac{1}{\varphi_s} \sum_{k\in\sigma,\sigma\in P_s} \frac{1}{2^{p_k}} M_k\Big(q\Big(||\frac{\Lambda_k \Delta^n_m y_k}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} \\
=\ & 0
\end{aligned}$$

Thus, we have $\alpha x + \beta y \in m\Big(\mathcal{M}, \varphi, q, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big)$. Hence $m\Big(\mathcal{M}, \varphi, q, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big)$ is a linear space. □

Theorem 2. *For any Musielak Orlicz function* $\mathcal{M} = (M_k)$ *and a bounded sequence* $p = (p_k)$ *of strictly positive real numbers* $m\Big(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, ||., \cdots, .||\Big)$ *is a topological linear space paranormed by*

$$g(x) = \inf\Big\{\rho^{\frac{p_r}{M}} : \Big(\lim_k \frac{1}{\varphi_s} \sum_{k\in\sigma, \sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k\Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}} \le 1,\ r = 1, 2, 3, \cdots\Big\},$$

where $M = \max(1, \sup_k p_k < \infty)$.

Proof. Clearly $g(x) \ge 0$ for $x = (x_k) \in m\Big(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, ||., \cdots, .||\Big)$.
Since $M_k(0) = 0$, we get $g(0) = 0$. Again, if $g(x) = 0$, then

$$g(x) = \inf\Big\{\rho^{\frac{p_r}{M}} : \Big(\lim_k \frac{1}{\varphi_s} \sum_{k\in\sigma, \sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k\Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}} \le 1\Big\} = 0.$$

This implies that for a given $\epsilon > 0$, there exist some $\rho_\epsilon(0 < \rho_\epsilon < \epsilon)$ such that

$$\Big(\lim_k \frac{1}{\varphi_s} \sum_{k\in\sigma, \sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k\Delta_m^n x_k}{\rho_\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}} \le 1.$$

Thus

$$\begin{aligned}
&\Big(\lim_k \frac{1}{\varphi_s} \sum_{k\in\sigma, \sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k\Delta_m^n x_k}{\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}} \\
&\le \Big(\lim_k \frac{1}{\varphi_s} \sum_{k\in\sigma, \sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k\Delta_m^n x_k}{\rho_\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}} \\
&\le 1,
\end{aligned}$$

Suppose that $x_k \ne 0$ for each $k \in N$. this implies that $\Lambda_k\Delta_m^n x_k \ne 0$ for each $k \in N$. Let $\epsilon \to 0$, then $\|\frac{\Lambda_k\Delta_m^n x_k}{\epsilon}, z_1, ..., z_{n-1}\| \to \infty$. It follows that

$$\Big(\lim_k \frac{1}{\varphi_s} \sum_{k\in\sigma, \sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k\Delta_m^n x_k}{\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}} \to \infty$$

Which is a contradiction. Therefore $\Lambda_n\Delta_m^n x_k = 0$ for each k and thus $x_k = 0$ for each $k \in N$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\Big(\lim_k \frac{1}{\varphi_s} \sum_{k\in\sigma, \sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k\Delta_m^n x_k}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}} \le 1$$

and

$$\Big(\lim_k \frac{1}{\varphi_s} \sum_{k\in\sigma, \sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k\Delta_m^n y_k}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}} \le 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by Minkowski's inequality, we have

$$\Big(\lim_k \tfrac{1}{\varphi_s}\sum_{k\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k\Delta_m^n x_k}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}}$$
$$\begin{aligned}
&\leq \Big(\lim_k \frac{1}{\varphi_s}\sum_{k\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k\Delta_m^n (x_k+y_k)}{\rho_1+\rho_2}, z_1,\cdots,z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}}\\
&\leq \Big(\lim_k \frac{1}{\varphi_s}\sum_{k\in\sigma,\sigma\in P_s} M_k\Big(\frac{\rho_1}{\rho_1+\rho_2}\|\frac{\Lambda_k\Delta_m^n x_k}{\rho_1}, z_1,...,z_{n-1}\| + \frac{\rho_2}{\rho_1+\rho_2}||\frac{\Lambda_k\Delta_m^n y_k}{\rho_2}, z_1,...,z_{n-1}\|\Big)\Big)^{p_k}\Big)^{\frac{1}{M}}\\
&\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\lim_k \frac{1}{\varphi_s}\sum_{k\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k\Delta_m^n x_k}{\rho_1}, z_1,\cdots,z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}}\\
&+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\lim_k \frac{1}{\varphi_s}\sum_{k\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k\Delta_m^n y_k}{\rho_2}, z_1,\cdots,z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}}\\
&\leq 1.
\end{aligned}$$

Since $\rho's$ are non-negative, so we have

$$\begin{aligned}
g(x+y) &= \inf\Big\{\rho^{\frac{p_r}{M}} : \Big(\lim_k \frac{1}{\varphi_s}\sum_{k\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k\Delta_m^n (x_k+y_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}} \leq 1\Big\}\\
&\leq \inf\Big\{\rho_1^{\frac{p_r}{M}} : \Big(\lim_k \frac{1}{\varphi_s}\sum_{k\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k\Delta_m^n x_k}{\rho_1}, z_1,\cdots,z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}} \leq 1\Big\}\\
&+ \inf\Big\{\rho_2^{\frac{p_r}{M}} : \Big(\lim_k \frac{1}{\varphi_s}\sum_{k\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k\Delta_m^n y_k}{\rho_2}, z_1,\cdots,z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}} \leq 1\Big\}.
\end{aligned}$$

Therefore, $g(x+y) \leq g(x) + g(y)$
Finally, we prove that the scalar multiplication is continuous. Let μ be any complex number. By definition,

$$g(\mu x) = \inf\Big\{\rho^{\frac{p_r}{M}} : \Big(\lim_k \frac{1}{\varphi_s}\sum_{k\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\mu\Lambda_k\Delta_m^n x_k}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}} \leq 1\Big\}.$$

Then

$$g(\mu x) = \inf\Big\{(|\mu|t)^{\frac{p_r}{M}} : \Big(\lim_k \frac{1}{\varphi_s}\sum_{k\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\mu\Lambda_k\Delta_m^n x_k}{t}, z_1,\cdots,z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}}\Big\},$$

where $t = \frac{\rho}{|\mu|}$. Since $|\mu|^{p_r} \leq \max(1, |\mu| \sup p_r)$, we have

$$g(\mu x) = \max(1, |\mu|\sup p_r)\inf\Big\{(t)^{\frac{p_r}{M}} : \Big(\lim_k \frac{1}{\varphi_s}\sum_{k\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k\Delta_m^n x_k}{t}, z_1,\cdots,z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}} \leq 1\Big\}$$

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. □

Theorem 3. *If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function. Then*

$$m\Big(\mathcal{M}, \varphi^*, q, \Lambda, \Delta_m^n, p, ||.,\cdots,.||\Big) \subset m\Big(\mathcal{M}, \varphi^{**}, q, \Lambda, \Delta_m^n, p, ||.,\cdots,.||\Big)$$

if and only if $\sup_{s\geq 1} \dfrac{\varphi_s^*}{\varphi_s^{**}} < \infty.$

Proof. Let $x \in m\Big(\mathcal{M}, \varphi^*, q, \Lambda, \Delta_m^n, p, ||., \cdots, .||\Big)$ and $N = \sup_{s\geq 1} \frac{\varphi_s^*}{\varphi_s^{**}} < \infty$. Then we get

$$\frac{1}{\varphi_s^{**}} \sum_{n\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k}$$

$$\begin{aligned} &\leq \sup_{s\geq 1} \frac{\varphi_s^*}{\varphi_s^{**}} \frac{1}{\varphi_s^*} \sum_{n\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} \\ &= N \frac{1}{\varphi_s^*} \sum_{n\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} \\ &= 0. \end{aligned}$$

Thus $x \in m\Big(\mathcal{M}, \varphi^{**}, q, \Lambda, \Delta_m^n, p, ||., \cdots, .||\Big)$. Conversely, suppose that

$$m\Big(\mathcal{M}, \varphi^*, q, \Lambda, \Delta_m^n, p, ||., \cdots, .||\Big) \subset m\Big(\mathcal{M}, \varphi^{**}, q, \Lambda, \Delta_m^n, p, ||., \cdots, .||\Big)$$

and $x \in m\Big(\mathcal{M}, \varphi^*, q, \Lambda, \Delta_m^n, p, ||., \cdots, .||\Big)$. Then there exists a $\rho > 0$ such that

$$\frac{1}{\varphi_s^*} \sum_{n\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} < \epsilon,$$

for every $\epsilon > 0$. Suppose that $\sup_{s\geq 1} \frac{\varphi_s^*}{\varphi_s^{**}} = \infty$, then there exists a sequence of numbers (s_j) such that $\lim_{j\to\infty} \frac{\varphi_{s_j}^*}{\varphi_{s_j}^{**}} = \infty$. Hence, we have

$$\frac{1}{\varphi_s^{**}} \sum_{n\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k}$$

$$\leq \sup_{j\geq 1} \frac{\varphi_{s_j}^*}{\varphi_s^{**}} \frac{1}{\varphi_{s_j}^*} \sum_{n\in\sigma,\sigma\in P_s} M_k\Big(q\Big(||\frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} = \infty.$$

Therefore, $x \notin m\Big(\mathcal{M}, \varphi^{**}, q, \Lambda, \Delta_m^n, p, ||., \cdots, .||\Big)$, which is a contradiction. This completes the proof of the theorem. □

Corollary 4. *If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function. Then*

$$m\Big(\mathcal{M}, \varphi^*, q, \Lambda, \Delta_m^n, p, ||., \cdots, .||\Big) = m\Big(\mathcal{M}, \varphi^{**}, q, \Lambda, \Delta_m^n, p, ||., \cdots, .||\Big)$$

if and only if $\sup_{s\geq 1} \frac{\varphi_s^*}{\varphi_s^{**}} < \infty, \quad \sup_{s\geq 1} \frac{\varphi_s^{**}}{\varphi_s^*} > \infty.$

Proof. It is easy to prove so we omit. □

Theorem 5. *For Musielak-Orlicz functions $\mathcal{M} = (M_k)$, $\mathcal{M}' = (M'_k)$ and $\mathcal{M}'' = (M''_k)$ which satisfy Δ_2-condition and q, q_1, q_2 are seminorms. Then the following relation holds:*

(i) $m\Big(\mathcal{M}', \varphi, q, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big) \subset m\Big(\mathcal{M} \circ \mathcal{M}', \varphi, q, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big)$

(ii) $m\Big(\mathcal{M}', \varphi, q, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big) \cap m\Big(\mathcal{M}'', \varphi, q, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big)$

$$\subset m\Big(\mathcal{M}' + \mathcal{M}'', \varphi, q, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big)$$

(iii) $m\Big(\mathcal{M}, \varphi, q_1, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big) \cap m\Big(\mathcal{M}, \varphi, q_2, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big) m$

$$\subset \Big(\mathcal{M}, \varphi, q_1 + q_2, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big).$$

Proof. The proof of theorem along the same lines as the proof of the Theorem 2.5 of [1]. □

Corollary 6. *If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function which satisfy Δ_2- condition. Then $m\Big(\mathcal{M}, \varphi, q, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big) \subset m\Big(\mathcal{M}, \varphi, q, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big)$.*

Theorem 7. *If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function. Then the sequence space $m\Big(\mathcal{M}, \varphi, q, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big)$ is solid.*

Proof. Let $x \in m\Big(\mathcal{M}, \varphi, q, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big)$. Then there exists $\rho > 0$ such that

$$\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k\Big(q\Big(||\frac{\Lambda_k \Delta^n_m x_k}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} < \epsilon,$$

for every $\epsilon > 0$. Let (α_n) be a sequence of scalars with $|\alpha_k| \leq 1$ for all $n \in \mathbb{N}$. Then, we have

$$\frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k\Big(q\Big(||\frac{\Lambda_k \Delta^n_m (\alpha_k x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k}$$

$$\begin{aligned} &\leq \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} |\alpha_k| M_k\Big(q\Big(||\frac{\Lambda_k \Delta^n_m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} \\ &\leq \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k\Big(q\Big(||\frac{\Lambda_k \Delta^n_m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k}, \end{aligned}$$

which proves that $m\Big(\mathcal{M}, \varphi, q, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big)$ is solid space. □

Corollary 8. *If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function. The sequence space $m\Big(\mathcal{M}, \varphi, q, \Lambda, \Delta^n_m, p, ||., \cdots, .||\Big)$ is monotone.*

Proof. It is obvious. □

3 Generalized sequence spaces

Let $A = (a_{ik})$ be an infinite matrix of complex numbers. Then, we have

$$A(x) = (A_i(x)) \text{ if } A_i(x) = \sum_{k=1}^{\infty} a_{ik} x_k$$

converges for each i. For more details see [1].

In this section we introduce the following sequence spaces which are actually the generalizations of sequence spaces defined by Altun and Bilgin [1]. Thus we have the spaces:

$$m(\mathcal{M}, A, \varphi, \Delta_m^n, p, ||., \cdots, .||) = \Big\{x = (x_k) \in s : \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} M_k\Big(||\frac{A_k\Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k}$$

$$< \infty, \text{ for some } \rho > 0\Big\}.$$

If $M_k(x) = x$, we have

$$m(A, \varphi, \Delta_m^n, p, ||., \cdots, .||) = \Big\{x = (x_k) \in s : \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} \Big(||\frac{A_k\Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k}$$

$$< \infty, \text{ for some } \rho > 0\Big\}.$$

In this section of the present paper we shall also study some topological properties and inclusion relations between the spaces $m(\mathcal{M}, A, \varphi, \Delta_m^n, p, ||., \cdots, .||)$.

Theorem 9. *For Musielak-Orlicz functions $\mathcal{M} = (M_k)$. Then the sequence space $m(\mathcal{M}, A, \varphi, \Delta_m^n, p, ||., \cdots, .||)$ is a linear space over the set of complex number $\mathbb{C}$.*

Proof. Let $x = (x_k)$ and $y = (y_k) \in m(\mathcal{M}, A, \varphi, \Delta_m^n, p, ||., \cdots, .||)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} M_k\Big(||\frac{A_k\Delta_m^n x_k}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)^{p_k} < \infty$$

and

$$\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} M_k\Big(||\frac{A_k\Delta_m^n y_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} < \infty.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\|., ..., .\|$ is a n-norm on X and (M_k) are non-decreasing and convex function so by using inequality (1.1), we have

$$\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} M_k\Big(||\frac{A_k\Delta_m^n(\alpha x_k + \beta y_k)}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)^{p_k}$$

$$\begin{aligned}
&\leq \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} M_k\Big[\Big(||\frac{A_k\Delta_m^n(\alpha x_k)}{\rho_3}, z_1, \cdots, z_{n-1}||\Big) + \Big(||\frac{A_k\Delta_m^n(\beta y_k)}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&\leq K \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} \frac{1}{2^{p_k}} M_k\Big(||\frac{A_k\Delta_m^n x_k}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
&+ K \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} \frac{1}{2^{p_k}} M_k\Big(q\Big(||\frac{A_k\Delta_m^n y_k}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} \\
&< \infty.
\end{aligned}$$

Thus, we have $\alpha x+\beta y \in m(\mathcal{M}, A, \phi, \Delta_m^n, p, ||., \cdots, .||)$. Hence $m(\mathcal{M}, A, \phi, \Delta_m^n, p, ||., \cdots, .||)$ is a linear space. □

Theorem 10. *For any Musielak Orlicz function $\mathcal{M} = (M_k)$ and a bounded sequence $p = (p_k)$ of strictly positive real numbers, the space $m\Big(\mathcal{M}, \varphi, A, \Delta^n_m, p, ||., \cdots, .||\Big)$ is a topological linear space paranormed by*

$$g(x) = \inf\Big\{\rho^{\frac{p_r}{M}} : \Big(\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s}\sum_{k\in\sigma} M_k\Big(||\frac{A_k\Delta^n_m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k}\Big)^{\frac{1}{M}} \leq 1,\ r = 1, 2, 3, \cdots\Big\},$$

where $M = \max(1, \sup_k p_k < \infty)$.

Proof. Clearly $g(x) \geq 0$ for $x = (x_k) \in m\Big(\mathcal{M}, \varphi, A, \Delta^n_m, p, ||., \cdots, .||\Big)$. Since $M_k(0) = 0$, we get $g(0) = 0$. Again, if $g(x) = 0$, then

$$g(x) = \inf\Big\{\rho^{\frac{p_r}{M}} : \Big(\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s}\sum_{k\in\sigma} M_k\Big(||\frac{A_k\Delta^n_m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k}\Big)^{\frac{1}{M}} \leq 1\Big\} = 0.$$

This implies that for a given $\epsilon > 0$, there exist some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\Big(\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s}\sum_{k\in\sigma} M_k\Big(||\frac{A_k\Delta^n_m x_k}{\rho_\epsilon}, z_1, \cdots, z_{n-1}||\Big)^{p_k}\Big)^{\frac{1}{M}} \leq 1.$$

Thus
$\Big(\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s}\sum_{k\in\sigma} M_k\Big(||\frac{A_k\Delta^n_m x_k}{\epsilon}, z_1, \cdots, z_{n-1}||\Big)^{p_k}\Big)^{\frac{1}{M}}$

$$\begin{aligned} &\leq \Big(\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s}\sum_{k\in\sigma} M_k\Big(||\frac{A_k\Delta^n_m x_k}{\rho_\epsilon}, z_1, \cdots, z_{n-1}||\Big)^{p_k}\Big)^{\frac{1}{M}} \\ &\leq 1, \end{aligned}$$

Suppose that $x_k \neq 0$ for each $k \in N$. this implies that $A_k\Delta^n_m x_k \neq 0$ for each $k \in N$. Let $\epsilon \to 0$, then $\|\frac{A_k\Delta^n_m x_k}{\epsilon}, z_1, ..., z_{n-1}\| \to \infty$. It follows that

$$\Big(\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s}\sum_{k\in\sigma} M_k\Big(||\frac{A_k\Delta^n_m x_k}{\epsilon}, z_1, \cdots, z_{n-1}||\Big)^{p_k}\Big)^{\frac{1}{M}} \to \infty$$

Which is a contradiction. Therefore, $A_k\Delta^n_m x_k = 0$ for each k and thus $x_k = 0$ for each $k \in N$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\Big(\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s}\sum_{k\in\sigma} M_k\Big(||\frac{A_k\Delta^n_m x_k}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)^{p_k}\Big)^{\frac{1}{M}} \leq 1$$

and

$$\Big(\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s}\sum_{k\in\sigma} M_k\Big(||\frac{A_k\Delta^n_m y_k}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)^{p_k}\Big)^{\frac{1}{M}} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by Minkowski's inequality, we have

$$\Big(\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\varphi_s}\sum_{k\in\sigma}M_k\Big(||\frac{A_k\Delta_m^n x_k}{\rho},z_1,\cdots,z_{n-1}||\Big)^{p_k}\Big)^{\frac{1}{M}}$$

$$\begin{aligned}
&\leq \Big(\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\varphi_s}\sum_{k\in\sigma}M_k\Big(||\frac{A_k\Delta_m^n x_k}{\rho_1+\rho_2},z_1,\cdots,z_{n-1}||\Big)^{p_k}\Big)^{\frac{1}{M}}\\
&\leq \Big(\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\varphi_s}\sum_{k\in\sigma}M_k\Big(\frac{\rho_1}{\rho_1+\rho_2}||\frac{A_k\Delta_m^n x_k}{\rho_1},z_1,...,z_{n-1}||+\frac{\rho_2}{\rho_1+\rho_2}||\frac{A_k\Delta_m^n y_k}{\rho_2},z_1,...,z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}}\\
&\leq (\frac{\rho_1}{\rho_1+\rho_2})\Big(\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\varphi_s}\sum_{k\in\sigma}M_k\Big(q\Big(||\frac{A_k\Delta_m^n x_k}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}}\\
&+ (\frac{\rho_2}{\rho_1+\rho_2})\Big(\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\varphi_s}\sum_{k\in\sigma}M_k\Big(q\Big(||\frac{A_k\Delta_m^n y_k}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big)^{p_k}\Big)^{\frac{1}{M}}\\
&\leq 1.
\end{aligned}$$

Since $\rho' s$ are non-negative, so we have

$$\begin{aligned}
g(x+y) &= \inf\Big\{\rho^{\frac{p_r}{M}}:\Big(\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\varphi_s}\sum_{k\in\sigma}M_k\Big(||\frac{A_k\Delta_m^n(x_k+y_k)}{\rho},z_1,\cdots,z_{n-1}||\Big)^{p_k}\Big)^{\frac{1}{M}}\leq 1\Big\}\\
&\leq \inf\Big\{\rho_1^{\frac{p_r}{M}}:\Big(\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\varphi_s}\sum_{k\in\sigma}M_k\Big(||\frac{A_k\Delta_m^n x_k}{\rho_1},z_1,\cdots,z_{n-1}||\Big)^{p_k}\Big)^{\frac{1}{M}}\leq 1\Big\}\\
&+ \inf\Big\{\rho_2^{\frac{p_r}{M}}:\Big(\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\varphi_s}\sum_{k\in\sigma}M_k\Big(||\frac{A_k\Delta_m^n y_k}{\rho_2},z_1,\cdots,z_{n-1}||\Big)^{p_k}\Big)^{\frac{1}{M}}\leq 1\Big\}.
\end{aligned}$$

Therefore, $g(x+y)\leq g(x)+g(y)$
Finally, we prove that the scalar multiplication is continuous. Let μ be any complex number. By definition,

$$g(\mu x)=\inf\Big\{\rho^{\frac{p_r}{M}}:\Big(\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\varphi_s}\sum_{k\in\sigma}M_k\Big(||\frac{A_k\Delta_m^n x_k}{\rho},z_1,\cdots,z_{n-1}||\Big)^{p_k}\Big)^{\frac{1}{M}}\leq 1\Big\}.$$

Then

$$g(\mu x)=\inf\Big\{(|\mu|t)^{\frac{p_r}{M}}:\Big(\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\varphi_s}\sum_{k\in\sigma}M_k\Big(||\frac{A_k\Delta_m^n(\mu x_k)}{\rho_3},z_1,\cdots,z_{n-1}||\Big)^{p_k}\Big)^{\frac{1}{M}}\Big\},$$

where $t=\frac{\rho}{|\mu|}$. Since $|\mu|^{p_r}\leq\max(1,|\mu|\sup p_r)$, we have

$$g(\mu x)=\max(1,|\mu|\sup p_r)\inf\Big\{(t)^{\frac{p_r}{M}}:\Big(\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\varphi_s}\sum_{k\in\sigma}M_k\Big(||\frac{A_k\Delta_m^n x_k}{t},z_1,\cdots,z_{n-1}||\Big)^{p_k}\Big)^{\frac{1}{M}}\leq 1\Big\}$$

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. □

Theorem 11. *If $\mathcal{M}=(M_k)$ be any Musielak Orlicz function. Then*

$$m\Big(\mathcal{M},\varphi^*,A,\Delta_m^n,p,||.,\cdots,.||\Big)\subset m\Big(\mathcal{M},\varphi^{**},A,\Delta_m^n,p,||.,\cdots,.||\Big)$$

if and only if $\sup_{s\geq 1}\frac{\varphi_s^*}{\varphi_s^{**}}<\infty$.

Proof. The proof is trivial so we omit it. □

Corollary 12. *If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function. Then*

$$m\Big(\mathcal{M}, \varphi^*, A, \Delta_m^n, p, ||., \cdots, .||\Big) = m\Big(\mathcal{M}, \varphi^{**}, A, \Delta_m^n, p, ||., \cdots, .||\Big)$$

if and only if $\sup_{s\geq 1} \frac{\varphi_s^*}{\varphi_s^{**}} < \infty, \quad \sup_{s\geq 1} \frac{\varphi_s^{**}}{\varphi_s^*} > \infty.$

Proof. It is easy to prove so we omit. □

Theorem 13. *For Musielak-Orlicz functions $\mathcal{M} = (M_k)$, $\mathcal{M}' = (M_k')$ and $\mathcal{M}'' = (M_k'')$ which satisfy Δ_2-condition. Then the following relation holds:*
(i) $m\Big(\mathcal{M}', \varphi, A, \Delta_m^n, p, ||., \cdots, .||\Big) \subset m\Big(\mathcal{M} \circ \mathcal{M}', \varphi, A, \Delta_m^n, p, ||., \cdots, .||\Big)$
(ii) $m\Big(\mathcal{M}', \varphi, A, \Delta_m^n, p, ||., \cdots, .||\Big) \cap m\Big(\mathcal{M}'', \varphi, A, \Delta_m^n, p, ||., \cdots, .||\Big)$

$$\subset m\Big(\mathcal{M}' + \mathcal{M}'', \varphi, A, \Delta_m^n, p, ||., \cdots, .||\Big).$$

Proof. The proof is along the same lines as the proof of the Theorem 2.5 of [1]. □

Corollary 14. *If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function which satisfy Δ_2- condition. Then* $m\Big(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, ||., \cdots, .||\Big) \subset m\Big(\mathcal{M}, \varphi, A, \Delta_m^n, p, ||., \cdots, .||\Big)$.

Theorem 15. *If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function. Then the sequence space* $m\Big(\mathcal{M}, \varphi, A, \Delta_m^n, p, ||., \cdots, .||\Big)$ *is solid.*

Proof. Let $x \in m\Big(\mathcal{M}, \varphi, A, \Delta_m^n, p, ||., \cdots, .||\Big)$. Then there exists $\rho > 0$ such that

$$\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} M_k\Big(||\frac{A_k\Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} < \epsilon,$$

for every $\epsilon > 0$. Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then, we have

$$\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} M_k\Big(||\frac{A_k\Delta_m^n(\alpha_k x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k}$$

$$\leq \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} |\alpha_k| M_k\Big(||\frac{A_k\Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k}$$

$$\leq \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} M_k\Big(||\frac{A_k\Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k},$$

which proves that $m\Big(\mathcal{M}, \varphi, A, \Delta_m^n, p, ||., \cdots, .||\Big)$ is solid space. □

Corollary 16. *If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function. Then the sequence space* $m\Big(\mathcal{M}, \varphi, A, \Delta_m^n, p, ||., \cdots, .||\Big)$ *is monotone.*

Proof. It is obvious. □

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