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New inequalities of Hermite-Hadamard type for functions whose derivatives in absolute value are convex with applications

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Abstract
In this paper new Hadamard-type inequalities, which estimate the difference between $\frac{1}{b-a}\int_a^b f(x)dx$ and $\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+4b}{4}\right)}{2}$, are established for functions whose derivatives in absolute values are convex. Our established results refine those results which have been established to estimate the difference between the middle and the leftmost terms of the celebrated Hermite-Hadamard inequality. We also give some applications of our obtained results to get some error bounds for the general quadrature formula. Finally, some applications to special means of real numbers are given as well.

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1 Introduction

The following definition for convex functions is well known in the mathematical literature:

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard's inequality, due to its rich geometrical significance and applications, which is stated as follow:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both the inequalities hold in reversed direction if f is concave. Since its discovery in 1883, Hermite-Hadamard's inequality [4] has been considered the most useful inequality in mathematical analysis. Some of the classical inequalities for means can be derived from (1.1) for particular choices of the function f.

In [3], S. S. Dragomir and R. P. Agarwal obtained the following results which give estimate between the middle and the rightmost terms in (1.1):

Theorem 1. *[3] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \left(\frac{b-a}{8}\right)\left[|f'(a)| + |f'(b)|\right]. \tag{1.2}$$

and

Theorem 2. *[3] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^p$ is convex on $[a,b]$ for some fixed $p > 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{1.3}$$

In [11], C. E. M. Pearce and J. E. Pečarić gave an improvement and simplification of the constant in Theorem 2 and consolidated this result with Theorem 1 as the following Theorem:

Theorem 3. *[11] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q \ge 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{\frac{1}{q}}. \tag{1.4}$$

In [11], C. E. M. Pearce and J. E. Pečarić also established the following result which gives the estimate between the middle and the leftmost terms in (1.1):

Theorem 4. *[11] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q \ge 1$, then the following inequality holds:*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{\frac{1}{q}}. \tag{1.5}$$

In [7, 8], U. S. Kirmaci et al. proved the following results connected with the left part of (1.1):

Theorem 5. *[8] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \left(\frac{b-a}{8}\right)\left[|f'(a)| + |f'(b)|\right]. \tag{1.6}$$

Theorem 6. *[7] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^p$ is convex on $[a,b]$ for some fixed $p > 1$, then the following inequality holds:*

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f\left(\frac{a+b}{2}\right) \right| \le \left(\frac{3^{1-\frac{1}{p}}}{8} \right) (b-a) \left[|f'(a)| + |f'(b)| \right]. \tag{1.7}$$

Theorem 7. *[7] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^p$ is concave on $[a,b]$ for $p \ge 1$ and $|f'|$ is a linear map, then the following inequality holds:*

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f\left(\frac{a+b}{2}\right) \right| \le \left(\frac{b-a}{8} \right) |f'(a+b)|. \tag{1.8}$$

For more results on Hermite-Hadamard-type inequality providing new proofs, noteworthy extensions, generalizations and numerous applications, see [1]-[16] and the references therein.

In a recent paper [14], K. L. Tseng et al., established the following result which gives a refinement of (1.1):

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} \le \frac{1}{b-a} \int_a^b f(x)dx \\ &\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] \le \frac{f(a)+f(b)}{2}, \end{aligned} \tag{1.9}$$

where $f : [a,b] \to \mathbb{R}$, is a convex function (see [12, Remark 2.11, page7.]).

The main aim of this paper is to establish some new Hermite-Hadamard type inequalities which give an estimate between $\frac{1}{b-a}\int_a^b f(x)dx$ and $\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}$ for functions whose derivatives in absolute value are convex and as a consequence we will get refinements of those results which have been established to estimate the difference between the middle and the leftmost terms in (1.1).

In Section 3, we will propose some new error bounds for the general quadrature formula based on our established results. Applications of our results to special means are also given in Section 4.

2 Main Results

To prove our results we need the following lemma:

Lemma 8. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° , the interior of I,* *where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx$$
$$=\frac{b-a}{16}\left[\int_0^1 tf'\left(t\frac{3a+b}{4}+(1-t)a\right)dt\right.$$
$$+\int_0^1 (t-1)f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)dt$$
$$+\int_0^1 tf'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)dt$$
$$\left.+\int_0^1 (t-1)f'\left(tb+(1-t)\frac{a+3b}{4}\right)dt\right]. \quad (2.1)$$

Proof. By integration by parts and by making use of the substitution $x = t\frac{3a+b}{4} + (1-t)a$, we have

$$\frac{b-a}{16}\int_0^1 tf'\left(t\frac{3a+b}{4}+(1-t)a\right)dt$$
$$=\frac{b-a}{16}\left[\left.\frac{4tf'\left(t\frac{3a+b}{4}+(1-t)a\right)}{b-a}\right|_0^1-\frac{4}{b-a}\int_0^1 f\left(t\frac{3a+b}{4}+(1-t)a\right)dt\right]$$
$$=\frac{1}{4}f\left(\frac{3a+b}{4}\right)-\frac{1}{b-a}\int_a^{\frac{3a+b}{4}} f(x)\,dx. \quad (2.2)$$

Analogously, we also have the following equalities:

$$\frac{b-a}{16}\int_0^1 (t-1)f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)dt$$
$$=\frac{1}{4}f\left(\frac{3a+b}{4}\right)-\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)\,dx, \quad (2.3)$$

$$\frac{b-a}{16}\int_0^1 tf'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)dt$$
$$=\frac{1}{4}f\left(\frac{a+3b}{4}\right)-\frac{1}{b-a}\int_{\frac{x+b}{2}}^{\frac{x+3b}{4}} f(x)\,dx \quad (2.4)$$

and

$$\frac{b-a}{16}\int_0^1 (t-1)f'\left(tb+(1-t)\frac{a+3b}{4}\right)dt$$
$$=\frac{1}{4}f\left(\frac{a+3b}{4}\right)-\frac{1}{b-a}\int_{\frac{x+3b}{4}}^{b} f(x)\,dx. \quad (2.5)$$

Adding (2.2)-(2.5), we get the desired equality. This completes the proof of the lemma. □

Using the Lemma 1 the following results can be obtained:

Theorem 9. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \left(\frac{b-a}{96}\right)\left[\left|f'(a)\right|+4\left|f'\left(\frac{3a+b}{4}\right)\right| +2\left|f'\left(\frac{a+b}{2}\right)\right|+4\left|f'\left(\frac{a+3b}{4}\right)\right|+\left|f'(b)\right|\right]. \quad (2.6)$$

Proof. Using Lemma 1 and taking the modulus, we have

$$\begin{aligned}&\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\leq \frac{b-a}{16}\left[\int_0^1 t\left|f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|dt\right.\\ &+\int_0^1 (1-t)\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|dt\\ &+\int_0^1 t\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|dt\\ &\left.+\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|dt\right]. \quad (2.7)\end{aligned}$$

Using the convexity of $|f'|$ on $[a,b]$, we observe that the following inequality holds:

$$\begin{aligned}&\int_0^1 t\left|f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|dt\\ &\leq \left|f'\left(\frac{3a+b}{4}\right)\right|\int_0^1 t^2dt+\left|f'(a)\right|\int_0^1 t(1-t)\,dt\\ &=\frac{1}{3}\left|f'\left(\frac{3a+b}{4}\right)\right|+\frac{1}{6}\left|f'(a)\right|. \quad (2.8)\end{aligned}$$

Similarly, we also have that the following inequalities hold:

$$\int_0^1 (1-t)\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|dt \leq \frac{1}{6}\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{1}{3}\left|f'\left(\frac{3a+b}{4}\right)\right|, \quad (2.9)$$

$$\int_0^1 t\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|dt \leq \frac{1}{3}\left|f'\left(\frac{a+3b}{4}\right)\right|+\frac{1}{6}\left|f'\left(\frac{a+b}{2}\right)\right|, \quad (2.10)$$

and

$$\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|dt \leq \frac{1}{6}\left|f'(b)\right|+\frac{1}{3}\left|f'\left(\frac{a+3b}{4}\right)\right|. \quad (2.11)$$

Utilizing the inequalities (2.8)-(2.11), we get (2.6).
This completes the proof of the theorem. □

Corollary 10. *Suppose all the conditions of Theorem 9 are satisfied. Then*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\left(\frac{b-a}{16}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{2.12}$$

Moreover, if $|f'(x)|\le M$, for all $x\in[a,b]$, then we have also the following inequality:

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\left(\frac{b-a}{8}\right)M. \tag{2.13}$$

Proof. It follows from Theorem 9 and using the convexity of $|f'|$. □

Theorem 11. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q>1$, then the following inequality holds:*

$$\begin{aligned}\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|&\le\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{b-a}{16}\right)\\ \times\Bigg\{\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}&+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)^{\frac{1}{q}}\\ +\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}&+\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}\Bigg\},\end{aligned} \tag{2.14}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and using the well-known Hölder integral inequality, we have

$$\begin{aligned}&\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\le\frac{b-a}{16}\Bigg[\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\\ &+\left(\int_0^1 (1-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\\ &+\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\\ &+\left(\int_0^1 (1-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned} \tag{2.15}$$

Since $|f'|^q$ is convex on $[a,b]$, we have

$$\begin{aligned}&\int_0^1\left|f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt\\ &\le\left|f'\left(\frac{3a+b}{4}\right)\right|^q\int_0^1 tdt+\left|f'(a)\right|^q\int_0^1(1-t)\,dt\\ &=\frac{1}{2}\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\frac{1}{2}\left|f'(a)\right|^q.\end{aligned}$$

Similarly,

$$\int_0^1 \left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt \le \frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \frac{1}{2}\left|f'\left(\frac{3a+b}{4}\right)\right|^q,$$

$$\int_0^1 \left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt \le \frac{1}{2}\left|f'\left(\frac{a+3b}{4}\right)\right|^q + \frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q$$

and

$$\int_0^1 \left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q dt \le \frac{1}{2}\left|f'\left(\frac{a+3b}{4}\right)\right|^q + \frac{1}{2}\left|f'(b)\right|^q.$$

Using the last four inequalities in (2.15), we get the inequality (2.14), which completes the proof of the theorem. □

Corollary 12. *Suppose all the conditions of Theorem 11 are satisfied. Then*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{3}{q}}\left[1+3^{\frac{1}{q}}+5^{\frac{1}{q}}+7^{\frac{1}{q}}\right]\left(\frac{b-a}{16}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \quad (2.16)$$

Proof. It follows from Theorem 11 using the convexity of $|f'|^q$ and the fact

$$\sum_{k=1}^n (u_k+v_k)^s \le \sum_{k=1}^n (u_k)^s + \sum_{k=1}^n (v_k)^s,\ u_k, v_k \ge 0, 1 \le k \le n, 0 \le s < 1.$$

□

Theorem 13. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q \ge 1$, then the following inequality holds:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}}\left(\frac{b-a}{16}\right)$$
$$\times\left\{\left(\left|f'(a)\right|^q+2\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+2\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+2\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)^{\frac{1}{q}}+\left(2\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\}. \quad (2.17)$$

Proof. Suppose that $q \ge 1$. From Lemma 1 and using the well-known power-mean inequality, we have

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{16}\left[\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(t\frac{3a+b}{4}+(1-t)\,a\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^1 (1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_0^1 (1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\right]. \quad (2.18)$$

Since $|f'|^q$ is convex on $[a,b]$, we have

$$\int_0^1 t\left|f'\left(t\frac{3a+b}{4}+(1-t)\,a\right)\right|^q dt$$
$$\leq \left|f'\left(\frac{3a+b}{4}\right)\right|^q\int_0^1 t^2dt+|f'(a)|^q\int_0^1 t(1-t)\,dt$$
$$=\frac{1}{3}\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\frac{1}{6}|f'(a)|^q.$$

Analogously, we also have that the following inequalities:

$$\int_0^1 (1-t)\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt$$
$$\leq \frac{1}{6}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{3}\left|f'\left(\frac{3a+b}{4}\right)\right|^q,$$

$$\int_0^1 t\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt\leq \frac{1}{3}\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\frac{1}{6}\left|f'\left(\frac{a+b}{2}\right)\right|^q$$

and

$$\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q\leq \frac{1}{3}\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\frac{1}{6}|f'(b)|^q.$$

By making use of the last four inequalities in (2.18), we get (2.17). Hence the proof of the theorem is complete. □

Corollary 14. *Suppose all the conditions of Theorem 11 are satisfied. Then using similar arguments as in Corollary 12, we get the following inequality:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\left(\frac{1}{3}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)\left[1+2^{\frac{1}{q}}+\left(\frac{1}{2}\right)^{\frac{1}{q}}+\left(\frac{5}{2}\right)^{\frac{1}{q}}\right]\left(\frac{b-a}{16}\right)[|f'(a)|+|f'(b)|]. \quad (2.19)$$

Theorem 15. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q > 1$, then the following inequality holds:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{b-a}{16}\right)\left\{\left|f'\left(\frac{7a+b}{8}\right)\right|+\left|f'\left(\frac{5a+3b}{8}\right)\right| + \left|f'\left(\frac{3a+5b}{8}\right)\right|+\left|f'\left(\frac{a+7b}{8}\right)\right|\right\}, \quad (2.20)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and using the well-known Hölder integral inequality for $q > 1$ and $p = \frac{q}{q-1}$, we have

$$\begin{aligned}&\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\le \left(\frac{b-a}{16}\right)\left[\left(\int_0^1 t^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\ &+\left(\int_0^1 (1-t)^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt\right)^{\frac{1}{q}} \\ &+\left(\int_0^1 t^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \\ &\left.+\left(\int_0^1 (1-t)^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\right]. \quad (2.21)\end{aligned}$$

Since $|f'|^q$ is concave on $[a,b]$ so by using the inequality (1.1), we obtain:

$$\int_0^1\left|f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt \le \left|f'\left(\frac{\frac{3a+b}{4}+a}{2}\right)\right|^q = \left|f'\left(\frac{7a+b}{8}\right)\right|^q$$

Analogously, we have that the following inequalities:

$$\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt \le \left|f'\left(\frac{5a+3b}{8}\right)\right|^q,$$

$$\int_0^1\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt \le \left|f'\left(\frac{3a+5b}{8}\right)\right|^q$$

and

$$\int_0^1\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q dt \le \left|f'\left(\frac{a+7b}{8}\right)\right|^q.$$

Using the last four inequalities in (2.21), we get (2.20). This completes the proof of the theorem. □

Corollary 16. *Suppose all the assumptions of Theorem 15 are satisfied and assume that $|f'|$ is a linear map, then we get the following inequality:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{b-a}{8}\right)\left|f'(a+b)\right|. \tag{2.22}$$

Proof. It is a direct consequence of Theorem 15 and using the linearity of $|f'|$. □

Remark 17. Since not all the convex functions are linear map, hence the inequality (2.22) can be used when $|f'|^q$ is concave on $[a,b]$ for some fixed $q>1$ and $|f'|$ is a linear map. Moreover, it can be observed that the error bound in (2.22) is more easier to calculate as compared to calculate it in (2.20) when $|f'|$ is a linear map.

Theorem 18. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q \ge 1$, then the following inequality holds:*

$$\begin{aligned}\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| &\le \left(\frac{b-a}{32}\right)\left[\left|f'\left(\frac{13a+3b}{12}\right)\right|+\left|f'\left(\frac{11a+5b}{12}\right)\right|\right.\\ &\quad\left.+\left|f'\left(\frac{5a+13b}{12}\right)\right|+\left|f'\left(\frac{3a+13b}{12}\right)\right|\right].\end{aligned} \tag{2.23}$$

Proof. First, by the concavity of $|f'|^q$ on $[a,b]$ and the power-mean inequality, we note that

$$\begin{aligned}|f(\lambda x+(1-\lambda)y)|^q &\ge \lambda|f(x)|^q+(1-\lambda)|f(y)|^q\\ &\ge (\lambda|f(x)|+(1-\lambda)|f(y)|)^q\end{aligned}$$

and hence

$$|f(\lambda x+(1-\lambda)y)| \ge \lambda|f(x)|+(1-\lambda)|f(y)|,$$

for all $\lambda \in [0,1]$ and $x, y \in [a,b]$. This shows that $|f'|$ is also concave on $[a,b]$.

Accordingly, using Lemma 1 and the Jensen's integral inequality, we have

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \left(\frac{b-a}{16}\right)\left[\int_0^1 t\left|f'\left(t\frac{3a+b}{4}+(1-t)\,a\right)\right|dt\right.$$
$$+\int_0^1 (1-t)\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|dt$$
$$\int_0^1 t\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|dt$$
$$\left.+\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|dt\right]$$
$$\leq \frac{b-a}{16}\left[\left(\int_0^1 tdt\right)\left|f'\left(\frac{\int_0^1 t\left(t\frac{3a+b}{4}+(1-t)\,a\right)dt}{\int_0^1 tdt}\right)\right|\right]$$
$$+\left(\int_0^1 (1-t)\,dt\right)\left|f'\left(\frac{\int_0^1 (1-t)\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)dt}{\int_0^1 (1-t)\,dt}\right)\right|$$
$$+\left(\int_0^1 tdt\right)\left|f'\left(\frac{\int_0^1 t\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)dt}{\int_0^1 tdt}\right)\right|$$
$$+\left(\int_0^1 (1-t)\,dt\right)\left|f'\left(\frac{\int_0^1 (1-t)\left(tb+(1-t)\frac{a+3b}{4}\right)dt}{\int_0^1 (1-t)\,dt}\right)\right|,$$

which is equivalent to (2.23) and the proof of the theorem is complete. □

Corollary 19. *Suppose all the assumptions of Theorem 18 are satisfied and assume that $|f'|$ is a linear map, then we have the following inequality:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\left(\frac{b-a}{12}\right)\left|f'(a+b)\right|. \tag{2.24}$$

Proof. It follows from Theorem 18 and using the linearity of $|f'|$. □

Remark 20. The error bound in (2.22) is more easier to calculate as compared to calculate it in (2.20) when $|f'|^q$ is concave on $[a,b]$ for some fixed $q \geq 1$ and $|f'|$ is a linear map.

3 Application to the General Quadrature Formula

Let $d : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$. Consider the general quadrature formula

$$\int_a^b f(x)dx = Q(f,d) + R(f,d), \tag{3.1}$$

where

$$Q(f,d) = \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right](x_{i+1}-x_i)$$

and $R(f,d)$ is the associated error. Here, we derive some estimates for the error $R(f,d)$ given in (3.1).

Proposition 21. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then for every division d of $[a,b]$, we have:*

$$|R(f,d)| \leq \frac{1}{96}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[|f'(x_i)| + 4\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right| + 2\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right| + 4\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right| + |f'(x_{i+1})|\right]. \quad (3.2)$$

Proof. By applying Theorem 9 on the subinterval $[x_i, x_{i+1}]$ $(i=0,1,\ldots,n-1)$ of the division d, we have

$$\left|\frac{1}{2}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right] - \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(x)\,dx\right| \leq \left(\frac{x_{i+1}-x_i}{96}\right)\left[|f'(x_i)| + 4\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right| + 2\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right| + 4\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right| + |f'(x_{i+1})|\right]. \quad (3.3)$$

Now

$$\begin{aligned}|R(f,d)| &= \left|\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)dx - \sum_{i=0}^{n-1}\frac{1}{2}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right](x_{i+1}-x_i)\right| \\ &\leq \sum_{i=0}^{n-1}(x_{i+1}-x_i)\left|\frac{1}{2}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right] - \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(x)dx\right|. \quad (3.4)\end{aligned}$$

Using (3.3) in (3.4), we get (3.2). This completes the proof of the proposition. □

Corollary 22. *Suppose all the assumptions of Proposition 21 are satisfied. Then*

$$|R(f,d)| \leq \frac{1}{16}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[|f'(x_i)| + |f'(x_{i+1})|\right]. \quad (3.5)$$

Proof. It follows from Proposition 21 and using the convexity of $|f'|$. □

Proposition 23. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q>1$,* *then for every division d of $[a,b]$, we have*

$$\begin{aligned}&|R(f,d)|\\&\leq\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}+4}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left\{\left(\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|^q+|f'(x_i)|^q\right)^{\frac{1}{q}}\right.\\&\qquad+\left(\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|^q\right)^{\frac{1}{q}}\\&\qquad+\left(\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|^q+\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q\right)^{\frac{1}{q}}\\&\qquad\left.+\left(\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|^q+|f'(x_{i+1})|^q\right)^{\frac{1}{q}}\right\},\qquad(3.6)\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. The proof is similar to that of Proposition 21 and using Theorem 11. □

Corollary 24. *Suppose all the conditions of Proposition 23 are satisfied. Then*

$$\begin{aligned}|R(f,d)|\leq&\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{3}{q}+4}\left[1+3^{\frac{1}{q}}+5^{\frac{1}{q}}+7^{\frac{1}{q}}\right]\\&\times\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[|f'(x_i)|+|f'(x_{i+1})|\right].\qquad(3.7)\end{aligned}$$

Proposition 25. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q\geq1$, then for every division d of $[a,b]$, we have*

$$\begin{aligned}|R(f,d)|\leq&\left(\frac{1}{32}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left\{\left(|f'(x_i)|^q+2\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|^q\right)^{\frac{1}{q}}\right.\\&+\left(\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+2\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|^q\right)^{\frac{1}{q}}\\&+\left(\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+2\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|^q\right)^{\frac{1}{q}}\\&\left.+\left(2\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|^q+|f'(x_{i+1})|^q\right)^{\frac{1}{q}}\right\}.\qquad(3.8)\end{aligned}$$

Proof. The proof is similar to that of Proposition 21 and using Theorem 13. □

Corollary 26. *Suppose all the conditions of Proposition 25 are satisfied. Then*

$$\begin{aligned}|R(f,d)|\leq&\left(\frac{1}{3}\right)^{\frac{1}{q}}\left(\frac{1}{32}\right)\left[1+2^{\frac{1}{q}}+\left(\frac{1}{2}\right)^{\frac{1}{q}}+\left(\frac{5}{2}\right)^{\frac{1}{q}}\right]\\&\times\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[|f'(x_i)|+|f'(x_{i+1})|\right].\qquad(3.9)\end{aligned}$$

Proposition 27. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q > 1$, then for every division d of $[a,b]$, we have*

$$|R(f,d)| \leq \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} \left(\frac{1}{16}\right) \sum_{i=0}^{n-1} (x_{i+1}-x_i)^2 \left\{\left|f'\left(\frac{7x_i+x_{i+1}}{8}\right)\right| + \left|f'\left(\frac{5x_i+3x_{i+1}}{8}\right)\right| + \left|f'\left(\frac{3x_i+5x_{i+1}}{8}\right)\right| + \left|f'\left(\frac{x_i+7x_{i+1}}{8}\right)\right|\right\}, \quad (3.10)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The proof is similar to that of Proposition 21 and it follows from Theorem 15. □

Corollary 28. *Suppose all the conditions of Proposition 27 are satisfied. If $|f'|$ is a linear mapping, then we have the following inequality:*

$$|R(f,d)| \leq \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} \left(\frac{1}{8}\right) \sum_{i=0}^{n-1} (x_{i+1}-x_i)^2 \left|f'\left(x_{i+1}+x_i\right)\right|. \quad (3.11)$$

Remark 29. It can be observed that the error bound in (3.11) for general quadrature formula is more easier to calculate as compared to calculate it in (3.10) when $|f'|^q$ is concave on $[a,b]$ for some fixed $q > 1$ and $|f'|$ is a linear map.

Proposition 30. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q \geq 1$ and $|f'|^q$ is a linear mapping, then for every division d of $[a,b]$, then the following inequality holds:*

$$|R(f,d)| \leq \left(\frac{1}{32}\right) \sum_{i=0}^{n-1} (x_{i+1}-x_i)^2 \left|f'\left(x_{i+1}+x_i\right)\right|. \quad (3.12)$$

Proof. The proof is similar to that of Proposition 21 and it follows from Theorem 18. □

4 Applications to Special Means

Now, we consider the applications of our Theorems to the special means. We consider the means for arbitrary real numbers a, $b \in \mathbb{R}$. We take

1. The arithmetic mean:
$$A(a,b) = \frac{a+b}{2};\ a,\ b \in \mathbb{R}.$$

2. The harmonic mean:
$$H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}};\ a,b \in \mathbb{R}\backslash\{0\}.$$

3. The logarithmic mean:
$$L(a,b) = \frac{\ln|b| - \ln|a|}{b-a};\ a,b \in \mathbb{R}, a \neq b, a,b \neq 0.$$

4. Generalized log-mean:

$$L_n(a,b) = \left[\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}}; \ a,\ b \in \mathbb{R},\ n \in \mathbb{Z} \setminus \{-1, 0\},\ a \neq b, a, b \neq 0.$$

Now using the results of Section 2, we give some applications to special means of real numbers.

Proposition 31. *Let $a,\ b \in \mathbb{R}$, $a < b$, $0 \notin [a,b]$ and $n \in \mathbb{Z}$, $|n| \geq 2$. Then*

$$\left| A\left(\left(\frac{3a+b}{4}\right)^n, \left(\frac{a+3b}{4}\right)^n\right) - L_n^n(a,b)\right| \leq |n| \left(\frac{b-a}{8}\right) A\left(|a|^{n-1}, |b|^{n-1}\right). \quad (4.1)$$

Proof. The assertion follows from Corollary 10 when applied to the function $f(x) = x^n$, $x \in [a,b]$, $n \in \mathbb{Z}$, $|n| \geq 2$. □

Proposition 32. *Let $a,\ b \in \mathbb{R}$, $a < b$, $0 \notin [a,b]$. Then*

$$\left| H^{-1}\left(\frac{3a+b}{4}, \frac{a+3b}{4}\right) - L(a,b)\right| \leq \left(\frac{b-a}{8}\right) A\left(|a|^{-2}, |b|^{-2}\right). \quad (4.2)$$

Proof. It is a direct consequence of Corollary 10 when applied to the function, $f(x) = \frac{1}{x}$, $x \in [a,b]$. □

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