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Acta Universitatis Matthiae Belii, ser. Mathematics
Institute of Mathematics, Faculty of Natural Sciences
Matej Bel University
Tajovského 40
974 01 Banská Bystrica
Slovakia
E-mail: actamath@savbb.sk

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On generalized asymptotically equivalent double sequences through $(V, \lambda, \mu)-$summability

Kavita
Department of Mathematics
Haryana College of Technology and Management
Kaithal-136027, Haryana, India
kukreja.kavita90@gmail.com

Archana Sharma
Department of Mathematics
Haryana College of Technology and Management
Kaithal-136027, Haryana, India
manjue_kaushik@yahoo.com

Vijay Kumar*
Department of Mathematics
Haryana College of Technology and Management
Kaithal-136027, Haryana, India
vjy_kaushik@yahoo.com

Abstract

The purpose of this work is to introduce new generalizations of asymptotically equivalent double sequences which we call $S_{(\lambda,\mu)}-$ equivalent, $V_{(\lambda,\mu)}-$ equivalent, $C_{(1,1)}-$ equivalent, through $(V, \lambda, \mu)-$summability, and obtain some relevant connections between these notions.

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1 The first section

Marouf [1] introduced notion of asymptotically equivalent sequences in order to comparing the rate of growth of two sequences. Later, the idea is applied on many problems arising in the field of summability theory. In 2003, Patterson [2] presented statistical analogue of asymptotically equivalent sequences and studied some of their properties via statistical summability. Subsequently, many authors have shown their interest on asymptotically equivalent sequences in different directions (see [3], [4], [5], [6] and [7]). In present work we extend the idea of asymptotically equivalent double sequences through $(V, \lambda, \mu)-$ summability and obtain some results. We begin by recalling some definitions and results which form the base for present study.

*corresponding author

Definition 1. [1] The two non-negative sequences $x=(x_k)$ and $y=(y_k)$ are said to be asymptotically equivalent to a number L (denoted by $x \sim y$) provided that

$$\lim_{k\to\infty}\left(\frac{x_k}{y_k}\right)=L$$

In case, $L=1$ we simply say x is equivalent to y.

Definition 2. [8] A number sequence $x=(x_k)$ is said to be statistically convergent to a number L provided that for every $\epsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}\ |\{k\le n : |x_k-L|\ge\epsilon\}|=0;$$

where $|\{A\}|$ denotes the cardinality of a sets A, and $|a|$ denotes the absolute value of a number a. In this case, we write $S-\lim_{k\to\infty} x_k = L$ or $x_k \to L(S)$.
The next definition is a natural combination of Definition 1.1 and 1.2.

Definition 3. [2] The two non-negative sequences $x=(x_k)$ and $y=(y_k)$ are said to be asymptotically statistically equivalent of multiple L provided that for every $\epsilon>0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\le n : \left|\left(\frac{x_k}{y_k}\right)-L\right|\ge\epsilon\right\}\right|=0,$$

(denoted by $x \sim^S y$) and simply asymptotically statistical equivalent if $L=1$.

Let $\lambda=(\lambda_n)$ be a non decreasing sequence of positive real numbers tending to ∞ with $\lambda_{n+1}\le\lambda_n+1, \lambda_1=1$. The generalized de la Vallée Pousin mean of $x=(x_k)$ is defined by

$$t_n(x)=\frac{1}{\lambda_n}\sum_{k\in I_n} x_k,$$

where $I_n=[n-\lambda_n+1,n]$.

A sequence $x=(x_k)$ is said to be (V, λ)-summable to a number L (see [9]) provided that $t_n(x)\to L$ as $n\to\infty$. It is being noted that if we take $\lambda_n=n$, then (V, λ)-summability reduces to $(C,1)$- summability.

Definition 4. [10] Let $\lambda=(\lambda_n)$ be a sequence as described above. A number sequence $x=(x_k)$ is said to be λ-statistically convergent to a number L provided that for every $\epsilon>0$

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\ |\{k\in I_n : |x_k-L|\ge\epsilon\}|=0.$$

In this case, we write $S_\lambda-\lim_{k\to\infty} x_k=L$ or $x_k\to L(S_\lambda)$.

The next definition is a natural combination of Definition 1.1 and 1.4.

Definition 5. [11] The two non-negative sequences $x=(x_k)$ and $y=(y_k)$ are said to be asymptotically λ-statistically equivalent of multiple L provided that for every$\epsilon>0$

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\left\{k\in I_n : \left|\left(\frac{x_k}{y_k}\right)-L\right|\ge\epsilon\right\}\right|=0,$$

(denoted by $x \sim^{S_\lambda} y$) and simply asymptotically λ-statistical equivalent if $L=1$.

Definition 6. [11] Let $\lambda = (\lambda_n)$ be a non decreasing sequence as described above. The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically λ-equivalent of multiple L provided that for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} |x_k - L| = 0,$$

(denoted by $x \sim^{(V,\lambda)} y$) and simply strongly asymptotically λ-equivalent if $L = 1$.

By convergence of a double sequence we mean convergence in the Pringsheim's sense [12] as given follow: A double sequence $x = (x_{ij})$ of numbers is said to be convergent to a number L (in the Pringsheim's sense) provided for each $\epsilon > 0$, there exists a positive integer m such that

$$|x_{ij} - L| < \epsilon \quad \text{whenever } i, j \geq m.$$

In this case, the number L is called the Pringsheim's limit of (x_{ij}) and we write $P - \lim_{i,j\to\infty} x_{ij} = L$. Further, a double sequence $x = (x_{ij})$ is said to be bounded if there exists a positive number M such that $|x_{ij}| \leq M$ for all i, j, i.e., if $\|x\|_{(2,\infty)} = \sup_{i,j} |x_{ij}| < \infty$. It is remarkable that, in contrast to the case for single sequences, a convergent double sequence need not be bounded. Let l_2^∞ denotes the space of all bounded sequences of numbers. Mursaleen *et al.* [13] presented extension of Pringsheim's limit in term of statistical convergence for double sequences as follows:

Definition 7. [13] A double sequence $x = (x_{ij})$ of numbers is said to be statistically convergent to a number L provided for each $\epsilon > 0$

$$P - \lim_{n,m\to\infty} \frac{1}{nm} |\{i \leq n, j \leq m : |x_{ij} - L| \geq \epsilon\}| = 0.$$

In this case, the number L is called the statistical limit of x and we write $S(P) - \lim_{i,j\to\infty} x_{ij} = L$.

Definition 8. [6] The two non-negative double sequences $x = (x_{ij})$ and $y = (y_{ij})$ of numbers are said to be asymptotically statistically equivalent of multiple L provided that for every $\epsilon > 0$,

$$P - \lim_{n,m\to\infty} \frac{1}{nm} \left| \left\{ i \leq n, j \leq m : \left| \frac{x_{ij}}{y_{ij}} - L \right| \geq \epsilon \right\} \right| = 0,$$

(denoted by $x \sim^{S(P)} y$) and simply asymptotically statistical equivalent if $L = 1$.

Let $S(P)$ denotes the set of all sequences $x = (x_{ij})$ and $y = (y_{ij})$ such that $x \sim^{S(P)} y$.

Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two non decreasing sequences of positive real numbers tending to ∞ with

$\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$ and

$\mu_{m+1} \leq \mu_m + 1, \mu_1 = 1$.

The generalized de la Vallée Pousin mean of $x = (x_{ij})$ is defined by

$$t_{mn}(x) = \frac{1}{\lambda_n \mu_m} \sum_{(i,j)\in I_n \times I_m} x_{ij},$$

where $I_n = [n - \lambda_n + 1, n]$ and $I_m = [m - \mu_m + 1, m]$.

A double sequence $x = (x_{ij})$ is said to be $(V, \lambda, \mu)-$summable to a number L provided that $t_{mn}(x) \to L$ as $m, n \to \infty$. As in case of single sequences, if we choose $\lambda_n = n$ and $\mu_m = m$, then $(V, \lambda, \mu)-$summability reduces to $(C, 1, 1)-$summability. Let,

$$[C, 1, 1] = \left\{ x = (x_{ij}) : \exists\, L \in \mathbb{R}, P - \lim_{m,n\to\infty} \frac{1}{mn} \sum_{i=1,j=1}^{m,n} |x_{ij} - L| = 0 \right\} \text{ and}$$

$$[V, \lambda, \mu] = \left\{ x = (x_{ij}) : \exists\, L \in \mathbb{R}, P - \lim_{m,n\to\infty} \frac{1}{\lambda_n \mu_m} \sum_{(i,j)\in I_n \times I_m} |x_{ij} - L| = 0 \right\}.$$

Definition 9. [14] Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ as described above. A double sequence $x = (x_{ij})$ of numbers is said to be $(\lambda, \mu)-$statistically convergent to a number L provided for every $\epsilon > 0$,

$$P - \lim_{n,m\to\infty} \frac{1}{\lambda_n \mu_m} |\{(i, j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| = 0.$$

In this case, the number L is called $(\lambda, \mu)-$statistical limit of the sequence $x = (x_{ij})$ and we write $S_{(\lambda,\mu)}(P) - \lim_{i,j\to\infty} x_{ij} = L$.

We now consider some new kind of asymptotically equivalent double sequences defined through $[V, \lambda, \mu]-$summability.

2 Main Results

Definition 10. The two double sequences $x = (x_{ij})$ and $y = (y_{ij})$ are said to be asymptotically Cesàro equivalent of multiple L (denoted by $x \sim^{C_{(1,1)}} y$) provided that

$$P - \lim_{n,m\to\infty} \frac{1}{nm} \sum_{i=1,j=1}^{m,n} \left| \frac{x_{ij}}{y_{ij}} - L \right| = 0.$$

In case $L = 1$, we simply say x is asymptotically Cesàro equivalent to y.

Let $C_{(1,1)}$ denotes the set of all sequences $x = (x_{ij})$ and $y = (y_{ij})$ such that $x \sim^{C_{(1,1)}} y$.

Definition 11. Let p be a positive real number. The two double sequences $x = (x_{ij})$ and $y = (y_{ij})$ are said to be strongly asymptotically $p-$Cesàro equivalent of multiple L (denoted by $x \sim^{C^p_{(1,1)}} y$) provided that

$$P - \lim_{n,m\to\infty} \frac{1}{nm} \sum_{i=1,j=1}^{m,n} \left| \frac{x_{ij}}{y_{ij}} - L \right|^p = 0.$$

In case $L = 1$, we simply say x is strongly asymptotically $p-$Cesàro equivalent to y.

Let $C^p_{(1,1)}$ denotes the set of all sequences $x = (x_{ij})$ and $y = (y_{ij})$ such that $x \sim^{C^p_{(1,1)}} y$.

Remark 12. If $0 < p \leq q < \infty$, then $C^q_{(1,1)} \subseteq C^p_{(1,1)}$ and

$$C^p_{(1,1)} \cap l_2^\infty = C^1_{(1,1)} \cap l_2^\infty = C_{(1,1)} \cap l_2^\infty.$$

Theorem 13. *Let p be a positive real number such that $p \in (0, \infty)$, then $C^p_{(1,1)} \subseteq S(P)$.*

Proof. Let $p \in (0, \infty)$ and $x = (x_{ij})$ and $y = (y_{ij})$ be two double sequences such that $x \sim^{C^p_{(1,1)}} y$. For any $\epsilon > 0$, if we take

$$K_{mn} = \left\{ (i,j), i \le n, j \le m : \left| \frac{x_{ij}}{y_{ij}} - L \right|^p \ge \epsilon \right\},$$

then we can write

$$\frac{1}{nm} \sum_{i=1,j=1}^{m,n} \left| \frac{x_{ij}}{y_{ij}} - L \right|^p = \frac{1}{nm} \left\{ \sum_{(i,j)\in K_{mn}} \left| \frac{x_{ij}}{y_{ij}} - L \right|^p + \sum_{(i,j)\notin K_{mn}} \left| \frac{x_{ij}}{y_{ij}} - L \right|^p \right\}$$

$$\ge \frac{1}{nm} \left\{ \sum_{(i,j)\in K_{mn}} \left| \frac{x_{ij}}{y_{ij}} - L \right|^p \right\}$$

$$\ge \frac{1}{nm} \left| \left\{ (i,j),\, i \le n, j \le m : \left| \frac{x_{ij}}{y_{ij}} - L \right|^p \ge \epsilon \right\} \right|.$$

Since $x \sim^{C^p_{(1,1)}} y$, it follows that $x \sim^{S^{(P)}} y$. □

Theorem 14. *Let p be a positive real number such that $p \in (0, \infty)$, then $S(P) \cap l_2^\infty \subseteq C^p_{(1,1)}$.*

Proof. Let $p \in (0, \infty)$ and $x = (x_{ij})$, $y = (y_{ij}) \in l_2^\infty$ such that $x \sim^{S(P)} y$. Since $x = (x_{ij})$, $y = (y_{ij}) \in l_2^\infty$ so there is a real number M (say) such that for every i and j we have

$$\left| \frac{x_{ij}}{y_{ij}} - L \right| \le M.$$

Since $x \sim^{S(P)} y$ so for given $\epsilon > 0$ and enough large m and n we can write

$$\begin{aligned} \frac{1}{nm} \sum_{i=1,j=1}^{m,n} \left| \frac{x_{ij}}{y_{ij}} - L \right|^p &= \frac{1}{nm} \left\{ \sum_{\substack{i=1,j=1: \\ \left|\frac{x_{ij}}{y_{ij}} - L\right| \le \epsilon}}^{m,n} \left| \frac{x_{ij}}{y_{ij}} - L \right|^p + \sum_{\substack{i=1,j=1: \\ \left|\frac{x_{ij}}{y_{ij}} - L\right| > \epsilon}}^{m,n} \left| \frac{x_{ij}}{y_{ij}} - L \right|^p \right\} \\ &\le (\epsilon)^p + \frac{M^p}{nm} \left| \left\{ i \le n, j \le m : \left| \frac{x_{ij}}{y_{ij}} - L \right| \ge \epsilon \right\} \right| \\ &\le 2\epsilon^p \end{aligned}$$

This shows that $x \sim^{C^p_{(1,1)}} y$. □

Definition 15. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two non decreasing sequences of positive real numbers such that each tending to ∞ with

$\lambda_{n+1} \le \lambda_n + 1, \lambda_1 = 1$ and

$\mu_{m+1} \leq \mu_m + 1, \mu_1 = 1.$
The two non-negative double sequences $x = (x_{ij})$ and $y = (y_{ij})$ are said to be asymptotically $(\lambda, \mu)-$statistically equivalent of multiple L (denoted by $x \sim^{S_{(\lambda,\mu)}} y$) provided that for every $\epsilon > 0$,

$$P - \lim_{n,m\to\infty} \frac{1}{\lambda_n \mu_m} \left| \left\{ (i,j) \in I_n \times I_m : \left| \frac{x_{ij}}{y_{ij}} - L \right| \geq \epsilon \right\} \right| = 0.$$

In case $L = 1$, we simply say x is asymptotically $(\lambda, \mu)-$statistically equivalent to y.

Let $S_{(\lambda,\mu)}$ denotes the set of all sequences $x = (x_{ij})$ and $y = (y_{ij})$ such that $x \sim^{S_{(\lambda,\mu)}} y$. For the choose $\lambda_n = n$ and $\mu_m = m$, Definition 2.6 coincides with Definition 1.8.

Definition 16. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two sequences as in Definition 2.6. The two double sequences $x = (x_{ij})$ and $y = (y_{ij})$ are said to be strongly asymptotically $(\lambda, \mu)-$equivalent of multiple L (denoted by $x \sim^{V_{(\lambda,\mu)}} y$) provided that

$$P - \lim_{n,m\to\infty} \frac{1}{\lambda_n \mu_m} \sum_{(i,j)\in I_n \times I_m} \left| \frac{x_{ij}}{y_{ij}} - L \right| = 0.$$

In case $L = 1$, we simply say x is strongly asymptotically $(\lambda, \mu)-$equivalent to y.

Let $V_{(\lambda,\mu)}$ denotes the set of all sequences $x = (x_{ij})$ and $y = (y_{ij})$ such that $x \sim^{V_{(\lambda,\mu)}} y$.

Theorem 17. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two sequences as describe above, then we have following:*

(i) $x \sim^{V_{(\lambda,\mu)}} y$ implies $x \sim^{S_{(\lambda,\mu)}} y$ and the inclusion $V_{(\lambda,\mu)} \subset S_{(\lambda,\mu)}$ is proper.

(ii) If $x = (x_{ij})$, $y = (y_{ij}) \in l_2^\infty$ such that $x \sim^{S_{(\lambda,\mu)}} y$, then $x \sim^{V_{(\lambda,\mu)}} y$ and hence $x \sim^{C_{(1,1)}} y$ provided $x = (x_{ij})$ is not eventually constant.

(iii) $S_{(\lambda,\mu)}) \cap l_2^\infty = V_{(\lambda,\mu)} \cap l_2^\infty$.

Proof.

(i) Suppose $x = (x_{ij})$ and $y = (y_{ij})$ be two double sequences such that $x \sim^{V_{(\lambda,\mu)}} y$. For any $\epsilon > 0$, we can write

$$\sum_{(i,j)\in I_n \times I_m} \left| \frac{x_{ij}}{y_{ij}} - L \right| \geq \sum_{\substack{(i,j)\in I_n \times I_m \\ |\frac{x_{ij}}{y_{ij}} - L| \geq \epsilon}} \left| \frac{x_{ij}}{y_{ij}} - L \right|$$
$$\geq \epsilon \left| \left\{ (i,j) \in I_n \times I_m : \left| \frac{x_{ij}}{y_{ij}} - L \right| \geq \epsilon \right\} \right|.$$

Since $x \sim^{V_{(\lambda,\mu)}} y$, so $P - \lim_{n,m\to\infty} \frac{1}{\lambda_n \mu_m} \left| \left\{ (i,j) \in I_n \times I_m : \left| \frac{x_{ij}}{y_{ij}} - L \right| \geq \epsilon \right\} \right| = 0$. This shows that $x \sim^{S_{(\lambda,\mu)}} y$.

We next give an example that shows the containment $V_{(\lambda,\mu)} \subset S_{(\lambda,\mu)}$ is proper. Define sequences $x = (x_{ij})$ and $y = (y_{ij})$ as follows:

$$x_{ij} = \begin{cases} ij, & \text{if } n - [\sqrt{\lambda_n}] + 1 \leq i \leq n \quad \text{and} \quad m - [\sqrt{\mu_m}] + 1 \leq j \leq m] \\ 0 & \text{otherwise} \end{cases}$$

and $y_{ij} = 1$ for all i and j.
Then $x = x_{ij} \notin l_2^\infty$ and for every $\epsilon (0 < \epsilon \leq 1)$ we have

$$\frac{1}{\lambda_n \mu_m} \left| \left\{ (i,j) \in I_n \times I_m : \left| \frac{x_{ij}}{y_{ij}} - 0 \right| \geq \epsilon \right\} \right|$$
$$= \frac{1}{\lambda_n \mu_m} \left| \left\{ (i,j) \in I_n \times I_m : {}^{n-[\sqrt{\lambda_n}]+1 \leq i \leq n \text{ and}}_{m-[\sqrt{\mu_m}]+1 \leq j \leq m} \right\} \right|$$
$$\leq \frac{[\sqrt{\lambda_n}\sqrt{\mu_m}]}{\lambda_n \mu_m}.$$

It follows that

$$P - \lim_{n,m\to\infty} \frac{1}{\lambda_n \mu_m} \left| \left\{ (i,j) \in I_n \times I_m : \left| \frac{x_{ij}}{y_{ij}} - 0 \right| \geq \epsilon \right\} \right| = P - \lim_{n,m\to\infty} \frac{[\sqrt{\lambda_n}\sqrt{\mu_m}]}{\lambda_n \mu_m} = 0.$$

This shows that $x \sim^{S_{(\lambda,\mu)}} y$. Also note that

$$P - \lim_{n,m\to\infty} \frac{1}{\lambda_n \mu_m} \sum_{(i,j)\in I_n \times I_m} \left| \frac{x_{ij}}{y_{ij}} - 0 \right|$$

does not exists. Thus the inclusion $V_{(\lambda,\mu)} \subset S_{(\lambda,\mu)}$ is proper.
(ii) Let $x = (x_{ij})$, $y = (y_{ij}) \in l_2^\infty$ such that $x \sim^{S_{(\lambda,\mu)}} y$. Since $x = (x_{ij})$, $y = (y_{ij}) \in l_2^\infty$ so there is a real number M (say) such that for every i and j we have

$$\left| \frac{x_{ij}}{y_{ij}} - L \right| \leq M.$$

Also for given $\epsilon > 0$ and enough large m and n we can write

$$\frac{1}{\lambda_n \mu_m} \sum_{(i,j)\in I_n \times I_m} \left| \frac{x_{ij}}{y_{ij}} - L \right| = \frac{1}{\lambda_n \mu_m} \sum_{\substack{(i,j)\in I_n \times I_m \\ \left| \frac{x_{ij}}{y_{ij}} - L \right| \geq \epsilon}} \left| \frac{x_{ij}}{y_{ij}} - L \right| + \frac{1}{\lambda_n \mu_m} \sum_{\substack{(i,j)\in I_n \times I_m \\ \left| \frac{x_{ij}}{y_{ij}} - L \right| < \epsilon}} \left| \frac{x_{ij}}{y_{ij}} - L \right|$$
$$\leq \frac{M}{\lambda_n \mu_m} \left| \left\{ (i,j) \in I_n \times I_m : \left| \frac{x_{ij}}{y_{ij}} - L \right| \geq \epsilon \right\} \right| + \epsilon.$$

Since $x \sim^{S_{(\lambda,\mu)}} y$, it follows that the first part on right side of the above expression is zero, which immediately gives $x \sim^{V_{(\lambda,\mu)}} y$.
Furthermore, using the fact $(\frac{\lambda_n}{n}) \leq 1$ and $(\frac{\mu_m}{m}) \leq 1$, we have

$$\frac{1}{nm} \sum_{i=1,j=1}^{n,m} \left| \frac{x_{ij}}{y_{ij}} - L \right| = \frac{1}{nm} \sum_{i=1,j=1}^{n-\lambda_n, m-\mu_m} \left| \frac{x_{ij}}{y_{ij}} - L \right| + \frac{1}{nm} \sum_{i=n-\lambda_n+1, j=m-\mu_m+1}^{n,m} \left| \frac{x_{ij}}{y_{ij}} - L \right|$$
$$\leq \frac{1}{\lambda_n \mu_m} \sum_{i=1,j=1}^{n-\lambda_n, m-\mu_m} \left| \frac{x_{ij}}{y_{ij}} - L \right| + \frac{1}{\lambda_n \mu_m} \sum_{(i,j)\in I_n \times I_m} \left| \frac{x_{ij}}{y_{ij}} - L \right|$$
$$\leq \frac{2}{\lambda_n \mu_m} \sum_{(i,j)\in I_n \times I_m} \left| \frac{x_{ij}}{y_{ij}} - L \right|.$$

Since $x \sim^{V_{(\lambda,\mu)}} y$, it follows that $x \sim^{C_{(1,1)}} y$.

(iii) This immediately follows from (i) and (ii). □

Theorem 18. *Let* $\lambda = (\lambda_n)$ *and* $\mu = (\mu_m)$ *be two sequences as describe above. Then* $S(P) \subset S_{(\lambda,\mu)}$ *if and only if,* $\liminf_{n\to\infty} \frac{\lambda_n}{n} > 0$ *and* $\liminf_{m\to\infty} \frac{\mu_m}{m} > 0$.

Proof. For given $\epsilon > 0$, we have

$$\left\{(i,j) \in I_n \times I_m : \left|\frac{x_{ij}}{y_{ij}} - L\right| \geq \epsilon\right\} \subseteq \left\{i \leq n, j \leq m : \left|\frac{x_{ij}}{y_{ij}} - L\right| \geq \epsilon\right\}.$$

Therefore,

$$\begin{aligned}\frac{1}{nm}\left|\left\{i \leq n, j \leq m : \left|\frac{x_{ij}}{y_{ij}} - L\right| \geq \epsilon\right\}\right| &\geq \frac{1}{\lambda_n\mu_m}\left|\left\{(i,j) \in I_n \times I_m : \left|\frac{x_{ij}}{y_{ij}} - L\right| \geq \epsilon\right\}\right| \\ &= \left(\frac{\lambda_n}{n}\right)\left(\frac{\mu_m}{m}\right)\frac{1}{\lambda_n\mu_m}\left|\left\{(i,j) \in I_n \times I_m : \left|\frac{x_{ij}}{y_{ij}} - L\right| \geq \epsilon\right\}\right|.\end{aligned}$$

Taking limit as $n, m \to \infty$ and using the assumption, we get $S(P) \subset S_{(\lambda,\mu)}$.

Conversely, suppose that $x = (x_{ij})$, $y = (y_{ij})$ be two double sequences such that $x \sim^{S(P)} y$. Assume, either $\liminf_{n\to\infty} \frac{\lambda_n}{n}$ or $\liminf_{m\to\infty} \frac{\mu_m}{m}$ or both are zero. Then we can choose two subsequences (n_p) and (m_q) such that $\frac{\lambda_{n_p}}{n_p} < \frac{1}{p}$and $\frac{\mu_{m_q}}{m_q} < \frac{1}{q}$. Define double sequences $x = (x_{ij})$ and $y = (y_{ij})$ as follows:

$$x_{ij} = \begin{cases} 1 & \text{if } i \in I_{n_p} \quad \text{and} \quad j \in I_{m_q} \qquad (p,q = 1,2,3,\ldots) \\ 0 & \text{otherwise.}\end{cases}$$

and $y_{ij} = 1$ for all i and j. Then clearly $x \sim^{C(1,1)} y$ and therefore by Theorem 2.4, $x \sim^{S(P)} y$ which implies $x \sim^{S_{(\lambda,\mu)}} y$ as $S(P) \subset S_{(\lambda,\mu)}$. On the other hand the sequences $x = (x_{ij})$ and $y = (y_{ij})$ do not satisfy $x \sim^{V_{(\lambda,\mu)}} y$ which contradicts Theorem 2.8 (ii). Hence, we have $\liminf_{n\to\infty} \frac{\lambda_n}{n} > 0$ and $\liminf_{m\to\infty} \frac{\mu_m}{m} > 0$. □

Definition 19. Let p be a positive real number. The two double sequences $x = (x_{ij})$ and $y = (y_{ij})$ are said to be strongly asymptotically $V^p_{(\lambda,\mu)}-$equivalent of multiple L (denoted by $x \sim^{V^p_{(\lambda,\mu)}} y$) provided that

$$P - \lim_{n,m\to\infty} \frac{1}{\lambda_n\mu_m} \sum_{(i,j)\in I_n\times I_m} \left|\frac{x_{ij}}{y_{ij}} - L\right|^p = 0.$$

In case $L = 1$, we simply say x is strongly asymptotically $V^p_{(\lambda,\mu)}-$equivalent to y.

Let $V^p_{(\lambda,\mu)}$ denotes the set of all sequences $x = (x_{ij})$ and $y = (y_{ij})$ such that $x \sim^{V^p_{(\lambda,\mu)}} y$.

Following Theorems are the analogue of Theorem 2.4 and 2.5, consequently their proofs can be obtained similarly.

Theorem 20. *Let* p *be a positive real number such that* $p \in (0,\infty)$*, then* $V^p_{(\lambda,\mu)} \subseteq S^p_{(\lambda,\mu)}$.

Theorem 21. *Let* p *be a positive real number such that* $p \in (0,\infty)$*, then* $S^p_{(\lambda,\mu)} \cap l^\infty_2 \subseteq V^p_{(\lambda,\mu)}$.

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Coincidence point theorems for a family of multivalued mappings in partially ordered metric spaces

Binayak S. Choudhury
Department of Mathematics, Bengal Engineering and Science University, Shibpur, Howrah - 711103, West Bengal, India
binayak12@yahoo.co.in, binayak@becs.ac.in

Nikhilesh Metiya *
Department of Mathematics, Bengal Institute of Technology, Kolkata - 700150, West Bengal, India
metiya.nikhilesh@gmail.com

Abstract

In this paper we establish certain multivalued coincidence point results of a family of multivalued mappings with a singlevalued mapping under the assumptions of certain almost contractive type inequalities. Our results are derived in metric spaces with a partial ordering. The corresponding singled valued cases are shown to extend a number of existing results. We have given one illustrative example. The methodology applied here is a blending of order theoretic and analytic methodologies.

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1 Introduction

In the fixed point theory of setvalued maps two types of distances are generally used. One is the Hausdorff distance. Nadler [22] had proved a multivalued version of the Banach's contraction mapping principle by using the Hausdorff metric. There are many other fixed point results using this Hausdorff metric, some instances of these works are in [9, 17, 29, 30, 31]. The another distance is the δ - distance. This is not metric like the Hausdorff distance, but shares most of the properties of a metric. It has been used in many problem on fixed point theory like those in [1, 2, 19, 33].

In recent times, fixed point theory has developed rapidly in partially ordered metric spaces; that is, metric spaces endowed with a partial ordering. References [10, 15, 23, 25, 27] are some recent instances of these works. A speciality of these problems is that they use both analytic and order theoretic methods. It is also one of the main reasons why they are considered interesting.

Khan et al. [21] initiated the use of a control function in metric fixed point theory which they called Altering distance function. Several works on fixed point theory like those noted in [12, 16, 26, 28] have utilized this control function.

*corresponding author

The concept of almost contractions were introduced by Berinde [5, 6]. It was shown in [5] that any strict contraction, the Kannan [20] and Zamfirescu [34] mappings, as well as a large class of quasi-contractions, are all almost contractions. Almost contractions and its generalizations were further considered in several works like [7, 11, 24].

The purpose of this paper is to establish some coincidence point results of a family of multivalued mappings with a single valued mapping under the assumptions of certain almost contractive type inequalities in partially ordered metric spaces. We have also utilized δ-compatible pairs in our theorems. In another theorem we have replaced the continuities of the functions with an order condition. We also give here the corresponding singlevalued versions of the theorems which generalize a number of existing works. An illustrative example for the multivalued case is given.

2 Mathematical Preliminaries

In the following we give some technical definitions which are used in our theorems.
Let $(X,\ d)$ be a metric space. We denote the class of nonempty and bounded subsets of X by $B(X)$. For $A,\ B \in B(X)$, functions $D(A,\ B)$ and $\delta(A,\ B)$ are defined as follows:

$$D(A,\ B) = \inf\,\{d(a,\ b) : a \in A,\ b \in B\}$$

and

$$\delta(A,\ B) = \sup\,\{d(a,\ b) : a \in A,\ b \in B\}.$$

If $A = \{a\}$, then we write $D(A,\ B) = D(a,\ B)$ and $\delta(A,\ B) = \delta(a,\ B)$. Also, in addition, if $B = \{b\}$, then $D(A,\ B) = d(a,\ b)$ and $\delta(A,\ B) = d(a,\ b)$. Obviously, $D(A,\ B) \leq \delta(A,\ B)$. For all $A,\ B,\ C \in B(X)$, the definition of $\delta(A,\ B)$ yields the following:

$$\delta(A,\ B) = \delta(B,\ A),$$

$$\delta(A,\ B) \leq \delta(A,\ C) + \delta(C,\ B),$$

$$\delta(A,\ B) = 0 \text{ iff } A = B = \{a\},$$

$$\delta(A,\ A) = \text{ diam } A. \ [13]$$

There are several works which have utilized δ - distance [2, 4, 13, 14, 19, 33].

Definition 1. ([13]) A sequence $\{A_n\}$ of subsets of metric space $(X,\ d)$ is said to be convergent to subset A of X if

(i) given $a \in A$, there is a sequence $\{a_n\}$ in X such that $a_n \in A_n$, for $n = 1, 2, 3, ...$, and $\{a_n\}$ converges to a.

(ii) given $\epsilon > 0$, there exists a positive integer N such that $A_n \subseteq A_\epsilon$, for all $n > N$, where A_ϵ is the union of all open sphere with centers in A and radius ϵ.

Lemma 2. *([13, 14]) If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$, where (X, d) is a complete metric space and $\{A_n\} \to A$ and $\{B_n\} \to B$ where $A, B \in B(X)$ then*

$$\delta(A_n, B_n) \to \delta(A,\ B) \quad as \quad n \to \infty.$$

Lemma 3. *([14]) If $\{A_n\}$ is a sequence of bounded subsets of a complete metric space (X, d) and if $\lim\limits_{n\to\infty} \delta(A_n, \{y\}) = 0$, for some $y \in X$, then $\{A_n\} \to \{y\}$ as $n \longrightarrow \infty$.*

Definition 4. ([14]) A set-valued mapping $T : X \longrightarrow B(X)$, where $(X,\ d)$ is a metric space, is continuous at a point $x \in X$ if $\{x_n\}$ is a sequence in X converging to x, then the sequence $\{Tx_n\}$ in $B(X)$ converges to Tx. T is said to be continuous in X if it is continuous at each point $x \in X$.

Lemma 5. *([14]) If $\{A_n\}$ is a sequence of nonempty subsets of X and $z \in X$ such that*

$$\lim_{n\to\infty} a_n = z,$$

where z is independent of the particular choice of each $a_n \in A_n$. If a self map g of X is continuous, $\{gz\}$ is the limit of the sequence $\{gA_n\}$.

Definition 6. ([18]) Two self maps g and T of a metric space (X, d) are said to be compatible mappings if $\lim\limits_{n\to\infty} d(gTx_n, Tgx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} gx_n = \lim\limits_{n\to\infty} Tx_n = t$, for some $t \in X$.

Definition 7. ([19]) The mappings $g : X \longrightarrow X$ and $T : X \longrightarrow B(X)$, where $(X,\ d)$ is a metric space, are δ- compatible if $\lim\limits_{n\to\infty} \delta(Tgx_n, gTx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $gTx_n \in B(X)$ and $Tx_n \to \{t\}$, $gx_n \to t$, for some t in X.

Definition 8. Let $(X,\ d)$ be a metric space and $g : X \longrightarrow X$ and $T : X \longrightarrow B(X)$. Then $u \in X$ is called a coincidence point of g and T if $\{gu\} = Tu$.

Definition 9. ([4]) Let A and B be two nonempty subsets of a partially ordered set $(X,\ \preceq)$. The relation between A and B is denoted and defined as follows:
$A \prec_1 B$, if for every $a \in A$ there exists $b \in B$ such that $a \preceq b$.

Definition 10. ([21]) A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is monotone increasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

3 Main Results

Lemma 11. *Let $(X,\ d)$ be a metric space and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_{n+1},\ x_n) = 0. \tag{3.1}$$

If $\{x_n\}$ is not a Cauchy sequence in $(X,\ d)$, then there exists $\epsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following four sequences tend to ϵ when $k \longrightarrow \infty$:

$$d(x_{m(k)},\ x_{n(k)}),\ d(x_{m(k)},\ x_{n(k)+1}),\ d(x_{n(k)},\ x_{m(k)+1}),\ d(x_{m(k)+1},\ x_{n(k)+1}). \tag{3.2}$$

Proof. Suppose that $\{x_n\}$ is a sequence in $(X,\ d)$ satisfying (3.1) which is not Cauchy. Then there exists $\epsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that for all positive integers k,

$$n(k) > m(k) > k,\ d(x_{m(k)},\ x_{n(k)-1}) < \epsilon,\ d(x_{m(k)},\ x_{n(k)}) \geq \epsilon.$$

Now,

$$\epsilon \le d(x_{m(k)},\ x_{n(k)}) \le d(x_{n(k)},\ x_{n(k)-1}) + d(x_{n(k)-1},\ x_{m(k)}) < d(x_{n(k)},\ x_{n(k)-1}) + \epsilon.$$

Letting $k \longrightarrow \infty$ in the above inequality and using (3.1), we have

$$\lim_{k\to\infty} d(x_{m(k)},\ x_{n(k)}) = \epsilon. \tag{3.3}$$

Again,

$$d(x_{m(k)},\ x_{n(k)}) \le d(x_{m(k)},\ x_{n(k)+1}) + d(x_{n(k)+1},\ x_{n(k)})$$

and

$$d(x_{m(k)},\ x_{n(k)+1}) \le d(x_{m(k)},\ x_{n(k)}) + d(x_{n(k)},\ x_{n(k)+1}).$$

Letting $k \longrightarrow \infty$ in the above inequalities and using (3.1) and (3.3), we have

$$\lim_{k\to\infty} d(x_{m(k)},\ x_{n(k)+1}) = \epsilon. \tag{3.4}$$

That the remaining two sequences in (3.2) tend to ϵ can be proved in a similar way. □

Theorem 12. *Let $\theta : [0,\ \infty) \longrightarrow [0,\ 1)$ be a continuous function and ψ be an altering distance function. Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that $(X,\ d)$ is a complete metric space. Let $\{T_\alpha : X \longrightarrow B(X) : \alpha \in \Lambda\}$ be a family of multivalued mappings. Let $g : X \longrightarrow X$ be a mapping such that $g(X)$ is closed in X. Suppose that there exists $\alpha_0 \in \Lambda$ such that*

(i) T_{α_0} and g are continuous,

(ii) $T_{\alpha_0}x \subseteq g(X)$ and $gT_{\alpha_0}x \in B(X)$, for every $x \in X$,

(iii) there exists $x_0 \in X$ such that $\{gx_0\} \prec_1 T_{\alpha_0}x_0$,

(iv) for $x,\ y \in X$, $gx \preceq gy$ implies $T_{\alpha_0}x \prec_1 T_{\alpha_0}y$,

(v) the pair $(g,\ T_{\alpha_0})$ is δ - compatible,

(vi) $\psi(\delta(T_{\alpha_0}x, T_\alpha y))$

$$\le \theta(d(gx, gy))\ max\ \{\psi(d(gx, gy)),\ \psi(D(gx, T_{\alpha_0}x)),\ \psi(D(gy, T_\alpha y)),$$
$$\sqrt{\psi(D(gx, T_\alpha y))\ .\ \psi(D(gy, T_{\alpha_0}x))}\ \}$$
$$+ L\ min\ \{\psi(D(gx,\ T_{\alpha_0}x)),\ \psi(D(gy,\ T_\alpha y)),\ \psi(D(gx,\ T_\alpha y)),\ \psi(D(gy,\ T_{\alpha_0}x))\},$$

where $x,\ y \in X$ such that gx and gy are comparable and $L \ge 0$.

Then g and $\{T_\alpha : \alpha \in \Lambda\}$ have a coincidence point.

Proof. First we establish that any coincidence point of g and T_{α_0} is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$ and conversely. Suppose that $z \in X$ be a coincidence point of g

and T_{α_0}. Then $\{gz\} = T_{\alpha_0}z$. From (vi) and using the properties of ψ, we have

$$\begin{aligned}
\psi(\delta(gz,\ T_\alpha z)) &= \psi(\delta(T_{\alpha_0}z,\ T_\alpha z))\\
&\le \theta(d(gz,gz))\ \max\{\psi(d(gz,gz)),\ \psi(D(gz,T_{\alpha_0}z)),\ \psi(D(gz,T_\alpha z)),\\
&\qquad \sqrt{\psi(D(gz,T_\alpha z))\ .\ \psi(D(gz,T_{\alpha_0}z))}\ \}\\
&\quad + L\ \min\{\psi(D(gz,\ T_{\alpha_0}z)),\ \psi(D(gz,\ T_\alpha z)),\ \psi(D(gz,\ T_\alpha z)),\ \psi(D(gz,\ T_{\alpha_0}z))\}\\
&= \theta(d(gz,gz))\ \max\{\psi(d(gz,gz)),\ \psi(d(gz,gz)),\ \psi(D(gz,T_\alpha z)),\\
&\qquad \sqrt{\psi(D(gz,T_\alpha z))\ .\ \psi(d(gz,gz))}\ \}\\
&\quad + L\ \min\{\psi(d(gz,\ gz)),\ \psi(D(gz,\ T_\alpha z)),\ \psi(D(gz,\ T_\alpha z)),\ \psi(d(gz,\ gz))\}\\
&= \theta(d(gz,gz))\psi(D(gz,T_\alpha z))\\
&< \psi(D(gz,T_\alpha z)), (\text{since } \theta(t) < 1,\ \text{for all } t \in [0,\ \infty)).
\end{aligned}$$

Again using the monotone property of ψ, we have

$$\delta(gz,\ T_\alpha z) < D(gz,\ T_\alpha z) \le \delta(gz,\ T_\alpha z),$$

which implies that $\delta(gz,\ T_\alpha z) = 0$, that is, $\{gz\} = T_\alpha z$, for all $\alpha \in \Lambda$. Hence z is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$. Converse part is trivial.

Now it is sufficient to prove that g and T_{α_0} have a coincidence point. Let $x_0 \in X$ be such that $\{gx_0\} \prec_1 T_{\alpha_0}x_0$. Then there exists $u \in T_{\alpha_0}x_0$ such that $gx_0 \preceq u$. Since $T_{\alpha_0}x_0 \subseteq g(X)$ and $u \in T_{\alpha_0}x_0$, there exists $x_1 \in X$ such that $gx_1 = u$. So $gx_0 \preceq gx_1$. Then by the assumption (iv), $T_{\alpha_0}x_0 \prec_1 T_{\alpha_0}x_1$. Since $u = gx_1 \in T_{\alpha_0}x_0$, there exists $v \in T_{\alpha_0}x_1$ such that $gx_1 \preceq v$. As $T_{\alpha_0}x_1 \subseteq g(X)$ and $v \in T_{\alpha_0}x_1$, there exists $x_2 \in X$ such that $gx_2 = v$. So $gx_1 \preceq gx_2$. Continuing this process we construct a sequence $\{x_n\}$ in X such that

$$gx_{n+1} \in T_{\alpha_0}x_n, \quad \text{for all} \quad n \ge 0, \tag{3.5}$$

and

$$gx_0 \preceq gx_1 \preceq gx_2 \preceq \ldots\ldots \preceq gx_n \preceq gx_{n+1}\ldots. \tag{3.6}$$

Let $\tau_n = d(gx_n, gx_{n+1})$.
Since $gx_n \preceq gx_{n+1}$, putting $\alpha = \alpha_0$, $x = x_n$ and $y = x_{n+1}$ in (vi) and using the properties of ψ, we have

$$\begin{aligned}
\psi(\tau_{n+1}) &\le \psi(\delta(T_{\alpha_0}x_n,\ T_{\alpha_0}x_{n+1}))\\
&\le \theta(\tau_n)\ \max\{\psi(\tau_n),\ \psi(D(gx_n,T_{\alpha_0}x_n)),\psi(D(gx_{n+1},T_{\alpha_0}x_{n+1})),\\
&\qquad \sqrt{\psi(D(gx_n,T_{\alpha_0}x_{n+1})).\ \psi(D(gx_{n+1},T_{\alpha_0}x_n))}\ \}\\
&\quad + L\ \min\{\psi(D(gx_n,T_{\alpha_0}x_n)),\psi(D(gx_{n+1},T_{\alpha_0}x_{n+1})),\\
&\qquad \psi(D(gx_n,T_{\alpha_0}x_{n+1})),\psi(D(gx_{n+1},T_{\alpha_0}x_n))\}\\
&\le \theta(\tau_n)\ \max\{\psi(\tau_n),\ \psi(d(gx_n,gx_{n+1})),\ \psi(d(gx_{n+1},gx_{n+2})),\\
&\qquad \sqrt{\psi(d(gx_n,gx_{n+2})).\ \psi(d(gx_{n+1},gx_{n+1}))}\ \}\\
&\quad + L\ \min\{\psi(d(gx_n,\ gx_{n+1})),\ \psi(d(gx_{n+1},\ gx_{n+2})),\\
&\qquad \psi(d(gx_n,\ gx_{n+2})),\ \psi(d(gx_{n+1},\ gx_{n+1}))\}\\
&= \theta(\tau_n)\ \max\{\psi(\tau_n),\ \psi(\tau_{n+1})\}.
\end{aligned} \tag{3.7}$$

Suppose that, $\max\{\psi(\tau_n),\ \psi(\tau_{n+1})\} = \psi(\tau_{n+1})$. Then from (3.7), it follows that

$$\psi(\tau_{n+1}) \le \theta(\tau_n)\ \psi(\tau_{n+1}) < \psi(\tau_{n+1}),\ (\text{since } \theta(\tau_n) < 1),$$

which is a contradiction. Hence

$$\psi(\tau_{n+1}) \leq \theta(\tau_n)\ \psi(\tau_n) < \psi(\tau_n),\ \text{(since } \theta(\tau_n) < 1).$$

By the monotone property of ψ, it follows that

$$\tau_{n+1} < \tau_n, \quad \text{for all} \ \ n \geq 0,$$

that is, $\{\tau_n\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists a $\tau \geq 0$ such that

$$\tau_n \longrightarrow \tau \quad \text{as} \quad n \longrightarrow \infty.$$

Taking $n \longrightarrow \infty$ in (3.7), using the continuities of θ and ψ, we have

$$\psi(\tau) \leq \theta(\tau)\ \psi(\tau) < \psi(\tau), \quad \text{(since} \ \ \theta(\tau) < 1),$$

which is a contradiction unless $\tau = 0$. Thus we have

$$\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} d(gx_n,\ gx_{n+1}) = 0. \tag{3.8}$$

Next we show that $\{gx_n\}$ is a Cauchy sequence. If $\{gx_n\}$ is not a Cauchy sequence, then following Lemma 11, there exists $\epsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that for all positive integers k, $n(k) > m(k) > k$ and

$$\lim_{k\to\infty} d(gx_{m(k)},\ gx_{n(k)}) = \epsilon, \tag{3.9}$$

$$\lim_{k\to\infty} d(gx_{m(k)},\ gx_{n(k)+1}) = \epsilon, \tag{3.10}$$

$$\lim_{k\to\infty} d(gx_{n(k)},\ gx_{m(k)+1}) = \epsilon, \tag{3.11}$$

and

$$\lim_{k\to\infty} d(gx_{m(k)+1},\ gx_{n(k)+1}) = \epsilon. \tag{3.12}$$

For each positive integer k, $gx_{m(k)}$ and $gx_{n(k)}$ are comparable. Then putting $\alpha = \alpha_0$, $x = x_{m(k)}$ and $y = x_{n(k)}$ in (vi) and using the monotone property of ψ, we have

$$\begin{aligned}
\psi(d(gx_{m(k)+1}, gx_{n(k)+1})) &\leq \psi(\delta(T_{\alpha_0}x_{m(k)},\ T_{\alpha_0}x_{n(k)})) \\
&\leq \theta(d(gx_{m(k)}, gx_{n(k)}))\ \max\ \{\psi(d(gx_{m(k)}, gx_{n(k)})), \psi(D(gx_{m(k)}, T_{\alpha_0}x_{m(k)})), \\
&\qquad \psi(D(gx_{n(k)}, T_{\alpha_0}x_{n(k)})), \\
&\qquad \sqrt{\psi(D(gx_{m(k)}, T_{\alpha_0}x_{n(k)})).\ \psi(D(gx_{n(k)}, T_{\alpha_0}x_{m(k)}))}\ \} \\
&\quad + L\ \min\ \{\psi(D(gx_{m(k)}, T_{\alpha_0}x_{m(k)})), \psi(D(gx_{n(k)}, T_{\alpha_0}x_{n(k)})), \\
&\qquad \psi(D(gx_{m(k)}, T_{\alpha_0}x_{n(k)})), \psi(D(gx_{n(k)}, T_{\alpha_0}x_{m(k)}))\} \\
&\leq \theta(d(gx_{m(k)}, gx_{n(k)}))\ \max\ \{\psi(d(gx_{m(k)}, gx_{n(k)})), \psi(d(gx_{m(k)}, gx_{m(k)+1})), \\
&\qquad \psi(d(gx_{n(k)}, gx_{n(k)+1})), \\
&\qquad \sqrt{\psi(d(gx_{m(k)}, gx_{n(k)+1})).\ \psi(d(gx_{n(k)}, gx_{m(k)+1}))}\ \} \\
&\quad + L\ \min\ \{\psi(d(gx_{m(k)},\ gx_{m(k)+1})),\ \psi(d(gx_{n(k)},\ gx_{n(k)+1})), \\
&\qquad \psi(d(gx_{m(k)},\ gx_{n(k)+1})),\ \psi(d(gx_{n(k)},\ gx_{m(k)+1}))\}.
\end{aligned}$$

Letting $k \longrightarrow \infty$ in the above inequality, using (3.8), (3.9), (3.10), (3.11) and (3.12) and using the properties of θ and ψ, we have

$$\psi(\epsilon) \leq \theta(\epsilon)\ \psi(\epsilon) < \psi(\epsilon),\ \text{(since}\ \ \theta(\epsilon) < 1),$$

which is a contradiction. Hence $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since X is complete and $g(X)$ is closed in X, there exists $u \in g(X)$ such that

$$gx_n \longrightarrow u \quad \text{as} \quad n \longrightarrow \infty.$$

Since $u \in g(X)$, there exists $z \in X$ such that $u = gz$. Then

$$gx_n \longrightarrow u = gz \quad \text{as} \quad n \longrightarrow \infty. \tag{3.13}$$

Since $\{\tau_n\}$ is monotone decreasing, from (3.7), we have

$$\psi(\tau_{n+1}) \leq \psi(\delta(T_{\alpha_0}x_n,\ T_{\alpha_0}x_{n+1})) \leq \theta(\tau_n)\psi(\tau_n).$$

As $\theta(\tau_n) < 1$, it follows that

$$\psi(\tau_{n+1}) \leq \psi(\delta(T_{\alpha_0}x_n,\ T_{\alpha_0}x_{n+1})) < \psi(\tau_n),$$

which, by the monotone property of ψ, implies that

$$\tau_{n+1} \leq \delta(T_{\alpha_0}x_n,\ T_{\alpha_0}x_{n+1}) < \tau_n.$$

Taking $n \to \infty$ in the above inequality, and using (3.8), we have

$$\lim_{n\to\infty} \delta(T_{\alpha_0}x_{n+1},\ T_{\alpha_0}x_n) = 0. \tag{3.14}$$

Now,

$$\delta(T_{\alpha_0}x_n,\ \{u\}) \leq \delta(T_{\alpha_0}x_n,\ gx_n) + \delta(gx_n,\ \{u\}) \leq \delta(T_{\alpha_0}x_n,\ T_{\alpha_0}x_{n-1}) + d(gx_n,\ u).$$

Taking $n \to \infty$ in the above inequality, and using (3.13) and (3.14), we have

$$\lim_{n\to\infty} \delta(T_{\alpha_0}x_n,\ \{u\}) = 0,$$

which, by Lemma 3, implies that

$$T_{\alpha_0}x_n \longrightarrow \{u\} \quad \text{as} \quad n \longrightarrow \infty. \tag{3.15}$$

Since the pair (g, T_{α_0}) is δ - compatible, from (3.13) and (3.15), we have

$$\lim_{n\to\infty} \delta(T_{\alpha_0}gx_n,\ gT_{\alpha_0}x_n) = 0.$$

As g and T_{α_0} are continuous, it follows by Lemma 5 that $\delta(T_{\alpha_0}u,\ gu) = 0$, that is, $T_{\alpha_0}u = \{gu\}$. Hence $u \in g(X) \subseteq X$ is a coincidence point of g and T_{α_0}. By what we have already proved, u is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$. □

In our next theorem we relax the continuity assumption on T_{α_0} and g by imposing an order condition. We also relax the δ - compatibility assumption of the pairs $(g,\ T_{\alpha_0})$ and the condition that $gT_{\alpha_0}x \in B(X)$, for every $x \in X$.

Theorem 13. *Let $\theta : [0,\ \infty) \longrightarrow [0,\ 1)$ be a continuous function and ψ be an altering distance function. Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that $(X,\ d)$ is a complete metric space. Assume that if $x_n \longrightarrow x$ is a nondecreasing sequence in X, then $x_n \preceq x$, for all n. Let $\{T_\alpha : X \longrightarrow B(X) : \alpha \in \Lambda\}$ be a family of multivalued mappings. Let $g : X \longrightarrow X$ be a mapping such that $g(X)$ is closed in X. Suppose that there exists $\alpha_0 \in \Lambda$ such that*

(i) $T_{\alpha_0}x \subseteq g(X)$, *for every* $x \in X$,

(ii) there exists $x_0 \in X$ *such that* $\{gx_0\} \prec_1 T_{\alpha_0}x_0$,

(iii) for $x,\ y \in X$, $gx \preceq gy$ *implies* $T_{\alpha_0}x \prec_1 T_{\alpha_0}y$,

(iv) $\psi(\delta(T_{\alpha_0}x,\ T_\alpha y))$

$$\le \theta(d(gx,gy))\ max\ \{\psi(d(gx,gy)),\psi(D(gx,T_{\alpha_0}x)),\psi(D(gy,T_\alpha y)),$$
$$\sqrt{\psi(D(gx,T_\alpha y))\ .\ \psi(D(gy,T_{\alpha_0}x))}\ \}$$
$$+L\ min\,\{\psi(D(gx,\ T_{\alpha_0}x)),\ \psi(D(gy,\ T_\alpha y)),\ \psi(D(gx,\ T_\alpha y)),\ \psi(D(gy,\ T_{\alpha_0}x))\},$$

where $x,\ y \in X$ *such that* gx *and* gy *are comparable and* $L \ge 0$.

Then g *and* $\{T_\alpha : \alpha \in \Lambda\}$ *have a coincidence point.*

Proof. We take the same sequence $\{gx_n\}$ as in the proof of Theorem 12. Then we have $gx_{n+1} \in T_{\alpha_0}x_n$, for all $n \ge 0$, $\{gx_n\}$ is monotonic nondecreasing and $gx_n \longrightarrow gz$ as $n \longrightarrow \infty$. Then by the order condition of the metric space, we have $gx_n \preceq gz$, for all n. Using the monotone property of ψ and the condition (iv), we have

$$\begin{aligned}\psi(\delta(gx_{n+1},\ T_\alpha z)) &\le \psi(\delta(T_{\alpha_0}x_n,\ T_\alpha z))\\ &\le \theta(d(gx_n,gz))\ \max\ \{\psi(d(gx_n,gz)),\psi(D(gx_n,T_{\alpha_0}x_n)),\psi(D(gz,T_\alpha z)),\\ &\qquad \sqrt{\psi(D(gx_n,T_\alpha z)).\ \psi(D(gz,T_{\alpha_0}x_n))}\ \}\\ &+ L\ \min\ \{\psi(D(gx_n,\ T_{\alpha_0}x_n)),\ \psi(D(gz,\ T_\alpha z)),\ \psi(D(gx_n,\ T_\alpha z)),\ \psi(D(gz,\ T_{\alpha_0}x_n))\}\\ &\le \theta(d(gx_n,gz))\ \max\ \{\psi(d(gx_n,gz)),\psi(d(gx_n,gx_{n+1})),\psi(D(gz,T_\alpha z)),\\ &\qquad \sqrt{\psi(D(gx_n,T_\alpha z)).\ \psi(d(gz,gx_{n+1}))}\ \}\\ &+ L\ \min\ \{\psi(d(gx_n,\ gx_{n+1})),\ \psi(D(gz,\ T_\alpha z)),\ \psi(D(gx_n,\ T_\alpha z)),\ \psi(d(gz,\ gx_{n+1}))\}.\end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality and using the properties of θ and ψ, we have

$$\psi(\delta(gz,T_\alpha z)) \le \theta(0)\psi(D(gz,T_\alpha z)) \le \theta(0)\psi(\delta(gz,T_\alpha z)) < \psi(\delta(gz,T_\alpha z))\ (\text{since}\ \theta(0) < 1),$$

which implies that $\delta(gz,\ T_\alpha z) = 0$, that is, $\{gz\} = T_\alpha z$, for all $\alpha \in \Lambda$. Hence z is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$. □

Considering $\{T_\alpha : X \longrightarrow B(X) : \alpha \in \Lambda\} = \{T\}$ in Theorem 12, we have the following corollary.

Corollary 14. *Let $\theta : [0,\ \infty) \longrightarrow [0,\ 1)$ be a continuous function and ψ be an altering distance function. Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a* *metric d on X such that $(X,\ d)$ is a complete metric space. Let $T : X \longrightarrow B(X)$ be a multivalued mapping and $g : X \longrightarrow X$ a mapping such that*

(i) T and g are continuous,

(ii) $Tx \subseteq g(X)$ and $gTx \in B(X)$, for every $x \in X$, and $g(X)$ is closed in X,

(iii) there exists $x_0 \in X$ such that $\{gx_0\} \prec_1 Tx_0$,

(iv) for $x,\ y \in X$, $gx \preceq gy$ implies $Tx \prec_1 Ty$,

(v) the pair $(g,\ T)$ is δ - compatible,

(vi) $\psi(\delta(Tx,Ty))$

$$\leq \theta(d(gx,gy))\ max\ \{\psi(d(gx,gy)),\psi(D(gx,Tx)),\psi(D(gy,Ty)),$$

$$\sqrt{\psi(D(gx,Ty))\ .\ \psi(D(gy,Tx))}\ \}$$

$$+\ L\ min\ \{\psi(D(gx,\ Tx)),\ \psi(D(gy,\ Ty)),\ \psi(D(gx,\ Ty)),\ \psi(D(gy,\ Tx))\},$$

where $x,\ y \in X$ such that gx and gy are comparable and $L \geq 0$.

Then g and T have a coincidence point.

Considering $\{T_\alpha : X \longrightarrow B(X) : \alpha \in \Lambda\} = \{T\}$ in Theorem 13, we have the following corollary.

Corollary 15. *Let $\theta : [0,\ \infty) \longrightarrow [0,\ 1)$ be a continuous function and ψ be an altering distance function. Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that $(X,\ d)$ is a complete metric space. Assume that if $x_n \longrightarrow x$ is a nondecreasing sequence in X, then $x_n \preceq x$, for all n. Let $T : X \longrightarrow B(X)$ be a multivalued mapping and $g : X \longrightarrow X$ a mapping such that*

(i) $Tx \subseteq g(X)$, for every $x \in X$, and $g(X)$ is closed in X,

(ii) there exists $x_0 \in X$ such that $\{gx_0\} \prec_1 Tx_0$,

(iii) for $x,\ y \in X$, $gx \preceq gy$ implies $Tx \prec_1 Ty$,

(iv) $\psi(\delta(Tx,Ty))$

$$\leq \theta(d(gx,gy))\ max\ \{\psi(d(gx,gy)),\psi(D(gx,Tx)),\psi(D(gy,Ty)),$$

$$\sqrt{\psi(D(gx,Ty))\ .\ \psi(D(gy,Tx))}\ \}$$

$$+\ L\ min\ \{\psi(D(gx,\ Tx)),\ \psi(D(gy,\ Ty)),\ \psi(D(gx,\ Ty)),\ \psi(D(gy,\ Tx))\},$$

where $x,\ y \in X$ such that gx and gy are comparable and $L \geq 0$.

Then g and T have a coincidence point.

The following theorems are single valued cases of the Theorems 12 and 13 respectively. Here we treat T as a multivalued mapping in which case Tx is a singleton set for every $x \in X$.

Theorem 16. *Let $\theta : [0,\ \infty) \longrightarrow [0,\ 1)$ be a continuous function and ψ be an altering distance function. Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that $(X,\ d)$ is a complete metric space. Let $\{T_\alpha : X \longrightarrow X : \alpha \in \Lambda\}$ be a family of mappings. Let $g : X \longrightarrow X$ be a mapping such that $g(X)$ is closed in X. Suppose that there exists $\alpha_0 \in \Lambda$ such that*

(i) T_{α_0} and g are continuous,

(ii) $T_{\alpha_0}(X) \subseteq g(X)$,

(iii) there exists $x_0 \in X$ such that $gx_0 \preceq T_{\alpha_0} x_0$,

(iv) for $x,\ y \in X$, $gx \preceq gy$ implies $T_{\alpha_0} x \preceq T_{\alpha_0} y$,

(v) the pair $(g,\ T_{\alpha_0})$ is compatible,

(vi) $\psi(d(T_{\alpha_0}x, T_\alpha y))$

$$\leq \theta(d(gx, gy))\ max\ \{\psi(d(gx, gy)), \psi(d(gx, T_{\alpha_0}x)), \psi(d(gy, T_\alpha y)), \sqrt{\psi(d(gx, T_\alpha y))\ .\ \psi(d(gy, T_{\alpha_0}x))}\ \}$$

$$+\ L\ min\ \{\psi(d(gx,\ T_{\alpha_0}x)),\ \psi(d(gy,\ T_\alpha y)),\ \psi(d(gx,\ T_\alpha y)),\ \psi(d(gy,\ T_{\alpha_0}x))\},$$

where $x,\ y \in X$ such that gx and gy are comparable and $L \geq 0$.

Then g and $\{T_\alpha : \alpha \in \Lambda\}$ have a coincidence point.

Theorem 17. *Let $\theta : [0,\ \infty) \longrightarrow [0,\ 1)$ be a continuous function and ψ be an altering distance function. Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that $(X,\ d)$ is a complete metric space. Assume that if $x_n \longrightarrow x$ is a nondecreasing sequence in X, then $x_n \preceq x$, for all n. Let $\{T_\alpha : X \longrightarrow X : \alpha \in \Lambda\}$ be a family of mappings. Let $g : X \longrightarrow X$ be a mapping such that $g(X)$ is closed in X. Suppose that there exists $\alpha_0 \in \Lambda$ such that*

(i) $T_{\alpha_0}(X) \subseteq g(X)$,

(ii) there exists $x_0 \in X$ such that $gx_0 \preceq T_{\alpha_0} x_0$,

(iii) for $x,\ y \in X$, $gx \preceq gy$ implies $T_{\alpha_0} x \preceq T_{\alpha_0} y$,

(iv) $\psi(d(T_{\alpha_0}x, T_\alpha y))$

$$\leq \theta(d(gx, gy))\ max\ \{\psi(d(gx, gy)), \psi(d(gx, T_{\alpha_0}x)), \psi(d(gy, T_\alpha y)), \sqrt{\psi(d(gx, T_\alpha y))\ .\ \psi(d(gy, T_{\alpha_0}x))}\ \}$$

$$+ \; L \; min \; \{\psi(d(gx, \; T_{\alpha_0}x)), \; \psi(d(gy, \; T_\alpha y)), \; \psi(d(gx, \; T_\alpha y)), \; \psi(d(gy, \; T_{\alpha_0}x))\},$$

where $x, \; y \in X$ such that gx and gy are comparable and $L \geq 0$.

Then g and $\{T_\alpha : \alpha \in \Lambda\}$ have a coincidence point.

Corollary 18. *Let $p, \; q, \; r, \; s$ be four continuous functions from $[0, \; \infty)$ into $[0, \; 1)$ which satisfy the property $p(t) + q(t) + r(t) + s(t) < 1$, for all $t \in [0, \; \infty)$ and ψ be an altering distance function. Let $(X, \; \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that $(X, \; d)$ is a complete metric space. Let $T : X \longrightarrow X$ and $g : X \longrightarrow X$ be two mappings such that*

(i) T and g are continuous,

(ii) $T(X) \subseteq g(X)$ and $g(X)$ is closed in X,

(iii) there exists $x_0 \in X$ such that $gx_0 \preceq Tx_0$,

(iv) for $x, \; y \in X$, $gx \preceq gy$ implies $Tx \preceq Ty$,

(v) the pair $(g, \; T)$ is compatible,

(vi) $\psi(d(Tx, Ty))$

$$\leq p(d(gx, gy))\psi(d(gx, gy)) + q(d(gx, gy))\psi(d(gx, Tx)) + r(d(gx, gy))\psi(d(gy, Ty))$$

$$+ s(d(gx, gy))\sqrt{\psi(d(gx, Ty)) \; . \; \psi(d(gy, Tx))},$$

where $x, \; y \in X$ such that gx and gy are comparable.

Then g and T have a coincidence point.

Proof. Starting with the inequality (vi), we have

$$\begin{aligned}\psi(d(Tx, Ty)) &\leq p(d(gx, gy))\psi(d(gx, gy)) + q(d(gx, gy))\psi(d(gx, Tx)) \\ &\quad + r(d(gx, gy))\psi(d(gy, Ty)) + s(d(gx, gy))\sqrt{\psi(d(gx, Ty)) \; . \; \psi(d(gy, Tx))}, \\ &\leq \theta(d(gx, gy)) \max\{\psi(d(gx, gy)), \psi(d(gx, Tx)), \psi(d(gy, Ty)) \\ &\qquad \sqrt{\psi(d(gx, Ty)) \; . \; \psi(d(gy, Tx))} \; \},\end{aligned}$$

where $\theta(d(gx, gy)) = p(d(gx, gy)) + q(d(gx, gy)) + r(d(gx, gy)) + s(d(gx, gy))$, which is a special case of the inequality (vi) of Theorem 16 obtained by considering $\{T_\alpha : X \longrightarrow X : \alpha \in \Lambda\} = \{T\}$ and $L = 0$. □

Corollary 19. *Let $p, \; q, \; r, \; s$ be four continuous functions from $[0, \; \infty)$ into $[0, \; 1)$ which satisfy the property $p(t) + q(t) + r(t) + s(t) < 1$, for all $t \in [0, \; \infty)$ and ψ be an altering distance function. Let $(X, \; \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that $(X, \; d)$ is a complete metric space. Assume that if $x_n \longrightarrow x$ is a nondecreasing sequence in X, then $x_n \preceq x$, for all n. Let $T : X \longrightarrow X$ and $g : X \longrightarrow X$ be two mappings such that* *(i)* $T(X) \subseteq g(X)$ *and* $g(X)$ *is closed in* X,

(ii) there exists $x_0 \in X$ *such that* $gx_0 \preceq Tx_0$,

(iii) for $x,\ y \in X$, $gx \preceq gy$ *implies* $Tx \preceq Ty$,

(iv) $\psi(d(Tx, Ty))$

$$\leq p(d(gx, gy))\psi(d(gx, gy)) + q(d(gx, gy))\psi(d(gx, Tx)) + r(d(gx, gy))\psi(d(gy, Ty))$$
$$+s(d(gx, gy))\sqrt{\psi(d(gx, Ty))\ .\ \psi(d(gy, Tx))},$$

where $x,\ y \in X$ *such that* gx *and* gy *are comparable.*

Then g *and* T *have a coincidence point.*

Proof. Like the proof of the Corollary 18, we can show that the inequality (iv) is a special case of the inequality (iv) of Theorem 17 obtained by considering $\{T_\alpha : X \longrightarrow X : \alpha \in \Lambda\} = \{T\}$ and $L = 0$. □

Example 20. Let $X = [1,\ \infty)$ with usual order $\preceq$ be a partially ordered set.
Let $d : X \times X \longrightarrow \mathbb{R}$ be given as

$$d(x,\ y) = |x - y|, \text{ for } \ x,\ y \in X.$$

Then $(X,\ d)$ is a complete metric space with the required properties mentioned in Theorems 12 and 13.
Let $g : X \to X$ be defined as follows:

$$gx = x^2, \text{ for } \ x \in X.$$

Then g has the properties mentioned in Theorems 12 and 13.
Let $\Lambda = \{1,\ 2,\ 3,\ ...\}$. Let the family of mappings $\{T_\alpha : X \to B(X) : \alpha \in \Lambda\}$ be defined as follows:

$$T_1x = \{1\}, \ \text{ for } \ x \in X \quad \text{and for } \alpha \geq 2, \ \ T_\alpha x = \begin{cases} \{1\}, & \text{if } 1 \leq x \leq 4, \\ \{1, \dfrac{2\ \alpha}{\alpha + 1}\}, & \text{if } x > 4. \end{cases}$$

For any sequence $\{x_n\}$ in X, $T_1x_n \to \{t\}$, $gx_n \to t$, for some t in X implies $t = 1$. Then clearly, the pair $(g,\ T_1)$ is δ - compatible. Also, g and T_1 satisfy required conditions mentioned in Theorems 12 and 13.
Let $\psi : [0,\ \infty) \longrightarrow [0,\ \infty)$ be defined as follows:

$$\psi(t) = t^2, \text{ for } \ t \in [0,\ \infty).$$

Then ψ has the properties mentioned in Theorems 12 and 13.
Let $\theta : [0,\ \infty) \longrightarrow [0,\ 1)$ be defined as follows:

$$\theta(t) = \frac{1}{2}, \text{ for all } \ t \in [0,\ \infty).$$

Then θ satisfies the required properties mentioned in Theorems 12 and 13.
The condition (vi) of Theorem 12 and the condition (iv) of Theorem 13 are satisfied for any $L \geq 0$. Hence all the condition of Theorems 12 and 13 are satisfied and it is seen that 1 is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$.

Note In the above example if one takes $g : X \to X$ to be function as follows:

$$gx = \begin{cases} \dfrac{x}{2}, & \text{if } 1 \leq x \leq 4, \\ 200, & \text{if } x > 4. \end{cases}$$

Then the above example is still applicable to Theorem 13 but not applicable to Theorem 12 because g is not continuous and hence does not satisfy required properties mentioned in Theorem 12.

Remark 21. Theorems 16 and 17 are generalizations of ordered versions of theorem 3.1 in [8] which generalizes the Banach contraction principle [3], theorem 2 of Khan et al [21], the theorem of Skof [32], and the theorem of Kannan [20]. Also, Theorems 16 and 17 generalize the ordered versions of the main result of Berinde [5].

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New inequalities of Hermite-Hadamard type for functions whose derivatives in absolute value are convex with applications

Muhammad Amer Latif
ICollege of Science, Department of Mathematics, University of Hail, Hail 2440, Saudi Arabia
m_amer_latif@hotmail.com

Sever S. Dragomir
School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia
sever.dragomir@vu.edu.au

Abstract
In this paper new Hadamard-type inequalities, which estimate the difference between $\frac{1}{b-a}\int_a^b f(x)dx$ and $\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+4b}{4}\right)}{2}$, are established for functions whose derivatives in absolute values are convex. Our established results refine those results which have been established to estimate the difference between the middle and the leftmost terms of the celebrated Hermite-Hadamard inequality. We also give some applications of our obtained results to get some error bounds for the general quadrature formula. Finally, some applications to special means of real numbers are given as well.

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1 Introduction

The following definition for convex functions is well known in the mathematical literature:

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard's inequality, due to its rich geometrical significance and applications, which is stated as follow:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both the inequalities hold in reversed direction if f is concave. Since its discovery in 1883, Hermite-Hadamard's inequality [4] has been considered the most useful inequality in mathematical analysis. Some of the classical inequalities for means can be derived from (1.1) for particular choices of the function f.

In [3], S. S. Dragomir and R. P. Agarwal obtained the following results which give estimate between the middle and the rightmost terms in (1.1):

Theorem 1. *[3] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \left(\frac{b-a}{8}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{1.2}$$

and

Theorem 2. *[3] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^p$ is convex on $[a,b]$ for some fixed $p > 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{\left|f'(a)\right|^{\frac{p}{p-1}}+\left|f'(b)\right|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{1.3}$$

In [11], C. E. M. Pearce and J. E. Pečarić gave an improvement and simplification of the constant in Theorem 2 and consolidated this result with Theorem 1 as the following Theorem:

Theorem 3. *[11] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q \ge 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{4}\left[\frac{\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{2}\right]^{\frac{1}{q}}. \tag{1.4}$$

In [11], C. E. M. Pearce and J. E. Pečarić also established the following result which gives the estimate between the middle and the leftmost terms in (1.1):

Theorem 4. *[11] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q \ge 1$, then the following inequality holds:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{4}\left[\frac{\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{2}\right]^{\frac{1}{q}}. \tag{1.5}$$

In [7, 8], U. S. Kirmaci et al. proved the following results connected with the left part of (1.1):

Theorem 5. *[8] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \left(\frac{b-a}{8}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{1.6}$$

Theorem 6. *[7] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^p$ is convex on $[a,b]$ for some fixed $p > 1$, then the following inequality holds:*

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f\left(\frac{a+b}{2}\right) \right| \leq \left(\frac{3^{1-\frac{1}{p}}}{8} \right) (b-a) \left[|f'(a)| + |f'(b)| \right]. \tag{1.7}$$

Theorem 7. *[7] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^p$ is concave on $[a,b]$ for $p \geq 1$ and $|f'|$ is a linear map, then the following inequality holds:*

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f\left(\frac{a+b}{2}\right) \right| \leq \left(\frac{b-a}{8} \right) \left| f'(a+b) \right|. \tag{1.8}$$

For more results on Hermite-Hadamard-type inequality providing new proofs, noteworthy extensions, generalizations and numerous applications, see [1]-[16] and the references therein.

In a recent paper [14], K. L. Tseng et al., established the following result which gives a refinement of (1.1):

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} \leq \frac{1}{b-a} \int_a^b f(x)dx \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] \leq \frac{f(a)+f(b)}{2}, \end{aligned} \tag{1.9}$$

where $f : [a,b] \to \mathbb{R}$, is a convex function (see [12, Remark 2.11, page7.]).

The main aim of this paper is to establish some new Hermite-Hadamard type inequalities which give an estimate between $\frac{1}{b-a}\int_a^b f(x)dx$ and $\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}$ for functions whose derivatives in absolute value are convex and as a consequence we will get refinements of those results which have been established to estimate the difference between the middle and the leftmost terms in (1.1).

In Section 3, we will propose some new error bounds for the general quadrature formula based on our established results. Applications of our results to special means are also given in Section 4.

2 Main Results

To prove our results we need the following lemma:

Lemma 8. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° , the interior of I,* *where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx$$
$$=\frac{b-a}{16}\left[\int_0^1 tf'\left(t\frac{3a+b}{4}+(1-t)a\right)dt\right.$$
$$+\int_0^1 (t-1)f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)dt$$
$$+\int_0^1 tf'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)dt$$
$$\left.+\int_0^1 (t-1)f'\left(tb+(1-t)\frac{a+3b}{4}\right)dt\right]. \quad (2.1)$$

Proof. By integration by parts and by making use of the substitution $x = t\frac{3a+b}{4} + (1-t)a$, we have

$$\frac{b-a}{16}\int_0^1 tf'\left(t\frac{3a+b}{4}+(1-t)a\right)dt$$
$$=\frac{b-a}{16}\left[\left.\frac{4tf'\left(t\frac{3a+b}{4}+(1-t)a\right)}{b-a}\right|_0^1-\frac{4}{b-a}\int_0^1 f\left(t\frac{3a+b}{4}+(1-t)a\right)dt\right]$$
$$=\frac{1}{4}f\left(\frac{3a+b}{4}\right)-\frac{1}{b-a}\int_a^{\frac{3a+b}{4}} f(x)\,dx. \quad (2.2)$$

Analogously, we also have the following equalities:

$$\frac{b-a}{16}\int_0^1 (t-1)f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)dt$$
$$=\frac{1}{4}f\left(\frac{3a+b}{4}\right)-\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)\,dx, \quad (2.3)$$

$$\frac{b-a}{16}\int_0^1 tf'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)dt$$
$$=\frac{1}{4}f\left(\frac{a+3b}{4}\right)-\frac{1}{b-a}\int_{\frac{x+b}{2}}^{\frac{x+3b}{4}} f(x)\,dx \quad (2.4)$$

and

$$\frac{b-a}{16}\int_0^1 (t-1)f'\left(tb+(1-t)\frac{a+3b}{4}\right)dt$$
$$=\frac{1}{4}f\left(\frac{a+3b}{4}\right)-\frac{1}{b-a}\int_{\frac{x+3b}{4}}^{b} f(x)\,dx. \quad (2.5)$$

Adding (2.2)-(2.5), we get the desired equality. This completes the proof of the lemma. □

Using the Lemma 1 the following results can be obtained:

Theorem 9. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \left(\frac{b-a}{96}\right)\left[\left|f'(a)\right|+4\left|f'\left(\frac{3a+b}{4}\right)\right|\right.$$
$$\left.+2\left|f'\left(\frac{a+b}{2}\right)\right|+4\left|f'\left(\frac{a+3b}{4}\right)\right|+\left|f'(b)\right|\right]. \quad (2.6)$$

Proof. Using Lemma 1 and taking the modulus, we have

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{16}\left[\int_0^1 t\left|f'\left(t\frac{3a+b}{4}+(1-t)\,a\right)\right|dt\right.$$
$$+\int_0^1 (1-t)\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|dt$$
$$+\int_0^1 t\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|dt$$
$$\left.+\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|dt\right]. \quad (2.7)$$

Using the convexity of $|f'|$ on $[a,b]$, we observe that the following inequality holds:

$$\int_0^1 t\left|f'\left(t\frac{3a+b}{4}+(1-t)\,a\right)\right|dt$$
$$\leq \left|f'\left(\frac{3a+b}{4}\right)\right|\int_0^1 t^2 dt+\left|f'(a)\right|\int_0^1 t(1-t)\,dt$$
$$=\frac{1}{3}\left|f'\left(\frac{3a+b}{4}\right)\right|+\frac{1}{6}\left|f'(a)\right|. \quad (2.8)$$

Similarly, we also have that the following inequalities hold:

$$\int_0^1 (1-t)\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|dt \leq \frac{1}{6}\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{1}{3}\left|f'\left(\frac{3a+b}{4}\right)\right|, \quad (2.9)$$

$$\int_0^1 t\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|dt \leq \frac{1}{3}\left|f'\left(\frac{a+3b}{4}\right)\right|+\frac{1}{6}\left|f'\left(\frac{a+b}{2}\right)\right|, \quad (2.10)$$

and

$$\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|dt \leq \frac{1}{6}\left|f'(b)\right|+\frac{1}{3}\left|f'\left(\frac{a+3b}{4}\right)\right|. \quad (2.11)$$

Utilizing the inequalities (2.8)-(2.11), we get (2.6).
This completes the proof of the theorem. □

Corollary 10. *Suppose all the conditions of Theorem 9 are satisfied. Then*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \left(\frac{b-a}{16}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{2.12}$$

Moreover, if $|f'(x)| \le M$, for all $x \in [a,b]$, then we have also the following inequality:

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \left(\frac{b-a}{8}\right)M. \tag{2.13}$$

Proof. It follows from Theorem 9 and using the convexity of $|f'|$. □

Theorem 11. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q > 1$, then the following inequality holds:*

$$\begin{aligned}\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| &\le \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{b-a}{16}\right)\\ &\times\left\{\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\},\end{aligned} \tag{2.14}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and using the well-known Hölder integral inequality, we have

$$\begin{aligned}&\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\le \frac{b-a}{16}\left[\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\ &+\left(\int_0^1 (1-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\\ &+\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\\ &\left.+\left(\int_0^1 (1-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\right].\end{aligned} \tag{2.15}$$

Since $|f'|^q$ is convex on $[a,b]$, we have

$$\begin{aligned}&\int_0^1\left|f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt\\ &\le \left|f'\left(\frac{3a+b}{4}\right)\right|^q\int_0^1 t\,dt+\left|f'(a)\right|^q\int_0^1(1-t)\,dt\\ &=\frac{1}{2}\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\frac{1}{2}\left|f'(a)\right|^q.\end{aligned}$$

Similarly,

$$\int_0^1 \left| f'\left(t\frac{a+b}{2} + (1-t)\frac{3a+b}{4} \right) \right|^q dt \le \frac{1}{2}\left| f'\left(\frac{a+b}{2} \right) \right|^q + \frac{1}{2}\left| f'\left(\frac{3a+b}{4} \right) \right|^q,$$

$$\int_0^1 \left| f'\left(t\frac{a+3b}{4} + (1-t)\frac{a+b}{2} \right) \right|^q dt \le \frac{1}{2}\left| f'\left(\frac{a+3b}{4} \right) \right|^q + \frac{1}{2}\left| f'\left(\frac{a+b}{2} \right) \right|^q$$

and

$$\int_0^1 \left| f'\left(tb + (1-t)\frac{a+3b}{4} \right) \right|^q dt \le \frac{1}{2}\left| f'\left(\frac{a+3b}{4} \right) \right|^q + \frac{1}{2}\left| f'(b) \right|^q.$$

Using the last four inequalities in (2.15), we get the inequality (2.14), which completes the proof of the theorem. □

Corollary 12. *Suppose all the conditions of Theorem 11 are satisfied. Then*

$$\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right|$$
$$\le \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{1}{2} \right)^{\frac{3}{q}} \left[1 + 3^{\frac{1}{q}} + 5^{\frac{1}{q}} + 7^{\frac{1}{q}} \right] \left(\frac{b-a}{16} \right) \left[|f'(a)| + |f'(b)| \right]. \quad (2.16)$$

Proof. It follows from Theorem 11 using the convexity of $|f'|^q$ and the fact

$$\sum_{k=1}^{n} (u_k + v_k)^s \le \sum_{k=1}^{n} (u_k)^s + \sum_{k=1}^{n} (v_k)^s, u_k, v_k \ge 0, 1 \le k \le n, 0 \le s < 1.$$

□

Theorem 13. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q \ge 1$, then the following inequality holds:*

$$\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}}\left(\frac{b-a}{16}\right)$$
$$\times \left\{ \left(|f'(a)|^q + 2\left| f'\left(\frac{3a+b}{4}\right) \right|^q \right)^{\frac{1}{q}} + \left(\left| f'\left(\frac{a+b}{2}\right) \right|^q + 2\left| f'\left(\frac{3a+b}{4}\right) \right|^q \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\left| f'\left(\frac{a+b}{2}\right) \right|^q + 2\left| f'\left(\frac{a+3b}{4}\right) \right|^q \right)^{\frac{1}{q}} + \left(2\left| f'\left(\frac{a+3b}{4}\right) \right|^q + |f'(b)|^q \right)^{\frac{1}{q}} \right\}. \quad (2.17)$$

Proof. Suppose that $q \ge 1$. From Lemma 1 and using the well-known power-mean inequality, we have

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{16}\left[\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^1 (1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 (1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}. \quad (2.18)$$

Since $|f'|^q$ is convex on $[a,b]$, we have

$$\int_0^1 t\left|f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt$$
$$\leq \left|f'\left(\frac{3a+b}{4}\right)\right|^q\int_0^1 t^2dt+|f'(a)|^q\int_0^1 t(1-t)\,dt$$
$$=\frac{1}{3}\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\frac{1}{6}|f'(a)|^q.$$

Analogously, we also have that the following inequalities:

$$\int_0^1 (1-t)\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt$$
$$\leq \frac{1}{6}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{3}\left|f'\left(\frac{3a+b}{4}\right)\right|^q,$$

$$\int_0^1 t\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt\leq \frac{1}{3}\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\frac{1}{6}\left|f'\left(\frac{a+b}{2}\right)\right|^q$$

and

$$\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q\leq \frac{1}{3}\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\frac{1}{6}|f'(b)|^q.$$

By making use of the last four inequalities in (2.18), we get (2.17). Hence the proof of the theorem is complete. □

Corollary 14. *Suppose all the conditions of Theorem 11 are satisfied. Then using similar arguments as in Corollary 12, we get the following inequality:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\left(\frac{1}{3}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)\left[1+2^{\frac{1}{q}}+\left(\frac{1}{2}\right)^{\frac{1}{q}}+\left(\frac{5}{2}\right)^{\frac{1}{q}}\right]\left(\frac{b-a}{16}\right)\left[|f'(a)|+|f'(b)|\right]. \quad (2.19)$$

Theorem 15. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q > 1$, then the following inequality holds:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{b-a}{16}\right)\left\{\left|f'\left(\frac{7a+b}{8}\right)\right|+\left|f'\left(\frac{5a+3b}{8}\right)\right| + \left|f'\left(\frac{3a+5b}{8}\right)\right|+\left|f'\left(\frac{a+7b}{8}\right)\right|\right\}, \quad (2.20)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and using the well-known Hölder integral inequality for $q > 1$ and $p = \frac{q}{q-1}$, we have

$$\begin{aligned}\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| &\leq \left(\frac{b-a}{16}\right)\Bigg[\left(\int_0^1 t^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} \\ &+\left(\int_0^1 (1-t)^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt\right)^{\frac{1}{q}} \\ &+\left(\int_0^1 t^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \\ &+\left(\int_0^1 (1-t)^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg]. \quad (2.21)\end{aligned}$$

Since $|f'|^q$ is concave on $[a,b]$ so by using the inequality (1.1), we obtain:

$$\int_0^1\left|f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt \leq \left|f'\left(\frac{\frac{3a+b}{4}+a}{2}\right)\right|^q = \left|f'\left(\frac{7a+b}{8}\right)\right|^q$$

Analogously, we have that the following inequalities:

$$\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt \leq \left|f'\left(\frac{5a+3b}{8}\right)\right|^q,$$

$$\int_0^1\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt \leq \left|f'\left(\frac{3a+5b}{8}\right)\right|^q$$

and

$$\int_0^1\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q dt \leq \left|f'\left(\frac{a+7b}{8}\right)\right|^q.$$

Using the last four inequalities in (2.21), we get (2.20). This completes the proof of the theorem. □

Corollary 16. *Suppose all the assumptions of Theorem 15 are satisfied and assume that $|f'|$ is a linear map, then we get the following inequality:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{b-a}{8}\right)\left|f'(a+b)\right|. \tag{2.22}$$

Proof. It is a direct consequence of Theorem 15 and using the linearity of $|f'|$. □

Remark 17. Since not all the convex functions are linear map, hence the inequality (2.22) can be used when $|f'|^q$ is concave on $[a,b]$ for some fixed $q>1$ and $|f'|$ is a linear map. Moreover, it can be observed that the error bound in (2.22) is more easier to calculate as compared to calculate it in (2.20) when $|f'|$ is a linear map.

Theorem 18. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q\ge 1$, then the following inequality holds:*

$$\begin{aligned}\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| &\\ \le \left(\frac{b-a}{32}\right)\Bigg[\left|f'\left(\frac{13a+3b}{12}\right)\right|+\left|f'\left(\frac{11a+5b}{12}\right)\right| &\\ +\left|f'\left(\frac{5a+13b}{12}\right)\right|+\left|f'\left(\frac{3a+13b}{12}\right)\right|\Bigg].\end{aligned} \tag{2.23}$$

Proof. First, by the concavity of $|f'|^q$ on $[a,b]$ and the power-mean inequality, we note that

$$\begin{aligned}|f(\lambda x+(1-\lambda)y)|^q &\ge \lambda|f(x)|^q+(1-\lambda)|f(y)|^q\\ &\ge (\lambda|f(x)|+(1-\lambda)|f(y)|)^q\end{aligned}$$

and hence

$$|f(\lambda x+(1-\lambda)y)|\ge\lambda|f(x)|+(1-\lambda)|f(y)|,$$

for all $\lambda\in[0,1]$ and $x,\ y\in[a,b]$. This shows that $|f'|$ is also concave on $[a,b]$.

Accordingly, using Lemma 1 and the Jensen's integral inequality, we have

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\le\left(\frac{b-a}{16}\right)\left[\int_0^1 t\left|f'\left(t\frac{3a+b}{4}+(1-t)\,a\right)\right|dt\right.$$
$$+\int_0^1 (1-t)\left|f'\left(t\frac{a+b}{2}+(1-t)\,\frac{3a+b}{4}\right)\right|dt$$
$$\int_0^1 t\left|f'\left(t\frac{a+3b}{4}+(1-t)\,\frac{a+b}{2}\right)\right|dt$$
$$\left.+\int_0^1 (1-t)\left|f'\left(tb+(1-t)\,\frac{a+3b}{4}\right)\right|dt\right]$$
$$\le\frac{b-a}{16}\left[\left(\int_0^1 tdt\right)\left|f'\left(\frac{\int_0^1 t\left(t\frac{3a+b}{4}+(1-t)\,a\right)dt}{\int_0^1 tdt}\right)\right|\right]$$
$$+\left(\int_0^1 (1-t)\,dt\right)\left|f'\left(\frac{\int_0^1 (1-t)\left(t\frac{a+b}{2}+(1-t)\,\frac{3a+b}{4}\right)dt}{\int_0^1 (1-t)\,dt}\right)\right|$$
$$+\left(\int_0^1 tdt\right)\left|f'\left(\frac{\int_0^1 t\left(t\frac{a+3b}{4}+(1-t)\,\frac{a+b}{2}\right)dt}{\int_0^1 tdt}\right)\right|$$
$$+\left(\int_0^1 (1-t)\,dt\right)\left|f'\left(\frac{\int_0^1 (1-t)\left(tb+(1-t)\,\frac{a+3b}{4}\right)dt}{\int_0^1 (1-t)\,dt}\right)\right|,$$

which is equivalent to (2.23) and the proof of the theorem is complete. □

Corollary 19. *Suppose all the assumptions of Theorem 18 are satisfied and assume that* $|f'|$ *is a linear map, then we have the following inequality:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\left(\frac{b-a}{12}\right)\left|f'(a+b)\right|. \tag{2.24}$$

Proof. It follows from Theorem 18 and using the linearity of $|f'|$. □

Remark 20. The error bound in (2.22) is more easier to calculate as compared to calculate it in (2.20) when $|f'|^q$ is concave on $[a,b]$ for some fixed $q\ge 1$ and $|f'|$ is a linear map.

3 Application to the General Quadrature Formula

Let $d: a=x_0<x_1<...<x_{n-1}<x_n=b$ be a division of the interval $[a,b]$. Consider the general quadrature formula

$$\int_a^b f(x)dx=Q(f,d)+R(f,d), \tag{3.1}$$

where

$$Q(f,d)=\frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right](x_{i+1}-x_i)$$

and $R(f,d)$ is the associated error. Here, we derive some estimates for the error $R(f,d)$ given in (3.1).

Proposition 21. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then for every division d of $[a,b]$, we have:*

$$|R(f,d)| \leq \frac{1}{96}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[|f'(x_i)| + 4\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|\right.$$
$$\left. +2\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right| + 4\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right| + |f'(x_{i+1})|\right]. \quad (3.2)$$

Proof. By applying Theorem 9 on the subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of the division d, we have

$$\left|\frac{1}{2}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right) + f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]\right.$$
$$\left. -\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(x)\,dx\right|$$
$$\leq \left(\frac{x_{i+1}-x_i}{96}\right)\left[|f'(x_i)| + 4\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|\right.$$
$$\left. +2\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right| + 4\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right| + |f'(x_{i+1})|\right]. \quad (3.3)$$

Now

$$|R(f,d)| = \left|\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)dx\right.$$
$$\left. -\sum_{i=0}^{n-1}\frac{1}{2}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right) + f\left(\frac{x_i+3x_{i+1}}{4}\right)\right](x_{i+1}-x_i)\right|$$
$$\leq \sum_{i=0}^{n-1}(x_{i+1}-x_i)\left|\frac{1}{2}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right) + f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]\right.$$
$$\left. -\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(x)dx\right|. \quad (3.4)$$

Using (3.3) in (3.4), we get (3.2). This completes the proof of the proposition. □

Corollary 22. *Suppose all the assumptions of Proposition 21 are satisfied. Then*

$$|R(f,d)| \leq \frac{1}{16}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[|f'(x_i)| + |f'(x_{i+1})|\right]. \quad (3.5)$$

Proof. It follows from Proposition 21 and using the convexity of $|f'|$. □

Proposition 23. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q > 1$,* *then for every division d of $[a,b]$, we have*

$$\begin{aligned}
&|R(f,d)| \\
&\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{1}{q}+4} \sum_{i=0}^{n-1} (x_{i+1}-x_i)^2 \left\{ \left(\left| f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|^q + |f'(x_i)|^q \right)^{\frac{1}{q}} \right. \\
&\quad + \left(\left| f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q + \left| f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|^q \right)^{\frac{1}{q}} \\
&\quad + \left(\left| f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|^q + \left| f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q \right)^{\frac{1}{q}} \\
&\quad \left. + \left(\left| f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|^q + |f'(x_{i+1})|^q \right)^{\frac{1}{q}} \right\}, \qquad (3.6)
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. The proof is similar to that of Proposition 21 and using Theorem 11. □

Corollary 24. *Suppose all the conditions of Proposition 23 are satisfied. Then*

$$\begin{aligned}
|R(f,d)| &\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{3}{q}+4} \left[1+3^{\frac{1}{q}}+5^{\frac{1}{q}}+7^{\frac{1}{q}}\right] \\
&\quad \times \sum_{i=0}^{n-1} (x_{i+1}-x_i)^2 \left[|f'(x_i)|+|f'(x_{i+1})|\right]. \qquad (3.7)
\end{aligned}$$

Proposition 25. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q \geq 1$, then for every division d of $[a,b]$, we have*

$$\begin{aligned}
|R(f,d)| &\leq \left(\frac{1}{32}\right) \left(\frac{1}{3}\right)^{\frac{1}{q}} \sum_{i=0}^{n-1} (x_{i+1}-x_i)^2 \left\{ \left(|f'(x_i)|^q + 2\left| f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|^q \right)^{\frac{1}{q}} \right. \\
&\quad + \left(\left| f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q + 2\left| f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|^q \right)^{\frac{1}{q}} \\
&\quad + \left(\left| f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q + 2\left| f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|^q \right)^{\frac{1}{q}} \\
&\quad \left. + \left(2\left| f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|^q + |f'(x_{i+1})|^q \right)^{\frac{1}{q}} \right\}. \qquad (3.8)
\end{aligned}$$

Proof. The proof is similar to that of Proposition 21 and using Theorem 13. □

Corollary 26. *Suppose all the conditions of Proposition 25 are satisfied. Then*

$$\begin{aligned}
|R(f,d)| &\leq \left(\frac{1}{3}\right)^{\frac{1}{q}} \left(\frac{1}{32}\right) \left[1+2^{\frac{1}{q}}+\left(\frac{1}{2}\right)^{\frac{1}{q}}+\left(\frac{5}{2}\right)^{\frac{1}{q}}\right] \\
&\quad \times \sum_{i=0}^{n-1} (x_{i+1}-x_i)^2 \left[|f'(x_i)|+|f'(x_{i+1})|\right]. \qquad (3.9)
\end{aligned}$$

Proposition 27. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q > 1$, then for every division d of $[a,b]$, we have*

$$|R(f,d)| \leq \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} \left(\frac{1}{16}\right) \sum_{i=0}^{n-1} (x_{i+1}-x_i)^2 \left\{\left|f'\left(\frac{7x_i+x_{i+1}}{8}\right)\right| + \left|f'\left(\frac{5x_i+3x_{i+1}}{8}\right)\right| + \left|f'\left(\frac{3x_i+5x_{i+1}}{8}\right)\right| + \left|f'\left(\frac{x_i+7x_{i+1}}{8}\right)\right|\right\}, \quad (3.10)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The proof is similar to that of Proposition 21 and it follows from Theorem 15. □

Corollary 28. *Suppose all the conditions of Proposition 27 are satisfied. If $|f'|$ is a linear mapping, then we have the following inequality:*

$$|R(f,d)| \leq \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} \left(\frac{1}{8}\right) \sum_{i=0}^{n-1} (x_{i+1}-x_i)^2 \left|f'(x_{i+1}+x_i)\right|. \quad (3.11)$$

Remark 29. It can be observed that the error bound in (3.11) for general quadrature formula is more easier to calculate as compared to calculate it in (3.10) when $|f'|^q$ is concave on $[a,b]$ for some fixed $q > 1$ and $|f'|$ is a linear map.

Proposition 30. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q \geq 1$ and $|f'|^q$ is a linear mapping, then for every division d of $[a,b]$, then the following inequality holds:*

$$|R(f,d)| \leq \left(\frac{1}{32}\right) \sum_{i=0}^{n-1} (x_{i+1}-x_i)^2 \left|f'(x_{i+1}+x_i)\right|. \quad (3.12)$$

Proof. The proof is similar to that of Proposition 21 and it follows from Theorem 18. □

4 Applications to Special Means

Now, we consider the applications of our Theorems to the special means. We consider the means for arbitrary real numbers a, $b \in \mathbb{R}$. We take

1. The arithmetic mean:
$$A(a,b) = \frac{a+b}{2};\ a,\ b \in \mathbb{R}.$$

2. The harmonic mean:
$$H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}};\ a, b \in \mathbb{R}\setminus\{0\}.$$

3. The logarithmic mean:
$$L(a,b) = \frac{\ln|b| - \ln|a|}{b-a}; a, b \in \mathbb{R}, a \neq b, a, b \neq 0.$$

4. Generalized log-mean:

$$L_n(a,b) = \left[\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}}; \ a,\ b \in \mathbb{R},\ n \in \mathbb{Z}\backslash\{-1,0\},\ a \neq b, a, b \neq 0.$$

Now using the results of Section 2, we give some applications to special means of real numbers.

Proposition 31. *Let* $a,\ b \in \mathbb{R},\ a < b,\ 0 \notin [a,b]$ *and* $n \in \mathbb{Z},\ |n| \geq 2$. *Then*

$$\left| A\left(\left(\frac{3a+b}{4}\right)^n, \left(\frac{a+3b}{4}\right)^n\right) - L_n^n(a,b)\right| \leq |n| \left(\frac{b-a}{8}\right) A\left(|a|^{n-1}, |b|^{n-1}\right). \quad (4.1)$$

Proof. The assertion follows from Corollary 10 when applied to the function $f(x) = x^n$, $x \in [a,b]$, $n \in \mathbb{Z}$, $|n| \geq 2$. □

Proposition 32. *Let* $a,\ b \in \mathbb{R},\ a < b,\ 0 \notin [a,b]$. *Then*

$$\left| H^{-1}\left(\frac{3a+b}{4}, \frac{a+3b}{4}\right) - L(a,b)\right| \leq \left(\frac{b-a}{8}\right) A\left(|a|^{-2}, |b|^{-2}\right). \quad (4.2)$$

Proof. It is a direct consequence of Corollary 10 when applied to the function, $f(x) = \frac{1}{x}$, $x \in [a,b]$. □

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Conjecture on Dinitz Problem and Improvement of Hrnčiar's Result

Miroslav Haviar *
Faculty of Natural Sciences, M Bel University,
Tajovského 40, 974 01 Banská Bystrica, Slovakia
miroslav.haviar@umb.sk

Michal Ivaška †
Faculty of Natural Sciences, M Bel University,
Tajovského 40, 974 01 Banská Bystrica, Slovakia
michal.ivaska@umb.sk

Abstract

The paper aims at contributing to a better understanding of the Dinitz Problem by dealing with the number of "good choices" of representatives on a board of $n \times n$ cells. We conjecture that the number of good choices on an arbitrary board of order n is at least the number of good choices on a homogeneous board of order n, that is, at least the number $\ell(n)$ of Latin squares of order n. The first steps towards this conjecture are provided by proving that there are at least two good choices on an arbitrary board of order 3. This is slightly improving the result of Pavel Hrnčiar from 1991.

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1 Introduction

A simple-sounding problem introduced by Jeff Dinitz in 1978 asks whether on a board of $n \times n$ cells with n numbers in each cell one can choose a representative from every cell such that the selected numbers in each row and in each column are distinct (see e.g. [1, Chapter 28]). For arbitrary n the problem had been unsolved until Fred Galvin [2] presented his brilliant proof in 1995. But already in 1991 Pavel Hrnčiar gave a positive answer to the Dinitz Problem in the special case for $n = 3$. He showed that it is always possible to find one "good choice" of representatives on a board of 3×3 cells. The aim of our work is to present a conjecture on the number of "good choices" of representatives on the board of $n \times n$ cells (Section 3) and to prove that there are always at least two "good choices" of representatives on the board of 3×3 cells (Section 5).

The major part of our work deals with the concept of a *kernel* of a directed graph (Section 4). It is a subset of vertices satisfying two special conditions and it is amazingly connected to "good choices" on a board via so-called *square graphs* corresponding

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to boards. We show that every nondiscrete induced subgraph of the square graph corresponding to some board of 3×3 cells possesses at least two different kernels for some possible edge orientations.

We also introduce a new concept of so-called *tame choices* on a diagonal of a square graph, which is also connected to the existence of "good choices" of representatives (Section 5). In Section 6 we present various conjectures and counterexamples that arose in the process of our investigation.

2 Preliminaries

2.1 Dinitz Problem

For $n \geq 1$ consider n^2 cells arranged in an $(n \times n)$-square, let us call it a *board of order* n, and let (i, j) denote the cell in row i and column j. Suppose that for every cell (i, j) we are given a set $C(i, j)$ of n colours.

By a *choice* we mean that for each cell (i, j), exactly one colour is picked up from the set $C(i, j)$. Let a *good choice* be every choice in which the colours in each row and each column are distinct.

Is it then always possible to find a good choice for any board?

This simple-sounding colouring problem was raised by Jeff Dinitz in 1978 and it defied all attacks until its solution by Fred Galvin [2].

Let $C := \bigcup_{i,j} C(i, j)$ be a set of all colours of a board and let $|C|$ be *size* of the board.

It is worth to mention a particular case as presented in [1, p. 185]. If all colour sets are the same, say $\{1, 2, \dots, n\}$, then the Dinitz problem reduces to the following task: fill in the $(n \times n)$-square with the numbers $1, 2, \dots, n$ in such a way that the numbers in any row and column are distinct. This means that size of the board is n and all choices on it are precisely Latin squares. Since this is so easy, why would it be so much harder in the general case when the size is greater than n? The difficulty derives from the fact that not every colour of C is available at each cell.

2.2 Galvin's Proof

All definitions and results in this subsection are taken from [1, Chapter 28].

Definition 2.1. Let $G = (V, E)$ be a graph. Let us assume that we are given a nonempty set $C(v)$ of colours for each vertex $v \in V$. A **list colouring** is a colouring $c : V \longrightarrow \bigcup_{v \in V} C(v)$ where $c(v) \in C(v)$ for each $v \in V$ (a colouring is an assignment of colors to each vertex such that no edge connects two identically coloured vertices). A **list chromatic number** $\chi_\ell(G)$ is the smallest number k such that for any list of colour sets $C(v)$ with $|C(v)| = k$ for all $v \in V$ there always exists a list colouring.

Consider the *square graph* S_n which has as a vertex set the n^2 cells of our board of order n and two cells are adjacent if and only if they lie in the same row or column (see Figure 1). The Dinitz problem can now be stated as

$$\chi_\ell(S_n) = n?$$

Definition 2.2. Let $\vec{G} = (V, E)$ be a directed graph (shortly, digraph), that is, a graph where every edge e has an orientation. The notation $e = (u, v)$ means that there is an edge e, also denoted by $u \longrightarrow v$, whose initial vertex is u and whose terminal vertex is v. Then **outdegree** $d^+(v)$ of a vertex v is the number of edges with v as initial vertex, similarly for the **indegree** $d^-(v)$.

Furthermore, $d^+(v) + d^-(v) = d(v)$, where $d(v)$ is the degree of v.

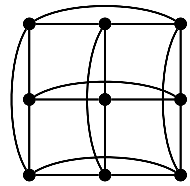

Figure 1. The graph S_3

Definition 2.3. For a graph $G = (V, E)$ and a non-empty subset $A \subseteq V$ we denote by $G[A]$ the subgraph which has A as vertex set and which contains all edges of G between vertices of A. We call $G[A]$ the **subgraph induced by** A, and say that H is an **induced subgraph** of G if $H = G[A]$ for some A.

Definition 2.4. Let $G = (V, E)$ be a graph without loops and multiple edges. A set $A \subseteq V$ is called **independent** if there are no edges within A.

Definition 2.5. Let $\vec{G} = (V, E)$ be a directed graph. A **kernel** $K \subseteq V$ is a subset of vertices such that

(1) K is independent in G, and

(2) for every $u \notin K$ there exists a vertex $v \in K$ with an edge $u \longrightarrow v$.

For example, vertices of a kernel of the subgraph of a graph $\vec{S_3}$ shown in the Figure 2 are encircled. (We remark that here, and often elsewhere, we use the term “graph" for “directed graph (digraph)" when no confusion arises.)

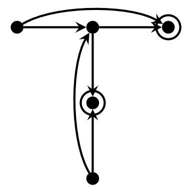

Figure 2. Kernel of the graph

In what follows, when we write G we mean the graph $\vec{G}$ without the orientations.

Lemma 2.6 ([1, Lemma 1])**.** Let $\vec{G} = (V, E)$ be a directed graph, and suppose that for each vertex $v \in V$ we have a color set that is larger than the outdegree, $|C(v)| \geq d^+(v)+1$. If every induced subgraph of $\vec{G}$ possesses a kernel, then there exists a list colouring of G with a colour from $C(v)$ for each v.

Denote the vertices of S_n by (i, j), $1 \leq i, j \leq n$. Thus (i, j) and (r, s) are adjacent if and only if $i = r$ or $j = s$. Take any Latin square L with letters from $\{1, 2, \ldots, n\}$ and denote by $L(i, j)$ the entry in cell (i, j). Next make S_n into a directed graph $\vec{S_n}$ by orienting the horizontal edges $(i, j) \longrightarrow (i, j')$ if $L(i, j) < L(i, j')$ and the vertical edges $(i, j) \longrightarrow (i', j)$ if $L(i, j) > L(i', j)$. Thus, horizontally we orient from the smaller to the larger element, and vertically the other way round. We shall denote this digraph $\vec{S_n}^L$ to emphasize that the orientation of edges is given by a Latin square L.

Notice that we obtain $d^+(i,j) = n-1$ for all (i,j). In fact, if $L(i,j) = k$, then $n-k$ cells in row i contain an entry larger than k, and $k-1$ cells in column j have an entry smaller than k.

The next result amazingly follows from the fact that a stable matching of a bipartite graph always exists (cf. [1, Lemma 2]).

Lemma 2.7 ([1, p. 189])**.** Every induced subgraph of $\vec{S_n}$ possesses a kernel.

Putting these two lemmas together with the fact that $d^+(i,j) = n-1$ for all (i,j), we get Galvin's solution [2] of the Dinitz Problem.

Theorem 2.8 ([1, p. 189])**.** We have $\chi_L(S_n) = n$ for all n.

2.3 Colouring Algorithm

Galvin's proof can tell us how to colour any board B of order n. We can use the following algorithm:

(1) choose any Latin square L (of the same order n as the board B);

(2) assign a digraph $\vec{S_n}^L$ with edge orientations given by L;

(3) choose any colour $c \in C$, where $C = \bigcup_{i,j} C(i,j)$;

(4) colour c generates a subgraph $\vec{S_n}^L[A]$, where $A = \{v \in V, c \in C(v)\}$;

(5) choose any kernel of this subgraph;

(6) colour the vertices of the kernel by a colour c;

(7) repeat steps 3–6 with colours c not previously used until the colouring is complete.

Galvin's proof implicitly says that after a finite number of steps (not more than s steps, where s is the size of the board) the colouring is complete and we obtain a good choice.

Notice that in this case every Latin square combined with any sequence of colours gives us a good choice according to the colouring algorithm. However, not all good choices are obtainable by this algorithm. It can even happen that two distinct Latin squares or two distinct sequences of colours can give us the same good choice (see Section 6).

3 Conjecture on the number of good choices

In this section we formulate a conjecture on the number of good choices on an arbitrary board which we find quite important with respect to a good understanding of the Dinitz Problem.

Let B_n be a board of order n. We shall denote $\sigma(B_n)$ the number of all distinct good choices on B_n. Galvin has shown that $\sigma(B_n) \geq 1$ for any board B_n.

By a *homogeneous* board of order n we shall mean a board of $n \times n$ cells with the same set $\{1, 2, \ldots, n\}$ of numbers in each cell. Hence the size of the homogeneous board is equal to its order.

Let $\ell(n)$ be the number of all Latin squares of order n. It is clear that $\ell(n)$ is the number of good choices on a homogeneous board of order n. So, $\sigma(B_n) = \ell(n)$ if the board B_n is homogeneous.

The following conjecture says that $\ell(n)$ is the optimal lower bound for the number of good choices on an *arbitrary* board of order n.

Conjecture. $\sigma(B_n) \geq \ell(n)$ for any board B_n.

The following table lists the values of $\ell(n)$, which are so far known for $1 \leq n \leq 11$ [4]. Thus, for given n, these are our conjectured optimal lower bounds for the number of good choices on an arbitrary board of order n.

n	$\ell(n)$
1	1
2	2
3	12
4	576
5	161280
6	812851200
7	61479419904000
8	108776032459082956800
9	5524751496156892842531225600
10	9982437658213039871725064756920320000
11	776966836171770144107444346734230682311065600000

4 Graph Kernel

In the next definition we introduce a new concept regarding graph kernels of induced subgraphs of a square graph.

Definition 4.1. Let $S_n[A]$ be a subgraph of a square graph S_n induced by some set A of vertices. We say that the graph $S_n[A]$ is k-**kerneled** if for some Latin squares $L_1, L_2, \ldots, L_m$ the digraphs $\vec{S_n}^{L_1}[A]$, $\vec{S_n}^{L_2}[A], \ldots$, $\vec{S_n}^{L_m}[A]$ have together at least k distinct kernels.

Lemma 4.2. Let $S_n[A]$ be a discrete graph. Then it is 1-kerneled and it is not k-kerneled for any $k > 1$.

Proof. It is easy to see that the only kernel in the discrete graph is the whole vertex set. □

In Figure 3 (on the left) we draw the same digraph as in Figure 2 and present the Latin square corresponding to its edge orientations. The kernel of this directed graph is encircled. Now we focus on the vertex u. If we orient all edges towards u and make a new digraph, then u must be in the kernel of this new digraph, because of the second condition of the graph kernel. We want to find some Latin square such that it will correspond to the orientation of this new graph. But it is easy, because we only need the entry in the cell corresponding to the vertex u to be lower than the entries in the cells corresponding to the vertices connected with u. So the entry in the cell corresponding to the vertex u will be 1. One of the possible Latin squares is shown on the right side of the Figure 3. The graph next to it is the graph with the new edge orientations and with a new kernel encircled.

The method described above can be simply generalised. It suffices to have some edge with end vertices whose degrees do not exceed 2, because in that case it is possible to orient all edges into one of its end vertices. By the first kernel condition, two end vertices of one edge cannot both belong to one kernel. We take the vertex which is not there and construct a new orientation of the graph, where this taken vertex will already be in some kernel.

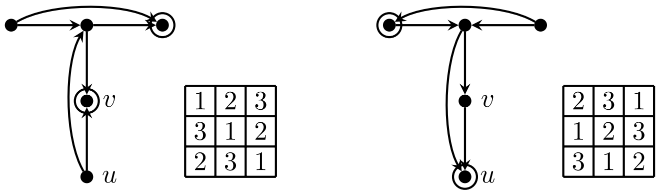

Figure 3. Two kernels of the same graph with changed edge orientations

Lemma 4.3. If there exists an edge $uv \in S_3[A]$ such that $d(u) \leq 2$ and $d(v) \leq 2$, then $S_3[A]$ is 2-kerneled.

Proof. From Lemma 2.7 we know that $\vec{S_3}^{L_1}[A]$ has a kernel, given by some Latin square L_1. We denote this kernel K_1. We need to show that there exists a kernel $K_2 \neq K_1$. We will do it the way that we change the orientations of edges in $\vec{S_3}^{L_1}[A]$ and obtain the new kernel K_2 of $\vec{S_3}^{L_2}[A]$, which will correspond to some Latin square L_2. Since a kernel is a set of independent vertices and $uv \in S_3[A]$, then $u \notin K_1$ or $v \notin K_1$. Without loss of generality we can assume that $u \notin K_1$. As $d(u) \leq 2$, we can orient all edges in $S_3^{L_2}[A]$ in such a way that $d^-(u) = d(u)$ and $d^+(u) = 0$. Note that a Latin square L_2 which will give such orientations always exists. Now $d^+(u) = 0$ implies that $u \in K_2$, because from the Definition 2.5 all vertices which are not in the kernel must be initial vertices of some edge with terminal vertex in the kernel (and so their outdegree must be at least 1). Thus $K_1 \neq K_2$ and the proof is complete. □

Definition 4.4. Let $S_n[A]$ be an induced subgraph of S_n and let $1 \leq r \leq n$. Then the r-th **row** R_r of the graph $S_n[A]$ is the set of vertices $\{(r, j) \in S_n[A],\ 1 \leq j \leq n\}$.

For the graph S_n and for every $i \in \{1, 2, \ldots, n\}$ we have $|R_i| = n$. For any induced subgraph $S_n[A]$ we have $|R_i| \leq n$.

Lemma 4.5. Let $S_3[A]$ be an induced subgraph such that $|R_i| = 3$ and $|R_j| = 0$ for some $1 \leq i, j \leq 3$, $i \neq j$. Then $S_3[A]$ is 2-kerneled.

Proof. Let $1 \leq k \leq 3$, $k \notin \{i, j\}$ (note that such k is unique). If $|R_k| = 0$ then by Lemma 4.3, $S_3[A]$ is 2-kerneled. So let $|R_k| \geq 1$. We can choose any vertex $w \in R_k$ (see Figure 4, in this case $i = 1$, $j = 2$ and $k = 3$). Then there exist vertices $u, v \in R_i$ such that u, w and v, w are independent. Now take any of these two pairs, for example take u, w and construct a Latin square which has an entry 3 in the cells corresponding to the vertices u, w. Since 3 is the biggest number in the Latin square of order 3, all vertices in the same line will be directed into vertices u, w. But this already means that $\{u, w\}$ is a kernel. For the pair v, w it can be showed analogously. Thus $\{u, w\}$ and $\{v, w\}$ are two distinct kernels of $S_3[A]$ and so $S_3[A]$ is 2-kerneled. □

Definition 4.6. A **diagonal** of a graph $S_n[A]$ is any set of n independent vertices.

Lemma 4.7. Every graph $S_n[A]$ containing k different diagonals is k-kerneled.

Proof. To a given diagonal we can take any Latin square of order n which has an entry n in all the cells corresponding to the vertices of the diagonal. Then the edge from all the other vertices will be oriented towards these vertices, so the diagonal will be a kernel. □

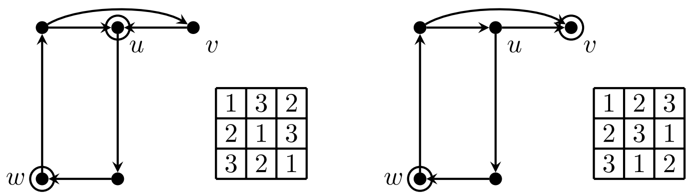

Figure 4. Two kernels of the same graph with an empty row

Clearly, the graph S_n has $n!$ diagonals. Certainly it can happen that some of its induced subgraphs contain no diagonals (for example the graph in Figure 4).

Lemma 4.8. Every graph $S_3[A]$ in which $|R_i| = 3$, for some $i \in \{1, 2, 3\}$, and which contains exactly one pair of independent vertices not contained in R_i, is 2-kerneled.

Proof. Note that such a graph always contains one diagonal, because the two independent vertices can be supplemented by an independent vertex from the i-th row. So by Lemma 4.7, this diagonal is already a kernel. We will show that there always exists a kernel K such that $|K| = 2$ (i.e. different from the first one).

So take any of the two independent vertices u and v. Say we take u. Then we take the vertex from the i-th row, which is independent with u, but not independent with v (there is exactly one such vertex). Up to isomorphism we can assume that we have one of the graphs in Figure 5. Let us first consider the graph on the left side. Its kernel corresponding to the Latin square is encircled. One can notice that if we add a vertex so that there will still be exactly one independent pair of vertices, up to isomorphism we obtain the graph on the right side of Figure 5. Now we can use the same Latin square as before to orient the edges, and the kernel will remain the same. □

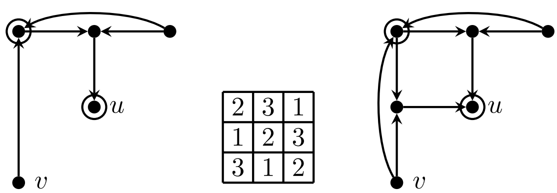

Figure 5. Graphs with one full row and one independent pair of vertices

Theorem 4.9. Every nondiscrete subgraph of S_3 is 2-kerneled.

Proof. Let $S_3[A]$ be a nondiscrete subgraph of S_3. If maximal degree of its vertices $\Delta(S_3[A]) \leq 2$, then $S_3[A]$ has two different kernels by Lemma 4.3. So now let there exist a vertex v with $d(v) > 2$. It is easy to see that then for the line R_i, where $v \in R_i$, we have $|R_i| = 3$ and $|R_j| \geq 1$ for some $1 \leq j \leq 3$, $i \neq j$. Without loss of generality we can assume that $|R_1| = 3$ and $|R_2| \geq 1$. Now we have these possibilities:

1. $|R_3| = 0$ — then the statement holds by Lemma 4.5.

2. $|R_2| = 1$ and $|R_3| = 1$ — if the two vertices in R_2 and R_3 are independent, the statement holds by Lemma 4.8, otherwise by Lemma 4.3.

3. $|R_2| = 2$ and $|R_3| = 1$ — if there is exactly one independent pair of vertices, the statement holds by Lemma 4.8. Otherwise there are two or more independent pairs in which case each of them can be completed to a diagonal with a vertex in R_1, so the statement holds by Lemma 4.7.

4. $|R_2| \geq 2$ and $|R_3| \geq 2$ — the statement holds by Lemma 4.7.

The proof is complete. □

5 From Kernels to Good Choices

Theorem 5.1. Let c be a colour of a board B of order n and let A be the set of all vertices of the graph S_n corresponding to those cells which contain the colour c. If the graph $S_n[A]$ is k-kerneled, then the board B has at least k distinct good choices.

Proof. Let $S_n[A]$ be a k-kerneled induced subgraph of the graph S_n, i.e. there exist distinct kernels $K_1, K_2, \dots, K_k$ of the digraphs $\vec{S_n}^{L_i}[A]$ with orientations given by some Latin squares $L_1, L_2, \dots, L_k$, respectively. According to the colouring algorithm from Galvin's proof (see Subsection 2.2), we take the Latin square L_1. The Latin square gives us an orientation of the graph S_n and in the first step we choose a colour c (in the final good choice the colour c will remain precisely in those cells corresponding to the vertices of the kernel K_1). After this, we continue with an arbitrary sequence of colours (see Subsection 2.3) until we obtain a good choice. We denote the obtained good choice by D_1. Similarly, when we choose in the first step the Latin square L_2 and the same colour c, we obtain a good choice that we can denote D_2. We do the same for all the other kernels so that we have good choices $D_1, D_2, \dots, D_k$. Note that the sets of cells with a colour c are distinct in all these good choices, since all the kernels were distinct. Thus we obtained k distinct good choices and the proof is complete. □

We recall that Galvin [2] has shown that $\sigma(B_n) \geq 1$ for any board B_n of order n.

Before Galvin, in 1991 Hrnčiar [3] showed that $\sigma(B_3) \geq 1$ for any board of order 3. We conjecture in Section 3 that $\sigma(B_3) \geq 12$. Our following result is improving the lower bound for $\sigma(B_3)$ and so can be understood as the first little step towards proving the conjecture.

Theorem 5.2. $\sigma(B_3) \geq 2$ for any board of order 3.

Proof. Let B_3 be a board of order 3 and let S_3 be its assigned square graph. We will distinguish two cases: (1) every colour of the board is only once in the same row or column, (2) there is a colour c such that it is at least twice in the same row or column.

(1) Let every colour be only once in the same row or column. We take any cell of the board, denote it by E. It contains 3 colours, say a, b, c. Note that in this case the induced subgraph of S_3 generated by any colour is discrete and so its kernel is the whole subgraph. Now according to the colouring algorithm we can take any Latin square and in the first step we take colour a. As the whole subgraph belongs to the kernel, we colour the cell E by the colour a. Then we continue with an arbitrary colour sequence to obtain a good choice. Similarly, when we take a colour b in the first step, we obtain a good choice with b in the cell E. So we constructed two distinct good choices.

(2) Let c be a colour of B_3 such that it is at least twice in the same row or column. Then the subgraph generated by this colour is nondiscrete, and, by Theorem 4.9, it is 2-kerneled. Now by Theorem 5.1, it has two distinct good choices. □

In the last part of this section we introduce some concepts corresponding to a board and present our final result.

Definition 5.3. Let B be a board of order n and let a **diagonal of a board** B be a set of any n cells such that none of them lie in the same row or column. Then any set of n colours from distinct cells of a diagonal will be called the **choice on a diagonal**.

Let B be a board and let D be a choice on some diagonal. We denote every colour in D as $d_{(a,b)}$ to express that this colour was chosen from the cell (a, b). Now for every cell (i, j) of B and for every colour $d_{(k,l)}$ of D we delete this colour from the cell (i, j) if $i = k$ or $j = l$, i.e. we delete every colour of the choice on a diagonal from all cells which lie in the same row or column.

Definition 5.4. We shall call the choice D **tame** if a reduced board obtained by the process above has at least $n - 1$ remaining colours in every cell.

For example, the choice $\{1, 2, 5\}$ on the diagonal $(1, 1), (2, 2), (3, 3)$ in the board on the left below is tame, because the reduced board on the right has at least 2 colours remaining in every cell.

$\{\mathbf{1}, 2, 3\}$	$\{3, 4, 5\}$	$\{2, 3, 5\}$
$\{4, 5, 6\}$	$\{\mathbf{2}, 3, 4\}$	$\{1, 4, 6\}$
$\{2, 3, 5\}$	$\{1, 2, 6\}$	$\{4, \mathbf{5}, 6\}$

$\{2, 3\}$	$\{3, 4, 5\}$	$\{2, 3\}$
$\{4, 5, 6\}$	$\{3, 4\}$	$\{1, 4, 6\}$
$\{2, 3\}$	$\{1, 6\}$	$\{4, 6\}$

As we shall see, the existence of tame choices guarantees the existence of good choices.

Theorem 5.5. Let B be a board. If there exist k tame choices on some diagonal of B, then $\sigma(B) \geq k$.

Proof. Denote this diagonal by D. We take any Latin square L such that it has an entry n in all cells corresponding to the diagonal D, where n is the order of L. We consider the square digraph $\vec{S_n}^L$ with edge orientations given by the Latin square L. The outdegree of every vertex will then be $n - 1$. Now we can delete all the vertices corresponding to D from $\vec{S_n}^L$. Since in every row there was an edge oriented from every vertex to the diagonal, now after we deleted it, the outdegree of every vertex will be $n - 2$.

For every tame choice we can now also delete all colours of a choice from all lists of colours for every vertex. By the definition, every vertex will still have at least $n - 1$ colours. So $|C(v)| \geq n - 1$ for every v. Now by Lemma 2.6, the subgraph with deleted diagonals can be list coloured with colours from $C(v)$ for every v. The deleted vertices can be coloured with colours of a choice on a diagonal and we obtain a good choice.

We can do the same for all tame choices and we obtain k distinct good choices on B. □

6 Misguided Conjectures and Counterexamples

In this section we present various conjectures arising in the process of our investigation and the counterexamples to these conjectures which we later found using (in almost all cases) self-developed computer programs. The aim of this section is to give a helpful hand to those who would follow similar steps as we did and come up with possibly the same conjectures in the process of their investigation.

Misguided conjecture 6.1. Let B be a board of order n. Then there exists a diagonal of B which has at least n tame choices.

Counterexample 6.1. The following board has exactly one tame choice in every diagonal:

$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$
$\{1, 2, 4\}$	$\{1, 2, 4\}$	$\{1, 2, 4\}$
$\{1, 3, 4\}$	$\{1, 3, 4\}$	$\{2, 3, 4\}$

Misguided conjecture 6.2. Let B be a board of order n. Then every diagonal of B has at least one tame choice.

Counterexample 6.2. The following board has no tame choices on the diagonal (1,1), (2,2), (3,3):

$\{1,2,3\}$	$\{1,3,4\}$	$\{3,4,6\}$
$\{2,3,4\}$	$\{3,4,5\}$	$\{2,3,4\}$
$\{1,2,4\}$	$\{2,5,6\}$	$\{2,4,6\}$

Colours of a board can be in general any natural numbers, but note that if a board has size s then we can replace all numbers greater than s with numbers smaller than or equal to s. So we can always obtain the board with colours $\{1,2,\ldots,s\}$. We shall call this process a *board normalisation* and the board obtained this way we shall call a *normalised board.*

Misguided conjecture 6.3. Let B be a normalised board of size s. Let L_1, L_2 be distinct Latin squares and let S be a sequence of colours from $\{1,2,\ldots,s\}$. Let D_1, D_2 be the good choices obtained from the Latin squares L_1, L_2, respectively, combined with the colour sequence S. Then D_1 and D_2 are distinct.

Counterexample 6.3. Consider the following board:

$\{1,2,3\}$	$\{2,3,4\}$	$\{3,4,5\}$
$\{2,3,5\}$	$\{1,3,4\}$	$\{1,3,4\}$
$\{1,4,5\}$	$\{1,2,3\}$	$\{1,3,5\}$

Now if we take the following two Latin squares

1	2	3
3	1	2
2	3	1

1	3	2
2	1	3
3	2	1

and we use the sequence of colours $1,2,3,4,5$, we will obtain the same good choice:

1	2	4
2	3	1
4	1	3

Misguided conjecture 6.4. Let B be a normalised board of size s. Let L be a Latin square and let S_1, S_2 be distinct sequences of colours from $\{1,2,\ldots,s\}$. Let D_1, D_2 be the good choices obtained from the Latin square L combined with the colour sequences S_1, S_2, respectively. Then D_1 and D_2 are distinct.

Counterexample 6.4. Take the same board as in Counterexample 6.3, the Latin square

1	2	3
2	3	1
3	1	2

and the sequences $2, 3, 4, 1, 5$ and $4, 3, 2, 5, 1$. Then for both sequences we obtain the same good choice:

2	3	4
3	4	1
4	2	3

Misguided conjecture 6.5. Let B be a normalised board of size s. Let L_1, L_2 be distinct Latin squares and let S_1, S_2 be distinct sequences of colours from $\{1, 2, \ldots, s\}$. Let D_1, D_2 be the good choices obtained from the Latin squares L_1, L_2 combined with the colour sequences S_1, S_2, respectively. Then D_1 and D_2 are distinct.

Counterexample 6.5. Consider the same board as in Counterexample 6.3, the Latin squares

2	1	3
1	3	2
3	2	1

3	1	2
1	2	3
3	2	1

and for these Latin squares take sequences of colours $1, 2, 4, 5, 3$ and $4, 2, 5, 1, 3$, respectively. In both cases we obtain the same good choice:

1	2	4
2	4	1
4	1	5

Misguided conjecture 6.6. Every good choice on a board can be obtained via the colouring algorithm (from Subsection 2.3) using some Latin square and some sequence of colours.

Counterexample 6.6. We can take the same board as in Counterexample 6.3. Then the following good choice can not be obtained via the colouring algorithm for any Latin square and sequence of colours:

3	4	5
2	1	3
4	2	1

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An application of generalized Bessel functions on certain analytic functions

Saurabh Porwal
Department of Mathematics
U.I.E.T. Campus, C.S.J.M. University, Kanpur-208024, (U.P.), India
saurabhjcb@rediffmail.com

K. K. Dixit
Department of Engineering Mathematics
Gwalior Institute of Information Technology, Gwalior-474015, (M.P.), India
kk.dixit@rediffmail.com

Abstract

The purpose of the present paper is to investigate some characterization for generalized Bessel functions of first kind to be in various subclasses of analytic functions. We also consider an integral operator related to the generalized Bessel function.

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1 Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Further, we denote by S the subclass of A consisting of functions of the form 1.1 which are also univalent in U and T be the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \tag{1.2}$$

Let $T(\lambda, \alpha)$ be the subclass of T consisting of functions which satisfy the condition

$$Re\left\{\frac{zf'(z)}{\lambda z f'(z) + (1-\lambda) f(z)}\right\} > \alpha, \tag{1.3}$$

for some $\alpha (0 \le \alpha < 1)$, $\lambda (0 \le \lambda < 1)$ and for all $z \in U$.

Also, we let $C(\lambda,\alpha)$ denote the subclass of T consisting of functions which satisfy the condition

$$Re\left\{\frac{f'(z)+zf''(z)}{f'(z)+\lambda zf''(z)}\right\}>\alpha, \tag{1.4}$$

for some $\alpha(0\leq\alpha<1)$, $\lambda(0\leq\lambda<1)$ and for all $z\in U$.

From 1.3 and 1.4 it is easy to verify that

$$f(z)\in C(\lambda,\alpha)\Leftrightarrow zf'(z)\in T(\lambda,\alpha).$$

The classes $T(\lambda,\alpha)$ and $C(\lambda,\alpha)$ were extensively studied by Altintas and Owa [1] and certain conditions for hypergeometric functions for these classes were studied by Mostafa [11].

It is worthy to note that $T(0,\alpha)\equiv T^*(\alpha)$, the class of starlike functions of order $\alpha(0\leq\alpha<1)$ and $C(0,\alpha)\equiv C(\alpha)$, the class of convex functions of order $\alpha(0\leq\alpha<1)$ (see [13]).

We recall that the generalized Bessel function of the first kind $w=w_{p,b,c}$ is defined as the particular solution of the second-order linear homogenous differential equation

$$z^2\omega''(z)+bz\omega'(z)+\left[cz^2-p^2+(1-b)p\right]\omega(z)=0, \tag{1.5}$$

where $b,p,c\in C$, which is a natural generalization of Bessel's equation. This function has the familiar representation

$$\omega(z)=\omega_{p,b,c}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n c^n}{n!\Gamma(p+n+\frac{b+1}{2})}\left(\frac{z}{2}\right)^{2n+p}, \quad z\in C. \tag{1.6}$$

The differential equation 1.5 permits the study of Bessel function, modified Bessel function, spherical Bessel function and modified spherical Bessel functions all together. Solutions of 1.5 are referred to as the generalized Bessel function of order p. The particular solution given by 1.6 is called the generalized Bessel function of the first kind of order p. Although the series defined above is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in U. It is worth mentioning that, in particular, when $b=c=1$, we reobtain the Bessel function $\omega_{p,1,1}=J_p$, and for $c=-1,b=1$ the function $\omega_{p,1,-1}$ becomes the modified Bessel function I_p. Now, consider the function $u_{p,b,c}$ defined by the transformation

$$u_{p,b,c}(z)=2^p\Gamma\left(p+\frac{b+1}{2}\right)z^{-p/2}\omega_{p,b,c}(z^{1/2}).$$

By using the well-known Pochhammer (or Appell) symbol, defined in terms of the Euler Gamma function for $a\neq 0,-1,-2,...$ by

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}1, & \text{if } n=0\\ a(a+1)...(a+n-1) & \text{if } n=1,2,3,...,\end{cases},$$

we obtain for the function $u_{p,b,c}$ the following representation

$$u_{p,b,c}(z)=\sum_{n=0}^{\infty}\frac{(-c/4)^n}{\left(p+\frac{(b+1)}{2}\right)_n}\frac{z^n}{n!}, \tag{1.7}$$

where $p+(b+1)/2\neq 0,-1,-2,....$ This function is analytic on C and satisfies the second-order linear differential equation

$$4z^2u''(z)+2\left(2p+b+1\right)zu'(z)+czu(z)=0.$$

The convolution (or Hadamard product) of two series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Now, we considered a linear operator $I(k,c) : A \to A$ defined by

$$\begin{aligned} I(k,c)f &= z u_{p,b,c}(z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} a_n z^n, \end{aligned}$$

where $k = p + \frac{b+1}{2}$. The generalized Bessel function is a recent topic of study in Geometric Function Theory (e.g. see the work of [2], [3], [4], [5] and [10]). Motivated by results on connections between various subclasses of analytic univalent functions by using hypergeometric functions (see [6], [7], [9], [12], [14], [15]) and by work of Baricz [2]-[5], we obtain sufficient condition for function $z(2 - u_p(z))$ belonging to the classes $T(\lambda, \alpha)$, $C(\lambda, \alpha)$ and connections between $R^{\tau}(A, B)$ and $C(\lambda, \alpha)$. Finally, we give a condition for an integral operator $G(k, c, z)$ belonging to the class $C(\lambda, \alpha)$.

For convenience throughout in the sequel, we use the following notations:

$$u_{p,b,c} = u_p, \quad k = p + \frac{b+1}{2}.$$

2 Main Results

To establish our main results, we shall require the following lemmas due to Dixit and Pal [8], Altintas and Owa [1] and Baricz [4].

Lemma 1. *([8]) If $f \in R^{\tau}(A, B)$ is of the form 1.1 then*

$$|a_n| \le \frac{(A-B)|\tau|}{n}, \quad (n \in N \setminus \{1\}). \tag{2.1}$$

The bounds given in 2.1 is sharp.

Lemma 2. *([1]) A function $f(z)$ defined by 1.2 is in the class $T(\lambda, \alpha)$, if and only if*

$$\sum_{n=2}^{\infty} [n - \lambda\alpha n - \alpha + \lambda\alpha] |a_n| \le 1 - \alpha.$$

Lemma 3. *([1]) A function $f(z)$ defined by 1.2 is in the class $C(\lambda, \alpha)$, if and only if*

$$\sum_{n=2}^{\infty} n [n - \lambda\alpha n - \alpha + \lambda\alpha] |a_n| \le 1 - \alpha.$$

Lemma 4. *([4]) If $b, p, c \in C$ and $k \ne 0, -1, -2, ...$ then the function u_p satisfies the recursive relation $4k u_p'(z) = -c u_{p+1}(z)$ for all $z \in C$.*

Theorem 5. *If $c < 0$, $k > 0 (k \ne 0, -1, -2, ...)$, then $z(2 - u_p(z))$ is in $T(\lambda, \alpha)$ if and only if*

$$(1 - \alpha\lambda) u_p'(1) + (1 - \alpha) u_p(1) \le 2(1 - \alpha), \tag{2.2}$$

Proof. Since

$$z\left(2-u_p(z)\right)=z-\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}z^n,$$

according to Lemma 2, we must show that

$$\sum_{n=2}^{\infty}\left[n(1-\lambda\alpha)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\leq 1-\alpha.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left[n(1-\alpha\lambda)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\
&=\sum_{n=0}^{\infty}\left[(n+2)(1-\alpha\lambda)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!}\\
&=(1-\alpha\lambda)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}n!}+(1-\alpha)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!}\\
&=(1-\alpha\lambda)u_p'(1)+(1-\alpha)\left[u_p(1)-1\right].
\end{aligned}$$

But this last expression is bounded above by $1-\alpha$ if and only if 2.2 holds. Thus the proof of Theorem 5 is established. □

Remark 6. In particular when $c=-1$ and $b=1$, the condition 2.2 becomes

$$2^{p-2}\Gamma(p+1)\left[(1-\alpha\lambda)I_{p+1}(1)+2(1-\alpha)I_p(1)\right]\leq 1-\alpha, \tag{2.3}$$

which is a necessary and sufficient condition for $z\left(2-\zeta_p(z^{1/2})\right)$ to be in $T(\lambda,\alpha)$, where

$$\zeta_p(z^{1/2})=2^p\Gamma(p+1)z^{-p/2}I_p(z^{1/2}). \tag{2.4}$$

Theorem 7. *If $c<0$, $k>0(k\neq 0,-1,-2,\ldots)$, then $z\left(2-u_p(z)\right)$ is in $C(\lambda,\alpha)$ if and only if*

$$(1-\alpha\lambda)u_p''(1)+(3-2\alpha\lambda-\alpha)u_p'(1)+(1-\alpha)u_p(1)\leq 2(1-\alpha), \tag{2.5}$$

Proof. Since

$$z\left(2-u_p(z)\right)=z-\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}z^n,$$

according to Lemma 3, we must show that

$$\sum_{n=2}^{\infty}n\left[n(1-\lambda\alpha)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\leq 1-\alpha.$$

Now

$$\sum_{n=2}^{\infty} n\left[n(1-\alpha\lambda)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}$$

$$=\sum_{n=2}^{\infty}\left\{(1-\alpha\lambda)(n-1)(n-2)+(3-2\alpha\lambda-\alpha)(n-1)+(1-\alpha)\right\}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}$$

$$=(1-\alpha\lambda)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-3)!}+(3-2\alpha\lambda-\alpha)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-2)!}+(1-\alpha)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}$$

$$=(1-\alpha\lambda)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}(n-1)!}+(3-2\alpha\lambda-\alpha)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}n!}+(1-\alpha)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!}$$

$$=(1-\alpha\lambda)\frac{(-c/4)^2}{k(k+1)}\sum_{n=0}^{\infty}\frac{(-c/4)^{n-1}}{(k+2)_{n-1}(n-1)!}+(3-2\alpha\lambda-\alpha)\frac{(-c/4)}{k}\sum_{n=0}^{\infty}\frac{(-c/4)^{n}}{(k+1)_{n}n!}+(1-\alpha)\left\{u_p(1)-1\right\}$$

$$=(1-\alpha\lambda)\frac{(-c/4)^2}{k(k+1)}u_{p+2}(1)+(3-2\alpha\lambda-\alpha)\frac{(-c/4)}{k}u_{p+1}(1)+(1-\alpha)\left\{u_p(1)-1\right\}$$

$$=(1-\alpha\lambda)u_p''(1)+(3-2\alpha\lambda-\alpha)u_p'(1)+(1-\alpha)\left\{u_p(1)-1\right\}$$

But this last expression is bounded above by $1-\alpha$ if and only if 2.5 holds. This completes the proof of Theorem 7. □

Theorem 8. *Let $c<0$, $k>0(k\neq 0,-1,-2,...)$. If $f\in R^{\tau}(A,B)$ and the inequality*

$$(A-B)|\tau|\left[(1-\alpha\lambda)u_p'(1)+(1-\alpha)\left\{u_p(1)-1\right\}\right]\leq 1-\alpha, \tag{2.6}$$

is satisfied then $I(k,c)f\in C(\lambda,\alpha)$.

Proof. By Lemma 3, it suffices to show that

$$P_1=\sum_{n=2}^{\infty} n\left[n-\lambda\alpha n-\alpha+\lambda\alpha\right]|a_n|\leq 1-\alpha.$$

Since $f\in R^{\tau}(A,B)$ then by Lemma 1 we have

$$|a_n|\leq\frac{(A-B)|\tau|}{n}.$$

Hence

$$P_1\leq(A-B)|\tau|\sum_{n=2}^{\infty}\left[n(1-\alpha\lambda)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}$$

$$=(A-B)|\tau|\sum_{n=0}^{\infty}\left[(n+2)(1-\alpha\lambda)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!}$$

$$=(A-B)|\tau|\left[(1-\alpha\lambda)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}n!}+(1-\alpha)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!}\right]$$

$$=(A-B)|\tau|\left[(1-\alpha\lambda)\frac{(-c/4)}{k}\sum_{n=0}^{\infty}\frac{(-c/4)^{n}}{(k+1)_{n}n!}+(1-\alpha)\left\{u_p(1)-1\right\}\right]$$

$$=(A-B)|\tau|\left[(1-\alpha\lambda)\frac{(-c/4)}{k}u_{p+1}(1)+(1-\alpha)\left\{u_p(1)-1\right\}\right]$$

$$=(A-B)|\tau|\left[(1-\alpha\lambda)u_p'(1)+(1-\alpha)\left\{u_p(1)-1\right\}\right].$$

But this last expression is bounded above by $1-\alpha$ if and only if 2.6 holds. □

3 An Integral Operator

In the following theorem, we obtain similar results in connection with a particular integral operator $G(k,c,z)$ as follows

$$G(k,c,z)=\int_0^z (2-u_p(t))\,dt \tag{3.1}$$

Theorem 9. *If $c<0$, $k>0(k\neq 0,-1,-2,...)$, then $G(k,c,z)$ defined by 3.1 is in $C(\lambda,\alpha)$ if and only if*

$$(1-\alpha\lambda)u_p'(1)+(1-\alpha)\left[u_p(1)-1\right]\leq(1-\alpha). \tag{3.2}$$

Proof. Since

$$\begin{aligned}G(k,c,z)&=z-\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\frac{z^n}{n}\\&=z-\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}}\frac{z^n}{n!}\end{aligned}$$

by Lemma 3, we need only to show that

$$\sum_{n=2}^{\infty}n\left[n(1-\lambda\alpha)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}n!}\leq 1-\alpha.$$

Now

$$\begin{aligned}&\sum_{n=2}^{\infty}n\left[n(1-\lambda\alpha)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}n!}\\=&\sum_{n=2}^{\infty}\left[n(1-\lambda\alpha)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\=&\sum_{n=0}^{\infty}\left[(n+2)(1-\lambda\alpha)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!}\\=&(1-\alpha\lambda)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}n!}+(1-\alpha)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!}\\=&(1-\alpha\lambda)u_p'(1)+(1-\alpha)\left[u_p(1)-1,\right]\end{aligned}$$

which is bounded above by $1-\alpha$, if and only if 3.2 holds. □

Remark 10. If we put $c=-1$ and $b=1$ in Theorem 7-9 we obtain analogues results of 2.3.

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$\mathcal{I}$-derivative

Vladislav Banas
Department of Algebra, Geometry, and Mathematical Education
Faculty of Mathematics, Physics and Informatics, Comenius University
Mlynská dolina, 842 48 Bratislava, Slovakia
vladislav.banas@gmail.com

Abstract
This paper deals with a derivative of a real function based on the notion of $\mathcal{I}$-convergence.

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1 Introduction

In the connection to some known results about functions preserving $\mathcal{I}$-convergence and $\mathcal{I}$-continuity (see [1, 2, 3, 4]) we introduce the $\mathcal{I}$-derivative of a real function, i.e. the derivative based on the notion of $\mathcal{I}$-convergence.

$\mathcal{I}$-convergence was introduced in [1] as a generalization of statistical convergence (see [5, 6]).

In this paper we will elucidate the relationship of $\mathcal{I}$-derivative to usual derivative with respect to the choice of ideals used in the definition of $\mathcal{I}$-derivative.

Definition 1. (see [7, p. 6]) A non-void family $\mathcal{I}$ of subsets of a given set X is called an *ideal* on X if it is hereditary and additive, i.e.

(1) $A \in \mathcal{I}$ and $B \subset A \Rightarrow B \in \mathcal{I}$,

(2) $A \in \mathcal{I}$ and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$.

An ideal $\mathcal{I}$ is called a *proper* ideal if $X \notin \mathcal{I}$.

A proper ideal $\mathcal{I}$ is said to be *admissible* (see [1]) if $\mathcal{I}$ contains every singleton.

The dual notion to the notion of an ideal is the notion of a filter.

Definition 2. (see [7, p. 6]) A non-void family $\mathcal{F}$ of subsets of a given set X is called a *filter* on X if

(1) $A \in \mathcal{F}$ and $A \subset B \Rightarrow B \in \mathcal{F}$,

(2) $A \in \mathcal{F}$ and $B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$.

A filter $\mathcal{F}$ is *proper* if $\emptyset \notin \mathcal{F}$.

Obviously, for each ideal $\mathcal{I}$ the system

$$\mathcal{F}(\mathcal{I}) = \{X \setminus A : A \in \mathcal{I}\}$$

is a filter on X.

Definition 3. (see [1]) Let $\mathcal{I}$ be a proper ideal on the set $\mathbb{N}$. A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-*convergent* to $\xi \in \mathbb{R}$ ($\mathcal{I}$-$\lim x_n = \xi$) if and only if for each $\varepsilon > 0$ the set

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - \xi| \geq \varepsilon\}$$

belongs to $\mathcal{I}$. The element ξ is called $\mathcal{I}$-limit of the sequence $(x_n)_{n\in\mathbb{N}}$.

In what follows we recall some basic properties of $\mathcal{I}$-convergence and of the notions $\mathcal{I}$-limit inferior and $\mathcal{I}$-limit superior(see [8, 9]).

Theorem 4. *(see [9]) Let $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ be sequences of real numbers such that $\mathcal{I}$-$\lim x_n = \xi$, $\mathcal{I}$-$\lim y_n = \eta$. Then*

(a) $\mathcal{I}$-$\lim(x_n \cdot y_n) = \xi \cdot \eta$,

(b) $\mathcal{I}$-$\lim(x_n + y_n) = \xi + \eta$.

Let $t \in \mathbb{R}$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Put

$$M_t = \{n : x_n > t\}, \qquad M^t = \{n : x_n < t\}.$$

Definition 5. (see [9])

(a) If there is a $t \in \mathbb{R}$ such that $M_t \notin \mathcal{I}$, we put

$$\mathcal{I}\text{-}\limsup x_n = \sup\{t \in \mathbb{R} : M_t \notin \mathcal{I}\}.$$

If $M_t \in \mathcal{I}$ for each $t \in \mathbb{R}$, then $\mathcal{I}\text{-}\limsup x_n = -\infty$.

(b) If there is a $t \in \mathbb{R}$ such that $M^t \notin \mathcal{I}$, we put

$$\mathcal{I}\text{-}\liminf x_n = \inf\{t \in \mathbb{R} : M^t \notin \mathcal{I}\}.$$

If $M^t \in \mathcal{I}$ for each $t \in \mathbb{R}$, then $\mathcal{I}\text{-}\liminf x_n = +\infty$.

Theorem 6. *(see [9]) The inequality*

$$\mathcal{I}\text{-}\liminf x_n \leq \mathcal{I}\text{-}\limsup x_n$$

holds for every sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers.

Theorem 7. *(see [9]) The sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is $\mathcal{I}$-convergent if and only if*

$$\mathcal{I}\text{-}\liminf x_n = \mathcal{I}\text{-}\limsup x_n.$$

If this equality holds, then

$$\mathcal{I}\text{-}\lim x_n = \mathcal{I}\text{-}\liminf x_n = \mathcal{I}\text{-}\limsup x_n.$$

We use various types of convergences in the article. We also often use sequences of real numbers with restrictions on their members. In order to keep the text comprehensible we introduce following notations.

Let $\mathbb{R}^{\mathbb{N}}$ denote the set of all sequences of real numbers. Let $x = (x_n)_{n\in\mathbb{N}}$. We put

$$\mathcal{S} = \{x \in \mathbb{R}^{\mathbb{N}} : x_n \neq 0, n \in \mathbb{N}\},$$

$$\mathcal{S}_{\mathcal{I}} = \{x : \mathcal{I}\text{-lim}\, x_n = 0, x_n \neq 0, n \in \mathbb{N}\}$$

and

$$\mathcal{S}_{\mathcal{I}}^{+} = \{x \in \mathcal{S}_{\mathcal{I}} : x_n > 0, n \in \mathbb{N}\},\ \mathcal{S}_{\mathcal{I}}^{-} = \{x \in \mathcal{S}_{\mathcal{I}} : x_n < 0, n \in \mathbb{N}\}.$$

Note that $\mathcal{S}_{\mathcal{I}_f}$, where $\mathcal{I}_f$ is the Fréchet ideal, contains only sequences convergent in the usual sense.

2 $\mathcal{I}$-derivative

Definition 8. Let $\mathcal{I}$ be an admissible ideal on the set $\mathbb{N}$. A function $f : \mathbb{R} \to \mathbb{R}$ has an *$\mathcal{I}$-derivative* $d \in \mathbb{R}$ at a point x_0, i.e. $\mathcal{I}\text{-}f'(x_0) = d$, if and only if

$$\mathcal{I}\text{-lim}\, \frac{f(x_0 + x_n) - f(x_0)}{x_n} = d \tag{2.1}$$

holds for each sequence $(x_n)_{n\in\mathbb{N}} \in \mathcal{S}_{\mathcal{I}}$.

In [2, Theorem 1] it is showed that if a function $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{I}$-continuous at a point x_0, i.e.

$$\mathcal{I}\text{-lim}\, x_n = x_0 \Rightarrow \mathcal{I}\text{-lim}\, f(x_n) = f(x_0) \tag{2.2}$$

holds for each sequence $(x_n)_{n\in\mathbb{N}}$, then it is continuous at the point x_0.

Proposition 9. *Let a function $f : \mathbb{R} \to \mathbb{R}$ have an $\mathcal{I}$-derivative $d \in \mathbb{R}$ at a point x_0 where $\mathcal{I}$ is an admissible ideal on the set $\mathbb{N}$. Then f is continuous at the point x_0.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \in \mathcal{S}_{\mathcal{I}}$. Obviously

$$f(x_0 + x_n) - f(x_0) = \frac{f(x_0 + x_n) - f(x_0)}{x_n} \cdot x_n$$

holds for each $n \in \mathbb{N}$. Thus

$$\mathcal{I}\text{-lim}(f(x_0 + x_n) - f(x_0)) = \mathcal{I}\text{-lim}\, \frac{f(x_0 + x_n) - f(x_0)}{x_n} \cdot x_n.$$

According to the assumption and Theorem 4 we have

$$\mathcal{I}\text{-lim}\, f(x_0 + x_n) = f(x_0)$$

what means that (2.2) holds for each sequence of real numbers and so f is continuous at the point x_0. □

For $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ put

$$D^{+} f(x_0) = \limsup_{h\to 0^{+}} \frac{f(x_0 + h) - f(x_0)}{h},$$

$$D_{+} f(x_0) = \liminf_{h\to 0^{+}} \frac{f(x_0 + h) - f(x_0)}{h},$$

$$D^- f(x_0) = \limsup_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h},$$

$$D_- f(x_0) = \liminf_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h}.$$

The numbers $D^+f(x_0), D_+f(x_0), D^-f(x_0), D_-f(x_0)$ are called Dini's derivatives (see e.g. [10, p. 27]) and it is well known that all Dini's derivatives at the point x_0 are equal to $d \in \mathbb{R}$ if and only if $f'(x_0) = d$.

For each $x \in \mathcal{S}^+_{\mathcal{I}}$ we put

$$D^+_x f(x_0) = \mathcal{I}\text{-}\limsup \frac{f(x_0+x_n)-f(x_0)}{x_n},$$

$$D^x_+ f(x_0) = \mathcal{I}\text{-}\liminf \frac{f(x_0+x_n)-f(x_0)}{x_n},$$

and for each $y \in \mathcal{S}^-_{\mathcal{I}}$

$$D^-_y f(x_0) = \mathcal{I}\text{-}\limsup \frac{f(x_0+y_n)-f(x_0)}{y_n},$$

$$D^y_- f(x_0) = \mathcal{I}\text{-}\liminf \frac{f(x_0+y_n)-f(x_0)}{y_n}.$$

Lemma 10. *Let $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$. Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers such that $\mathcal{I}$-$\lim x_n = x_0$, $x_n \neq x_0$ for each $n \in \mathbb{N}$ where $\mathcal{I}$ is an admissible ideal on the set $\mathbb{N}$. Then*

$$\liminf_{x\to x_0} f(x) \le \mathcal{I}\text{-}\liminf f(x_n), \tag{2.3}$$

$$\mathcal{I}\text{-}\limsup f(x_n) \le \limsup_{x\to x_0} f(x). \tag{2.4}$$

Proof. We will prove (2.3), the proof of (2.4) is quite similar.

Let $\lambda > 0$. Put $O_\lambda = \{f(x) : 0 < |x - x_0| < \lambda\}$. Because $(x_n)_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent, the set $\{n : |x_n - x_0| \ge \lambda\}$ belongs to $\mathcal{I}$ what implies $\{n : f(x_n) \in \mathbb{R} \setminus O_\lambda\} \in \mathcal{I}$. Let $s = \inf O_\lambda$. Case $s = -\infty$ is trivial so suppose that $s \in \mathbb{R}$. It is obvious that $\{f(x_n) : f(x_n) < s\} \subset \mathbb{R} \setminus O_\lambda$ and so $\{n : f(x_n) < s\}$ belongs to $\mathcal{I}$ what implies $s \notin \{t \in \mathbb{R} : \{n : f(x_n) < t\} \notin \mathcal{I}\}$. From the arbitrariness of the choice of λ we have

$$\sup_{\lambda>0}\{\inf O_\lambda\} \le \inf\{t \in \mathbb{R} : \{n : f(x_n) < t\} \notin \mathcal{I}\}$$

and the statement holds. □

Theorem 11. *Let $\mathcal{I}$ be an admissible ideal on the set $\mathbb{N}$. A function $f : \mathbb{R} \to \mathbb{R}$ has a derivative $d \in \mathbb{R}$ at a point x_0 if and only if $\mathcal{I}$-$f'(x_0) = d$.*

Proof. Let $f'(x_0) = d$. According to Theorem 6, Theorem 7 and Lemma 10 for each $x \in \mathcal{S}^+_{\mathcal{I}}$ we have

$$D_+ f(x_0) \le D^x_+ f(x_0) \le D^+_x f(x_0) \le D^+ f(x_0)$$

and for each $y \in \mathcal{S}^-_{\mathcal{I}}$

$$D_- f(x_0) \le D^y_- f(x_0) \le D^-_y f(x_0) \le D^- f(x_0).$$

Therefore (2.1) holds for each sequence from $\mathcal{S}_{\mathcal{I}}^{+} \cup \mathcal{S}_{\mathcal{I}}^{-}$. It is sufficient to show, that (2.1) holds for each $z \in \mathcal{S}_{\mathcal{I}} \setminus (\mathcal{S}_{\mathcal{I}}^{+} \cup \mathcal{S}_{\mathcal{I}}^{-})$. By contradiction. Let there exists $z \in \mathcal{S}_{\mathcal{I}} \setminus (\mathcal{S}_{\mathcal{I}}^{+} \cup \mathcal{S}_{\mathcal{I}}^{-})$ such that (2.1) does not hold. For $\eta > 0$ we put

$$K_\eta = \left\{ n : \frac{f(x_0 + z_n) - f(x_0)}{z_n} \notin B(d, \eta) \right\},$$

where $B(d, \eta)$ is open ball with center d and radius η. There is η_0 such that $K_{\eta_0} \notin \mathcal{I}$. Hence at least one of sets

$$A = \{n : n \in K_{\eta_0}, z_n < 0\}, B = \{n : n \in K_{\eta_0}, z_n > 0\}$$

does not belong to $\mathcal{I}$ and it is infinite. Let it be the set B. Define the sequence $s = (s_n)_{n \in \mathbb{N}}$ as follows. For $n \in B$ put $s_n = z_n$ and $s_n = \frac{1}{n}$ if $n \in \mathbb{N} \setminus B$. Obviously $s \in \mathcal{S}_{\mathcal{I}}^{+} \cup \mathcal{S}_{\mathcal{I}}^{-}$, a contradiction. If the set A does not belong to $\mathcal{I}$ for $n \in A$ we put $s_n = z_n$ and $s_n = -\frac{1}{n}$ if $n \in \mathbb{N} \setminus A$.

Suppose now, that f does not have the derivative at the point x_0. This implication follows immediately from the fact, that if one of Dini's derivatives is equal to $s \in \mathbb{R}$, there is a sequence $(x_n)_{n \in \mathbb{N}}$ convergent to x_0 such that the sequence $(y_n)_{n \in \mathbb{N}}$ of numbers

$$y_n = \frac{f(x_0 + x_n) - f(x_0)}{x_n}$$

converges to s or if $s = +\infty(-\infty)$ there is a sequence $(x_n)_{n \in \mathbb{N}}$ convergent to x_0 such that the sequence $(y_n)_{n \in \mathbb{N}}$ is increasing (decreasing) with limit $+\infty(-\infty)$. Hence for each admissible ideal $\mathcal{I}$ the function f does not have $\mathcal{I}$-derivative at the point x_0. □

In Definition 8 we have used the same ideal $\mathcal{I}$ for $\mathcal{I}$-convergence in $\mathcal{S}_{\mathcal{I}}$ and in (2.1). It is quite natural to ask what (if any) difference in our results will be reached by using various ideals.

Definition 12. Let $\mathcal{I}_1$, $\mathcal{I}_2$ be admissible ideals on the set $\mathbb{N}$. A function $f : \mathbb{R} \to \mathbb{R}$ has a $(\mathcal{I}_1, \mathcal{I}_2)$-*derivative* $d \in \mathbb{R}$ at a point x_0, i.e. $(\mathcal{I}_1, \mathcal{I}_2)$-$f'(x_0) = d$, if and only if

$$\mathcal{I}_1\text{-lim}\, x_n = 0 \Rightarrow \mathcal{I}_2\text{-lim}\, \frac{f(x_0 + x_n) - f(x_0)}{x_n} = d \tag{2.5}$$

holds for each sequence $(x_n)_{n \in \mathbb{N}} \in \mathcal{S}$.

Remark 13. Studying the proof of Theorem 11 we find that in case $\mathcal{I}_1 \subset \mathcal{I}_2$ we get the same results for $(\mathcal{I}_1, \mathcal{I}_2)$-derivative as well. The difference is reached if $\mathcal{I}_1 \setminus \mathcal{I}_2 \neq \emptyset$. The following theorem says that in this case $(\mathcal{I}_1, \mathcal{I}_2)$-derivative is no longer only local property of a real function at a point.

Theorem 14. *Let $\mathcal{I}_1$, $\mathcal{I}_2$ be admissible ideals on the set $\mathbb{N}$ such that $\mathcal{I}_1 \setminus \mathcal{I}_2 \neq \emptyset$. Then a function $f : \mathbb{R} \to \mathbb{R}$ has a $(\mathcal{I}_1, \mathcal{I}_2)$-derivative $d \in \mathbb{R}$ at a point x_0 if and only if f is a linear function.*

Proof. The case if f is a linear function is trivial.

Suppose that f is not linear and $(\mathcal{I}_1, \mathcal{I}_2)$-$f'(x_0) = d$. So there exists $z \in \mathbb{R} \setminus \{0\}$ such that

$$\frac{f(x_0 + z) - f(x_0)}{z} = d' \neq d. \tag{2.6}$$

Because $\mathcal{I}_1$, $\mathcal{I}_2$ are both admissible, there is an infinite set $A \in \mathcal{I}_1 \setminus \mathcal{I}_2$. Define sequence $(x_n)_{n\in\mathbb{N}}$ as follows. If $n \in A$ put $x_n = z$ else $x_n = \frac{1}{n}$. Obviously $\mathcal{I}_1$-$\lim x_n = 0$. Let $\eta > 0$ be such that $d' \notin B(d,\eta)$. Then

$$\left\{n : \frac{f(x_0 + x_n) - f(x_0)}{x_n} \notin B(d,\eta)\right\} \notin \mathcal{I}_2.$$

That is a contradiction because for the sequence $(x_n)_{n\in\mathbb{N}}$ the implication (2.5) does not hold. □

Simultaneously with $\mathcal{I}$-convergence another closely related kind of convergence called $\mathcal{I}^*$-convergence was introduced and investigated in [1, 9, 3], later generalized as $\mathcal{I}^{\mathcal{K}}$-convergence in [11].

Let $\mathcal{I}$ be an ideal on a set S and X be a topological space. A function $f : S \to X$ is said to be $\mathcal{I}$-convergent to $x \in X$ if

$$f^{-1}(U) = \{s \in S : f(s) \in U\} \in \mathcal{F}(\mathcal{I})$$

holds for each neighborhood of x.

Definition 15. (see [11]) Let $\mathcal{I}$, $\mathcal{K}$ be ideals on a set S. Let X be a topological space and $x \in X$. A function $f : S \to X$ is said to be $\mathcal{I}^{\mathcal{K}}$*-convergent* to x if there exists a set $M \in \mathcal{F}(\mathcal{I})$ such that the function $g : S \to X$ given by

$$g(s) = \begin{cases} f(s), & \text{if } s \in M \\ x, & \text{otherwise} \end{cases}$$

is $\mathcal{K}$-convergent to x.

Proposition 16. *Let $\mathcal{I}$ be an arbitrary and $\mathcal{K}$ an admissible ideal on the set $\mathbb{N}$. Let $(y_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, $\lim y_n = x$. Then a real sequence $(x_n)_{n\in\mathbb{N}}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to x if and only if there is a set $M \in \mathcal{F}(\mathcal{I})$ such that the sequence $(z_n)_{n\in\mathbb{N}}$, $z_n = x_n$ for $n \in M$ otherwise $z_n = y_n$, is $\mathcal{K}$-convergent to x.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be $\mathcal{I}^{\mathcal{K}}$-convergent to x, $\eta > 0$. Let M be the corresponding set belonging to $\mathcal{F}(\mathcal{I})$. Then $U = \{n : z_n \notin B(x,\eta) \wedge n \in M\}$ belongs to $\mathcal{K}$. The set $V = \{n : z_n \notin B(x,\eta) \wedge n \notin M\}$ is always finite or empty and since $\mathcal{K}$ is admissible the union $U \cup V$ belongs to $\mathcal{K}$ as well.

The converse implication can be proved using similar consideration. □

In [11] it is showed that $\mathcal{I}^{\mathcal{K}}$-convergence implies $\mathcal{I}$-convergence if $\mathcal{K} \subset \mathcal{I}$. The converse implication holds (assuming that X is a first countable topological space) if $\mathcal{I}$ has additive property with respect to $\mathcal{K}$, or more briefly that the condition $AP(\mathcal{I},\mathcal{K})$ holds. The condition $AP(\mathcal{I},\mathcal{K})$ holds, if for every sequence of mutually disjoint sets $(A_n)_{n\in\mathbb{N}}$ belonging to $\mathcal{I}$ there is a sequence $(B_n)_{n\in\mathbb{N}}$ of sets belonging to $\mathcal{I}$ such that $A_n \triangle B_n \in \mathcal{K}$ for $n \in \mathbb{N}$ and $B = \bigcup_{n\in\mathbb{N}} B_n \in \mathcal{I}$.

Definition 17. Let $\mathcal{I}$, $\mathcal{K}$ be admissible ideals on the set $\mathbb{N}$.
A function $f : \mathbb{R} \to \mathbb{R}$ has an $\mathcal{I}^{\mathcal{K}}$*-derivative* $d \in \mathbb{R}$ at a point x_0, i.e. $\mathcal{I}^{\mathcal{K}}$-$f'(x_0) = d$, if and only if

$$\mathcal{I}^{\mathcal{K}}\text{-}\lim x_n = 0 \Rightarrow \mathcal{I}^{\mathcal{K}}\text{-}\lim \frac{f(x_0 + x_n) - f(x_n)}{x_n} = d$$

holds for each sequence $(x_n)_{n\in\mathbb{N}} \in \mathcal{S}$.

Theorem 18. *Let $\mathcal{I}$, $\mathcal{K}$ be admissible ideals on the set $\mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$.*

(a) If $f'(x_0) = d$, then $\mathcal{I}^{\mathcal{K}}$-$f'(x_0) = d$.

(b) If $\mathcal{I}$ fulfils condition $AP(\mathcal{I}, \mathcal{K})$, $\mathcal{K} \subset \mathcal{I}$ and $\mathcal{I}^{\mathcal{K}}$-$f'(x_0) = d$, then $f'(x_0) = d$.

Proof. (a) Let $(x_n)_{n\in\mathbb{N}} \in \mathcal{S}_{\mathcal{I}^{\mathcal{K}}}$, $(y_n)_{n\in\mathbb{N}} \in \mathcal{S}_{\mathcal{I}_f}$. Thus there is a set $M \in \mathcal{F}(I)$ such that the sequence $(z_n)_{n\in\mathbb{N}}$, $z_n = x_n$ if $n \in M$ else $z_n = y_n$, is $\mathcal{K}$-convergent to the point 0. According to assumption, Proposition 16 and Theorem 11 we have

$$\mathcal{I}^{\mathcal{K}}\text{-}\lim \frac{f(x_0 + x_n) - f(x_n)}{x_n} = d.$$

(b) Let $(x_n)_{n\in\mathbb{N}} \in \mathcal{S}_{\mathcal{I}}$. Because $\mathcal{I}$ fulfils condition $AP(\mathcal{I}, \mathcal{K})$ and $\mathcal{K} \subset \mathcal{I}$ the implications

$$\mathcal{I}\text{-}\lim x_n = 0 \Rightarrow \mathcal{I}^{\mathcal{K}}\text{-}\lim x_n = 0,$$

$$\mathcal{I}^{\mathcal{K}}\text{-}\lim \frac{f(x_0 + x_n) - f(x_n)}{x_n} = d \Rightarrow \mathcal{I}\text{-}\lim \frac{f(x_0 + x_n) - f(x_n)}{x_n} = d$$

hold. Therefore $\mathcal{I}$-$f'(x_0) = d$ what is equivalent (Theorem 11) with $f'(x_0) = d$. □

According to Definition 12 we can introduce next definition.

Definition 19. Let $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{K}_1$, $\mathcal{K}_2$ be admissible ideals on the set $\mathbb{N}$.
A function $f : \mathbb{R} \to \mathbb{R}$ has a $(\mathcal{I}_1^{\mathcal{K}_1}, \mathcal{I}_2^{\mathcal{K}_2})$*-derivative* $d \in \mathbb{R}$ at a point x_0, i.e. $(\mathcal{I}_1^{\mathcal{K}_1}, \mathcal{I}_2^{\mathcal{K}_2})$-$f'(x_0) = d$, if and only if

$$\mathcal{I}_1^{\mathcal{K}_1}\text{-}\lim x_n = 0 \Rightarrow \mathcal{I}_2^{\mathcal{K}_2}\text{-}\lim \frac{f(x_0 + x_n) - f(x_n)}{x_n} = d$$

holds for each sequence $(x_n)_{n\in\mathbb{N}} \in \mathcal{S}$.

Theorem 20. *Let $f : \mathbb{R} \to \mathbb{R}$ and $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{K}_1$, $\mathcal{K}_2$ be admissible ideals on the set $\mathbb{N}$. Let $\mathcal{I}_1$, $\mathcal{I}_2$ possess conditions $AP(\mathcal{I}_1, \mathcal{K}_1)$, $AP(\mathcal{I}_2, \mathcal{K}_2)$ and $\mathcal{K}_1 \subset \mathcal{I}_1$, $\mathcal{K}_2 \subset \mathcal{I}_2$.*

(a) If $\mathcal{I}_1 \subset \mathcal{I}_2$ then $(\mathcal{I}_1^{\mathcal{K}_1}, \mathcal{I}_2^{\mathcal{K}_2})$-$f'(x_0) = d$ if and only if $f'(x_0) = d$.

(b) If $\mathcal{I}_1 \setminus \mathcal{I}_2 \neq \emptyset$ then $(\mathcal{I}_1^{\mathcal{K}_1}, \mathcal{I}_2^{\mathcal{K}_2})$-$f'(x_0) = d$ if and only if f is a linear function.

Proof. According to assumption the following equivalences

$$\mathcal{I}_1\text{-}\lim x_n = 0 \Leftrightarrow \mathcal{I}_1^{\mathcal{K}_1}\text{-}\lim x_n = 0,$$

$$\mathcal{I}_2\text{-}\lim \frac{f(x_0 + x_n) - f(x_n)}{x_n} = d \Leftrightarrow \mathcal{I}_2^{\mathcal{K}_2}\text{-}\lim \frac{f(x_0 + x_n) - f(x_n)}{x_n} = d$$

hold. The statement of Theorem 20 follows from Remark 13 and Theorem 14. □

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Bounded linear functionals on the n-normed space of p-summable sequences

Harmanus Batkunde
Department of Mathematics, Bandung Institute of Technology, Bandung 40132, Indonesia
batkunde@yahoo.com

Hendra Gunawan*
Department of Mathematics, Bandung Institute of Technology, Bandung 40132, Indonesia
hgunawan@math.itb.ac.id

Yosafat E.P. Pangalela
Department of Mathematics, Bandung Institute of Technology, Bandung 40132, Indonesia
matrix.yepp@gmail.com

Abstract

Let $(X, \|\cdot, \cdots, \cdot\|)$ be a real n-normed space, as introduced by S. Gähler in 1969. We shall be interested in bounded linear functionals on X, using the n-norm as our main tool. We study the duality properties and show that the space X' of bounded linear functionals on X also forms an n-normed space. We shall present more results on bounded multilinear n-functionals on the space of p-summable sequences being equipped with an n-norm. Open problems are also posed.

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1 Introduction

Let n be a nonnegative integer and X be a real vector space of dimension $d \geq n$. A real-valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the following four properties:

N.1 $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent,

N.2 $\|x_1, \dots, x_n\|$ is invariant under permutation,

N.3 $\|\alpha x_1, \dots, x_n\| = |\alpha| \; \|x_1, \dots, x_n\|$ for any $\alpha \in \mathbb{R}$,

N.4 $\|x_1 + x_1', x_2, \dots, x_n\| \leq \|x_1, x_2, \dots, x_n\| + \|x_1', x_2, \dots, x_n\|$,

is called an *n-norm* on X, and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called an n-normed space.

In an n-normed space $(X, \|\cdot, \dots, \cdot\|)$, one may observe that $\|x_1, \dots, x_n\| \geq 0$ and

$$\|x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, x_2, \dots, x_n\| = \|x_1, x_2, \dots, x_n\| \tag{1.1}$$

for every $x_1, \dots, x_n \in X$ and $\alpha_2, \dots, \alpha_n \in \mathbb{R}$.

*corresponding author

If $(X, \|\cdot\|)$ is a normed space and X' is its dual (consisting of bounded linear functionals on X), the following function defines an n-norm on X:

$$\|x_1, \ldots, x_n\|^G := \sup_{f_i \in X',\ \|f_i\| \leq 1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}. \tag{1.2}$$

Note that the determinant on the right hand side may be negative for certain f_i's, but in such a case we may replace one of the f_i's by its negative, so that the supremum of these determinants is always nonnegative.

For another example, if $(X, \langle \cdot, \cdot \rangle)$ is an inner product space, we can define the standard n-norm on X by

$$\|x_1, \ldots, x_n\|^S := \begin{vmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_n \rangle) \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \cdots & \langle x_n, x_n \rangle \end{vmatrix}^{1/2}. \tag{1.3}$$

The determinant above is known as Gram's determinant, whose value is always nonnegative. Geometrically, the value of $\|x_1, \ldots, x_n\|^S$ represents the volume of the n-dimensional parallelepiped spanned by $x_1, \ldots, x_n$ (see [5]).

The concept of n-normed spaces was initially introduced by Gähler [1, 2, 3, 4] in the 1960's. Recent results and related topics may be found in [8, 9, 10, 7, 11].

In this paper, we shall be interested in studying bounded linear functionals on X, using the n-norm as our main tool. We prove an analog of the Riesz-Fréchet Theorem and show that the dual space X', consisting of all bounded linear functionals on X, also forms an n-normed space. We shall present more results when X is the space of p-summable sequences being equipped with an n-norm. In addition, some open problems will be posed.

2 Bounded Linear Functionals

Let $(X, \|\cdot, \ldots, \cdot\|)$ be a real n-normed space and $f : X \to \mathbb{R}$ be a linear functional on X. We may define bounded linear functionals on X by using the n-norm in several ways as follows.

2.1 Bounded linear functionals (of 1st index)

Fix a linearly independent set $Y := \{y_1, \ldots, y_n\}$ in X. We say that f is *bounded with respect to* Y if and only if there exists $K > 0$ such that

$$|f(x)| \leq K \sum \|x, y_{i_2}, \ldots, y_{i_n}\| \tag{2.1}$$

for all $x \in X$, where the sum is taken over $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$ with $i_2 < \cdots < i_n$. [One might ask why we do not just take a linearly independent set $\{y_2, \ldots, y_n\}$ in X and put $|f(x)| \leq K \|x, y_2, \ldots, y_n\|$ for all $x \in X$. The drawback with this is that for a nonzero vector x in the linear span of $\{y_2, \ldots, y_n\}$, we have $\|x, y_2, \ldots, y_n\| = 0$ while $f(x) \neq 0$. This problem is overcome by taking a set of n linearly independent vectors and form the sum as in (2.1). Indeed, one might observe that the sum is equal to 0 if and only if $x = 0$.]

For simplicity, we shall say 'bounded' instead of 'bounded with respect to Y'. Clearly the set X_1' of all linear functionals which are bounded on X forms a vector space. Now, for $f \in X_1'$, we define

$$\|f\|_1 := \inf\{K > 0 : (2.1) \text{ holds}\}. \tag{2.2}$$

It is easy to see that

$$\|f\|_1 = \sup\{|f(x)| \,:\, \sum\|x, y_{i_2}, \dots, y_{i_n}\| \le 1\}$$

Moreover, the formula (2.2) defines a norm on X_1'.

To give an example, we invoke the notion of n-inner product spaces [11]. Assume that X is of dimension $d \ge n+1$. A real-valued function $\langle\cdot,\cdot|\cdot,\dots,\cdot\rangle$ on X^{n+1} satisfying the following properties:

I.1 $\langle x_1, x_1|x_2,\dots,x_n\rangle \ge 0$ and it is equal to 0 if and only if $x_1,\dots,x_n$ are linearly dependent,

I.2 $\langle x_{i_1}, x_{i_1}|x_{i_2},\dots,x_{i_n}\rangle = \langle x_1, x_1|x_2,\dots,x_n\rangle$ for any permutation $\{i_1,\dots,i_n\}$ of $\{1,\dots,n\}$,

I.3 $\langle x, y|x_2,\dots,x_n\rangle = \langle y, x|x_2,\dots,x_n\rangle$,

I.4 $\langle \alpha x, y|x_2,\dots,x_n\rangle = \alpha\langle x, y|x_2,\dots,x_n\rangle$ for any $\alpha \in \mathbb{R}$,

I.5 $\langle x+x', y|x_2,\dots,x_n\rangle = \langle x, y|x_2,\dots,x_n\rangle + \langle x', y|x_2,\dots,x_n\rangle$,

is called an n*-inner product* on X, and the pair $(X, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$ is called an n*-inner product space*.

Note that if $(X, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$ is an n-inner product space, then we can define an n-norm $\|\cdot,\dots,\cdot\|$ on X by

$$\|x_1, x_2, \dots, x_n\| := \langle x_1, x_1|x_2,\dots,x_n\rangle^{1/2}.$$

Here we have the Cauchy-Schwarz inequality:

$$|\langle x, y|x_2,\dots,x_n\rangle| \le \|x, x_2,\dots,x_n\|\,\|y, x_2,\dots,x_n\|.$$

Now we give an example of bounded linear functionals on X. Let $(X, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$ be an n-inner product space, and $\|\cdot,\dots,\cdot\| := \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle^{1/2}$ be the induced n-norm on X. With respect to the set $Y = \{y_1,\dots,y_n\}$, define $f : X \to \mathbb{R}$ by

$$f(x) := \sum\langle x, y_{i_1}|y_{i_2},\dots,y_{i_n}\rangle, \tag{2.3}$$

where the sum is taken over $\{i_2,\dots,i_n\} \subset \{1,\dots,n\}$ with $i_2 < \dots < i_n$ and $i_1 \in \{1,\dots,n\} \setminus \{i_2,\dots,i_n\}$. Clearly f is linear. Furthermore, we have:

Fact 1. The linear functional f defined by (2.3) is bounded with $\|f\|_1 = \|y_1,\dots,y_n\|$.

Proof. We observe that for every $x \in X$, we have

$$\begin{aligned}|f(x)| &\le \sum|\langle x, y_{i_1}|y_{i_2},\dots,y_{i_n}\rangle|\\ &\le \sum\|x, y_{i_2},\dots,y_{i_n}\|\,\|y_{i_1}, y_{i_2},\dots,y_{i_n}\|\\ &= \|y_1,\dots,y_n\|\sum\|x, y_{i_2},\dots,y_{i_n}\|\end{aligned}$$

where the sum is taken over $\{i_2,\dots,i_n\} \subset \{1,\dots,n\}$ with $i_2 < \dots < i_n$. Thus f is bounded with $\|f\|_1 \le \|y_1,\dots,y_n\|$.

To show that $\|f\|_1 = \|y_1,\dots,y_n\|$, just take $x := \|y_1,\dots,y_n\|^{-1}y_1$. Then we see that $\sum\|x, y_{i_2},\dots,y_{i_n}\| = 1$ and

$$\begin{aligned}|f(x)| &= \|y_1,\dots,y_n\|^{-1}\,f(y_1)\\ &= \|y_1,\dots,y_n\|^{-1}\sum\langle y_1, y_{i_1}|y_{i_2},\dots,y_{i_n}\rangle\\ &= \|y_1,\dots,y_n\|^{-1}\,\langle y_1, y_1|y_2,\dots,y_n\rangle\\ &= \|y_1,\dots,y_n\|^{-1}\,\|y_1,\dots,y_n\|^2\\ &= \|y_1,\dots,y_n\|.\end{aligned}$$

[Note that when $i_1 \neq 1$ and $\{i_2, \ldots, i_n\} = \{1, \ldots, n\} \setminus \{i_1\}$, we have

$$|\langle y_1, y_{i_1} | y_{i_2}, \ldots, y_{i_n} \rangle| \leq \|y_1, y_{i_2}, \ldots, y_{i_n}\| \, \|y_{i_1}, y_{i_2}, \ldots, y_{i_n}\| = 0$$

because one of $y_{i_2}, \ldots, y_{i_n}$ must be equal to y_1.] □

2.2 Bounded linear functionals of p-th index

Fix a linearly independent set $Y := \{y_1, \ldots, y_n\}$ in X and $1 \leq p \leq \infty$. We say that f is *bounded of p-th index* (with respect to Y) if and only if there exists $K > 0$ such that

$$|f(x)| \leq K \left(\sum \|x, y_{i_2}, \ldots, y_{i_n}\|^p\right)^{1/p} \tag{2.4}$$

where the sum is taken over $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$ with $i_2 < \cdots < i_n$. [If $p = \infty$, then the sum is the maximum of all possible values of $\|x, y_{i_2}, \ldots, y_{i_n}\|$.]

As in the case where $p = 1$, the set X'_p of all linear functionals which are bounded of p-index on X forms a vector space. Now, for $f \in X'_p$, we define

$$\|f\|_p := \inf\{K > 0 \, : \, (2.4) \text{ holds}\}. \tag{2.5}$$

One then has

$$\|f\|_p = \sup\{|f(x)| \, : \, \sum \|x, y_{i_2}, \ldots, y_{i_n}\|^p \leq 1\}.$$

Moreover, the formula (2.5) defines a norm on X'_p.

Fact 2. The linear functional f defined by (2.3) is bounded of p-th index with $\|f\|_p = n^{1/p'} \|y_1, \ldots, y_n\|$, where p' is the dual exponent of p (that is, $\frac{1}{p} + \frac{1}{p'} = 1$).

Proof. For every $x \in X$, it follows from Hölder's inequality that

$$|f(x)| \leq \sum \|x, y_{i_2}, \ldots, y_{i_n}\| \, \|y_1, \ldots, y_n\| \leq n^{1/p'} \|y_1, \ldots, y_n\| \left(\sum \|x, y_{i_2}, \ldots, y_{i_n}\|^p\right)^{1/p},$$

whence $\|f\|_p \leq n^{1/p'} \|y_1, \ldots, y_n\|$.

To obtain the equality, take $x := n^{-1/p} \|y_1, \ldots, y_n\|^{-1}(y_1 + \cdots + y_n)$. Then, using (1.1), one may verify that $\sum \|x, y_{i_2}, \ldots, y_{i_n}\|^p = 1$. Moreover, we have

$$\begin{aligned} f(x) &= n^{-1/p} \|y_1, \ldots, y_n\|^{-1} \sum \langle y_1 + \cdots + y_n, y_{i_1} | y_{i_2}, \ldots, y_{i_n} \rangle \\ &= n^{-1/p} \|y_1, \ldots, y_n\|^{-1} \sum \langle y_{i_1}, y_{i_1} | y_{i_2}, \ldots, y_{i_n} \rangle \\ &= n^{-1/p} \|y_1, \ldots, y_n\|^{-1} \cdot n \, \|y_1, \ldots, y_n\|^2 \\ &= n^{1/p'} \|y_1, \ldots, y_n\|. \end{aligned}$$

This convinces us that $\|f\|_p = n^{1/p'} \|y_1, \ldots, y_n\|$. □

The following theorem tells us that X'_1 and X'_p are identical as a set.

Theorem 3. *Let f be a linear functional on X. If f is bounded of 1st index, then f is bounded of p-th index; and vice versa. In other words, $X'_1 = X'_p$.*

Proof. Suppose that f is bounded of p-index (with respect to $Y = \{y_1, \ldots, y_n\}$). If x satisfies $\sum \|x, y_{i_2}, \ldots, y_{i_n}\| \leq 1$, then each term of the sum is less than 1, i.e., $\|x, y_{i_2}, \ldots, y_{i_n}\| \leq 1$. Hence $\|x, y_{i_2}, \ldots, y_{i_n}\|^p \leq \|x, y_{i_2}, \ldots, y_{i_n}\|$, and so

$$\sum \|x, y_{i_2}, \ldots, y_{i_n}\|^p \leq \sum \|x, y_{i_2}, \ldots, y_{i_n}\| \leq 1.$$

Consequently, $|f(x)| \leq \|f\|_p$, and thus f is bounded of 1st index with $\|f\|_1 \leq \|f\|_p$.

Conversely, suppose that f is bounded of 1st index. If x satisfies $\sum \|x, y_{i_2}, \ldots, y_{i_n}\|^p \leq 1$, then $\sum \|x, y_{i_2}, \ldots, y_{i_n}\| \leq n^{1/p'}$, where p' is the dual exponent of p. Hence

$$\sum \| \tfrac{x}{n^{1/p'}}, y_{i_2}, \ldots, y_{i_n}\| \leq 1,$$

and so $\left|f\left(\frac{x}{n^{1/p'}}\right)\right| \leq \|f\|_1$ or $|f(x)| \leq n^{1/p'}\|f\|_1$. We therefore conclude that f is bounded of p-th index with $\|f\|_p \leq n^{1/p'}\|f\|_1$. □

Remark 4. Unless we need to specify the index explicitly, we may simply use the word 'bounded' instead of 'bounded of p-th index'. We also denote by X' the set of all bounded linear functionals on X and call it the *dual space* of X (with respect to Y). Theorem 3 states further that, on X', the norms $\|\cdot\|_p$ are all equivalent to $\|\cdot\|_1$, with

$$\|f\|_1 \leq \|f\|_p \leq n^{1/p'} \|f\|_1,$$

for every $f \in X'$.

2.3 Duality properties for $p = 2$

Let us now discuss another example of bounded linear functionals on the n-inner product space X, using the linearly independent set $Y = \{y_1, \ldots, y_n\}$. Let $y \neq y_i$ for $i = 1, \ldots, n$. Define $f_y : X \to \mathbb{R}$ by

$$f_y(x) := \sum \langle x, y | y_{i_2}, \ldots, y_{i_n}\rangle, \tag{2.6}$$

where the sum is taken over $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$ with $i_2 < \cdots < i_n$. Then f_y is linear. Moreover, we have:

Fact 5. The linear functional f_y defined by (2.6) is bounded of 2nd index with $\|f_y\|_2 = \left(\sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2\right)^{1/2}$.

Proof. For every $x \in X$, it follows from Cauchy-Schwarz inequalities that

$$\begin{aligned} |f_y(x)| &\leq \sum |\langle x, y | y_{i_2}, \ldots, y_{i_n}\rangle| \\ &\leq \sum \|x, y_{i_2}, \ldots, y_{i_n}\| \, \|y, y_{i_2}, \ldots, y_{i_n}\| \\ &\leq \left(\sum \|x, y_{i_2}, \ldots, y_{i_n}\|^2\right)^{1/2} \left(\sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2\right)^{1/2}, \end{aligned}$$

whence $\|f_y\|_2 \leq \left(\sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2\right)^{1/2}$.

Now, if we take $x := \left(\sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2\right)^{-1/2} y$, we get

$$\begin{aligned} f_y(x) &= \left(\sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2\right)^{-1/2} f_y(y) \\ &= \left(\sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2\right)^{-1/2} \sum \langle y, y | y_{i_2}, \ldots, y_{i_n}\rangle \\ &= \left(\sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2\right)^{-1/2} \sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2 \\ &= \left(\sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2\right)^{1/2}. \end{aligned}$$

We must therefore have $\|f_y\|_2 = \left(\sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2\right)^{1/2}$. □

It is desirable to have an analog of the Riesz-Fréchet Theorem for linear functionals which are bounded of 2nd index on an n-inner product space. For that, we import the following theorem from [9].

Theorem 6 ([9]). *Let $(X, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$ be an n-inner product space and $\|\cdot,\dots,\cdot\| = \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle^{1/2}$ be the induced n-norm on X. With respect to the linearly independent set $Y = \{y_1,\dots,y_n\}$, the mapping $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{R}$ given by*

$$\langle x,y\rangle := \sum \langle x,y|y_{i_2},\dots,y_{i_n}\rangle \tag{2.7}$$

defines an inner product on X, and its induced norm $\|\cdot\|_2 : X \to \mathbb{R}$ is given by

$$\|x\|_2 := \left(\sum \|x,y_{i_2},\dots,y_{i_n}\|^2\right)^{1/2}. \tag{2.8}$$

Corollary 7. *If $(X, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$ is complete with respect to the norm $\|\cdot\|_2$ in* (2.8)*, then for every linear functional f which is bounded of 2nd index on X there exists a unique $y \in X$ such that*

$$f(x) = \langle x,y\rangle, \quad x \in X,$$

where $\langle\cdot,\cdot\rangle$ is the inner product in (2.7)*. Moreover, we have $\|y\|_2 = \|f\|_2$.*

Theorem 8. *Let $(X, \|\cdot,\dots,\cdot\|)$ be an n-normed space, X' be the dual space of X (with respect to Y), and $\|\cdot\|_2$ be the derived norm on X given by*

$$\|x\|_2 := \left(\sum \|x,y_{i_2},\dots,y_{i_n}\|^2\right)^{1/2}.$$

Then, the function $\|\cdot,\dots,\cdot\|' : (X')^n \to \mathbf{R}$ given by

$$\|f_1,\dots,f_n\|' := \sup_{x_i\in X,\ \|x_i\|_2\le 1} \begin{vmatrix} f_1(x_1) & \dots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \dots & f_n(x_n) \end{vmatrix}$$

defines an n-norm on X'.

Proof. Similar to the proof of Fact 2 in [6]. □

3 Bounded Multilinear n-Functionals on ℓ^p

In this section, we shall focus on the space of p-summable sequences of real numbers, denoted by $\ell^p = \ell^p_{\mathbb{N}}(\mathbb{R})$, where $1 \le p < \infty$. Recall that a sequence $u := \{u_k\}_{k=1}^\infty$ (of real numbers) belongs ℓ^p space if $\|u\|_p := \left(\sum_{k=1}^\infty |u_k|^p\right)^{1/p} < \infty$. It is known that the dual space of ℓ^p is $\ell^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$.

3.1 Several n-norms on ℓ^p

Using the formula (1.2), ℓ^p may be equipped with the following n-norm:

$$\|x_1,\dots,x_n\|_p^G := \sup_{y_i\in\ell^{p'},\ \|y_i\|_{p'}\le 1} \begin{vmatrix} \sum_{k=1}^\infty x_{1k}y_{1k} & \cdots & \sum_{k=1}^\infty x_{1k}y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^\infty x_{nk}y_{1k} & \cdots & \sum_{k=1}^\infty x_{nk}y_{nk} \end{vmatrix}, \tag{3.1}$$

where p' denotes the dual exponent of p. But there is another formula of n-norm that we can define on ℓ^p, namely

$$\|x_1,\dots,x_n\|_p^H := \left[\frac{1}{n!}\sum_{k_1=1}^\infty \cdots \sum_{k_n=1}^\infty \left|\begin{vmatrix} x_{1k_1} & \cdots & x_{1k_n} \\ \vdots & \ddots & \vdots \\ x_{nk_1} & \cdots & x_{nk_n} \end{vmatrix}\right|^p\right]^{\frac{1}{p}}, \tag{3.2}$$

where $x_i = \{x_{ik}\}_{k=1}^{\infty}$, $i = 1, \ldots, n$. As shown in [12], the two n-norms are equivalent:

$$(n!)^{(1/p)-1}\|x_1,\ldots,x_n\|_p^H \leq \|x_1,\ldots,x_n\|_p^G \leq (n!)^{1/p}\|x_1,\ldots,x_n\|_p^H.$$

On ℓ^2, both n-norms coincide with the standard n-norm given by (1.3) [6].

Next, one may observe that, by taking the sums and like terms out of the determinant and knowing that there are $n!$ possible ways to do so (see [7]), the determinant on the right hand side of (3.1) can be rewritten as

$$\frac{1}{n!}\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\begin{vmatrix} x_{1k_1} & \cdots & x_{1k_n} \\ \vdots & \ddots & \vdots \\ x_{nk_1} & \cdots & x_{nk_n}\end{vmatrix}\begin{vmatrix} y_{1k_1} & \cdots & y_{1k_n} \\ \vdots & \ddots & \vdots \\ y_{nk_1} & \cdots & y_{nk_n}\end{vmatrix}.$$

By Hölder's inequality, we find that this sum is dominated by

$$\|x_1,\ldots,x_n\|_p^H\|y_1,\ldots,y_n\|_{p'}^H.$$

This inspires us to define another n-norm on ℓ^p, namely

$$\|x_1,\ldots,x_n\|_p^I := \sup_{y_i\in\ell^{p'},\, \|y_1,\ldots,y_n\|_{p'}^H\leq 1}\begin{vmatrix} \sum_{k=1}^{\infty} x_{1k}y_{1k} & \cdots & \sum_{k=1}^{\infty} x_{1k}y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{\infty} x_{nk}y_{1k} & \cdots & \sum_{k=1}^{\infty} x_{nk}y_{nk}\end{vmatrix}. \tag{3.3}$$

Theorem 9. *The three n-norms on ℓ^p, namely $\|\cdot,\ldots,\cdot\|_p^I$, $\|\cdot,\ldots,\cdot\|_p^H$, and $\|\cdot,\ldots,\cdot\|_p^G$, are equivalent.*

Proof. By the observation above, we have $\|x_1,\ldots,x_n\|_p^I \leq \|x_1,\ldots,x_n\|_p^H$. By Theorem 2.3 of [12], we have $\|x_1,\ldots,x_n\|_p^H \leq (n!)^{1/p'}\|x_1,\ldots,x_n\|_p^G$. Now, using the inequality

$$\|y_1,\ldots,y_n\|_{p'}^H \leq (n!)^{1/p}\|y_1\|_{p'}\cdots\|y_n\|_{p'}$$

(see Fact 3.1 of [7]), we see that if $\|y_i\|_{p'} \leq 1$ for $i = 1,\ldots,n$, then $\|y_1,\ldots,y_n\|_{p'}^H \leq (n!)^{1/p}$. Hence we obtain

$$\|x_1,\ldots,x_n\|_p^G \leq (n!)^{1/p}\|x_1,\ldots,x_n\|_p^I.$$

The chain of these inequalities shows that the three n-norms are equivalent. □

3.2 Multilinear n-functionals on ℓ^p

By a *multilinear n-functional* on a real vector space X we mean a mapping $F : X^n \to \mathbb{R}$ which is linear in each variable. A multilinear n-functional F is *bounded* on an n-normed space $(X, \|\cdot,\ldots,\cdot\|)$ if and only if there exists $K > 0$ such that

$$|F(x_1,\ldots,x_n)| \leq K\,\|x_1,\ldots,x_n\| \tag{3.4}$$

for every $x_1,\ldots,x_n \in X$. Note that for a bounded multilinear n-functional F on an n-normed space $(X, \|\cdot,\ldots,\cdot\|)$, we have $F(x_1,\ldots,x_n) = 0$ when $x_1,\ldots,x_n$ are linearly dependent. Moreover, we have the following proposition.

Proposition 10. *If F is a bounded multilinear n-functional on an n-normed space $(X, \|\cdot,\ldots,\cdot\|)$, then F is antisymmetric, that is*

$$F(x_1,\ldots,x_n) = \operatorname{sgn}(\sigma)\,F(x_{\sigma(1)},\ldots,x_{\sigma(n)})$$

for any $x_1,\ldots,x_n \in X$ and any permutation σ of $(1,\ldots,n)$. [Here $\operatorname{sgn}(\sigma) = 1$ *if σ is an even permutation and* $\operatorname{sgn}(\sigma) = -1$ *if σ is an odd permutation.]*

Proof. We give the proof for the case where $n = 2$ and leave the other case to the reader. Here, F is antisymmetric if and only if $F(x_1, x_2) = -F(x_2, x_1)$ for every $x_1, x_2 \in X$. To see this, we observe that

$$F(x_1 + x_2, x_1 + x_2) = F(x_1, x_1) + F(x_1, x_2) + F(x_2, x_1) + F(x_2, x_2).$$

But $F(x, x) = 0$ for every $x \in X$, and so we are done. □

We note that the set X^* of all bounded multilinear n-functionals on $(X, \|\cdot, \dots, \cdot\|)$ forms a vector space. Next, for a bounded multilinear n-functional F, we may define

$$\|F\| := \inf\{K > 0 \,:\, (3.4) \text{ holds}\},$$

or equivalently

$$\|F\| := \sup\{|F(x_1, \dots, x_n)| \,:\, \|x_1, \dots, x_n\| \le 1\}.$$

This formula defines a norm on X^*.

We shall now discuss some multilinear n-functionals on ℓ^p (where $1 \le p < \infty$). Let $Y := \{y_1, \dots, y_n\}$ in $\ell^{p'}$, where p' is the dual exponent of p. We define

$$F_Y(x_1, \dots, x_n) := \frac{1}{n!} \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \begin{vmatrix} x_{1k_1} & \cdots & x_{1k_n} \\ \vdots & \ddots & \vdots \\ x_{nk_1} & \cdots & x_{nk_n} \end{vmatrix} \begin{vmatrix} y_{1k_1} & \cdots & y_{1k_n} \\ \vdots & \ddots & \vdots \\ y_{nk_1} & \cdots & y_{nk_n} \end{vmatrix}, \tag{3.5}$$

for $x_1, \dots, x_n \in \ell^p$. Clearly F_Y is linear in each variable. Further, we have

$$|F_Y(x_1, \dots, x_n)| \le \|x_1, \dots, x_n\|_p^H \|y_1, \dots, y_n\|_{p'}^H,$$

and so F_Y is bounded on $(\ell^p, \|\cdot, \dots, \cdot\|_p^H)$ with $\|F_Y\| \le \|y_1, \dots, y_n\|_{p'}^H$.

For $p = 2$, we have the following fact.

Fact 11 ([6])**.** Consider the n-normed space $(\ell^2, \|\cdot, \dots, \cdot\|_2^H)$. For fixed linearly independent $Y := \{y_1, \dots, y_n\}$ in ℓ^2, let F_Y be the multilinear n-functional defined as in (3.5). Then F_Y is bounded on $(\ell^2, \|\cdot, \dots, \cdot\|_2^H)$ with

$$\|F_Y\| = \|y_1, \dots, y_n\|_2^H.$$

Proof. From the inequality

$$|F_Y(x_1, \dots, x_n)| \le \|x_1, \dots, x_n\|_2^H \; \|y_1, \dots, y_n\|_2^H,$$

we see that F_Y is bounded with $\|F_Y\| \le \|y_1, \dots, y_n\|_2^H$. Next, if we take

$$x_i := \frac{y_i}{\sqrt[n]{\|y_1, \dots, y_n\|_2^H}}, \quad i = 1, \dots, n,$$

then $\|x_1, \dots, x_n\|_2^H = 1$ and $F_Y(x_1, \dots, x_n) = \|y_1, \dots, y_n\|_2^H$. Hence we conclude that $\|F_Y\| = \|y_1, \dots, y_n\|_2^H$. □

Regarding the n-functional F_Y on $(\ell^p, \|\cdot, \dots, \cdot\|_p^H)$, we have an open problem.

Problem 1. *Compute the exact norm of F_Y in* (3.5)*, especially for $p \ne 2$.*

Problem 2. *Can every bounded multilinear n-functional on ℓ^p be identified by $(y_1, \dots, y_n)$ where $y_i \in \ell^{p'}$, $i = 1, \dots, n$?*

Note that the multilinear n-functional F_Y may be reformulated as

$$F_Y(x_1,\dots,x_n) = \begin{vmatrix} \sum_{k=1}^{\infty} x_{1k}y_{1k} & \cdots & \sum_{k=1}^{\infty} x_{1k}y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{\infty} x_{nk}y_{1k} & \cdots & \sum_{k=1}^{\infty} x_{nk}y_{nk} \end{vmatrix}.$$

From this expression, we get the following result.

Fact 12. Let $e_k := (0,\dots,0,1,0,\dots)$ where the k-th term is the only term with value 1. Then, for $k_1,\dots,k_n \in \mathbb{N}$, we have

$$F_Y(e_{k_1},\dots,e_{k_n}) = \begin{vmatrix} y_{1k_1} & \cdots & y_{1k_n} \\ \vdots & \ddots & \vdots \\ y_{nk_1} & \cdots & y_{nk_n} \end{vmatrix}.$$

Accordingly, the multiindex sequence $\{F_Y(e_{k_1},\dots,e_{k_n})\}_{k_1,\dots,k_n}$ is p'-summable, in the sense that

$$\left[\frac{1}{n!}\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\left|\begin{vmatrix} y_{1k_1} & \cdots & y_{1k_n} \\ \vdots & \ddots & \vdots \\ y_{nk_1} & \cdots & y_{nk_n} \end{vmatrix}\right|^{p'}\right]^{\frac{1}{p'}} < \infty.$$

Proof. The first part is straightforward, while the second part follows from the fact that $y_1,\dots,y_n \in \ell^{p'}$ and that the sum is actually equal to $\|y_1,\dots,y_n\|_{p'}^{H}$. □

The following problem is still open.

Problem 3. *Let F be a bounded multilinear n-functional on ℓ^p. Must the multiindex sequence $\{F(e_{k_1},\dots,e_{k_n})\}_{k_1,\dots,k_n}$ be p'-summable?*

In general, the converse of Fact 11 holds, as follows. (We leave the proof to the reader.)

Proposition 13. *Let $c := \{c_{k_1\cdots k_n}\}_{k_1,\dots,k_n}$ be a multiindex sequence which is antisymmetric and p'-summable. Then, the n-functional F_c given by*

$$F_c(x_1,\dots,x_n) := \sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty} x_{1k_1}\cdots x_{nk_n}c_{k_1\cdots k_n}, \tag{3.6}$$

where $x_i := (x_{ik_i})_{k_i=1}^{\infty} \in \ell^p$ $(i = 1,\dots,n)$, is linear in each variable, and is bounded on $(\ell^p, \|\cdot,\dots,\cdot\|_p^H)$ with

$$\|F_c\| \le \left[\frac{1}{n!}\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}|c_{k_1\cdots k_n}|^{p'}\right]^{1/p'}.$$

Remark 14. Similar to Problem 1, we do not know the exact norm of the n-functional F_c in (3.6)

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