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Conjecture on Dinitz Problem and Improvement of Hrnčiar's Result

Miroslav Haviar *
Faculty of Natural Sciences, M Bel University,
Tajovského 40, 974 01 Banská Bystrica, Slovakia
miroslav.haviar@umb.sk

Michal Ivaška †
Faculty of Natural Sciences, M Bel University,
Tajovského 40, 974 01 Banská Bystrica, Slovakia
michal.ivaska@umb.sk

Abstract

The paper aims at contributing to a better understanding of the Dinitz Problem by dealing with the number of "good choices" of representatives on a board of $n \times n$ cells. We conjecture that the number of good choices on an arbitrary board of order n is at least the number of good choices on a homogeneous board of order n, that is, at least the number $\ell(n)$ of Latin squares of order n. The first steps towards this conjecture are provided by proving that there are at least two good choices on an arbitrary board of order 3. This is slightly improving the result of Pavel Hrnčiar from 1991.

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1 Introduction

A simple-sounding problem introduced by Jeff Dinitz in 1978 asks whether on a board of $n \times n$ cells with n numbers in each cell one can choose a representative from every cell such that the selected numbers in each row and in each column are distinct (see e.g. [1, Chapter 28]). For arbitrary n the problem had been unsolved until Fred Galvin [2] presented his brilliant proof in 1995. But already in 1991 Pavel Hrnčiar gave a positive answer to the Dinitz Problem in the special case for $n = 3$. He showed that it is always possible to find one "good choice" of representatives on a board of 3×3 cells. The aim of our work is to present a conjecture on the number of "good choices" of representatives on the board of $n \times n$ cells (Section 3) and to prove that there are always at least two "good choices" of representatives on the board of 3×3 cells (Section 5).

The major part of our work deals with the concept of a *kernel* of a directed graph (Section 4). It is a subset of vertices satisfying two special conditions and it is amazingly connected to "good choices" on a board via so-called *square graphs* corresponding

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to boards. We show that every nondiscrete induced subgraph of the square graph corresponding to some board of 3×3 cells possesses at least two different kernels for some possible edge orientations.

We also introduce a new concept of so-called *tame choices* on a diagonal of a square graph, which is also connected to the existence of “good choices" of representatives (Section 5). In Section 6 we present various conjectures and counterexamples that arose in the process of our investigation.

2 Preliminaries

2.1 Dinitz Problem

For $n \geq 1$ consider n^2 cells arranged in an $(n \times n)$-square, let us call it a *board of order* n, and let (i, j) denote the cell in row i and column j. Suppose that for every cell (i, j) we are given a set $C(i, j)$ of n colours.

By a *choice* we mean that for each cell (i, j), exactly one colour is picked up from the set $C(i, j)$. Let a *good choice* be every choice in which the colours in each row and each column are distinct.

Is it then always possible to find a good choice for any board?

This simple-sounding colouring problem was raised by Jeff Dinitz in 1978 and it defied all attacks until its solution by Fred Galvin [2].

Let $C := \bigcup_{i,j} C(i, j)$ be a set of all colours of a board and let $|C|$ be *size* of the board.

It is worth to mention a particular case as presented in [1, p. 185]. If all colour sets are the same, say $\{1, 2, \dots, n\}$, then the Dinitz problem reduces to the following task: fill in the $(n \times n)$-square with the numbers $1, 2, \dots, n$ in such a way that the numbers in any row and column are distinct. This means that size of the board is n and all choices on it are precisely Latin squares. Since this is so easy, why would it be so much harder in the general case when the size is greater than n? The difficulty derives from the fact that not every colour of C is available at each cell.

2.2 Galvin's Proof

All definitions and results in this subsection are taken from [1, Chapter 28].

Definition 2.1. Let $G = (V, E)$ be a graph. Let us assume that we are given a nonempty set $C(v)$ of colours for each vertex $v \in V$. A **list colouring** is a colouring $c : V \longrightarrow \bigcup_{v \in V} C(v)$ where $c(v) \in C(v)$ for each $v \in V$ (a colouring is an assignment of colors to each vertex such that no edge connects two identically coloured vertices). A **list chromatic number** $\chi_\ell(G)$ is the smallest number k such that for any list of colour sets $C(v)$ with $|C(v)| = k$ for all $v \in V$ there always exists a list colouring.

Consider the *square graph* S_n which has as a vertex set the n^2 cells of our board of order n and two cells are adjacent if and only if they lie in the same row or column (see Figure 1). The Dinitz problem can now be stated as

$$\chi_\ell(S_n) = n?$$

Definition 2.2. Let $\vec{G} = (V, E)$ be a directed graph (shortly, digraph), that is, a graph where every edge e has an orientation. The notation $e = (u, v)$ means that there is an edge e, also denoted by $u \longrightarrow v$, whose initial vertex is u and whose terminal vertex is v. Then **outdegree** $d^+(v)$ of a vertex v is the number of edges with v as initial vertex, similarly for the **indegree** $d^-(v)$.

Furthermore, $d^+(v) + d^-(v) = d(v)$, where $d(v)$ is the degree of v.

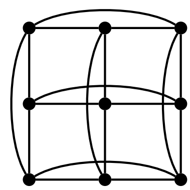


Figure 1. The graph S_3

Definition 2.3. For a graph $G = (V, E)$ and a non-empty subset $A \subseteq V$ we denote by $G[A]$ the subgraph which has A as vertex set and which contains all edges of G between vertices of A. We call $G[A]$ the **subgraph induced by** A, and say that H is an **induced subgraph** of G if $H = G[A]$ for some A.

Definition 2.4. Let $G = (V, E)$ be a graph without loops and multiple edges. A set $A \subseteq V$ is called **independent** if there are no edges within A.

Definition 2.5. Let $\vec{G} = (V, E)$ be a directed graph. A **kernel** $K \subseteq V$ is a subset of vertices such that

(1) K is independent in G, and

(2) for every $u \notin K$ there exists a vertex $v \in K$ with an edge $u \longrightarrow v$.

For example, vertices of a kernel of the subgraph of a graph $\vec{S_3}$ shown in the Figure 2 are encircled. (We remark that here, and often elsewhere, we use the term "graph" for "directed graph (digraph)" when no confusion arises.)

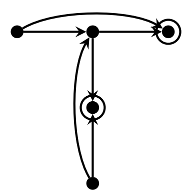


Figure 2. Kernel of the graph

In what follows, when we write G we mean the graph $\vec{G}$ without the orientations.

Lemma 2.6 ([1, Lemma 1])**.** Let $\vec{G} = (V, E)$ be a directed graph, and suppose that for each vertex $v \in V$ we have a color set that is larger than the outdegree, $|C(v)| \geq d^+(v)+1$. If every induced subgraph of $\vec{G}$ possesses a kernel, then there exists a list colouring of G with a colour from $C(v)$ for each v.

Denote the vertices of S_n by (i, j), $1 \leq i, j \leq n$. Thus (i, j) and (r, s) are adjacent if and only if $i = r$ or $j = s$. Take any Latin square L with letters from $\{1, 2, \ldots, n\}$ and denote by $L(i, j)$ the entry in cell (i, j). Next make S_n into a directed graph $\vec{S_n}$ by orienting the horizontal edges $(i, j) \longrightarrow (i, j')$ if $L(i, j) < L(i, j')$ and the vertical edges $(i, j) \longrightarrow (i', j)$ if $L(i, j) > L(i', j)$. Thus, horizontally we orient from the smaller to the larger element, and vertically the other way round. We shall denote this digraph $\vec{S_n}^L$ to emphasize that the orientation of edges is given by a Latin square L.

Notice that we obtain $d^+(i,j) = n-1$ for all (i,j). In fact, if $L(i,j) = k$, then $n-k$ cells in row i contain an entry larger than k, and $k-1$ cells in column j have an entry smaller than k.

The next result amazingly follows from the fact that a stable matching of a bipartite graph always exists (cf. [1, Lemma 2]).

Lemma 2.7 ([1, p. 189])**.** Every induced subgraph of $\vec{S_n}$ possesses a kernel.

Putting these two lemmas together with the fact that $d^+(i,j) = n-1$ for all (i,j), we get Galvin's solution [2] of the Dinitz Problem.

Theorem 2.8 ([1, p. 189])**.** We have $\chi_L(S_n) = n$ for all n.

2.3 Colouring Algorithm

Galvin's proof can tell us how to colour any board B of order n. We can use the following algorithm:

(1) choose any Latin square L (of the same order n as the board B);

(2) assign a digraph $\vec{S_n}^L$ with edge orientations given by L;

(3) choose any colour $c \in C$, where $C = \bigcup_{i,j} C(i,j)$;

(4) colour c generates a subgraph $\vec{S_n}^L[A]$, where $A = \{v \in V, c \in C(v)\}$;

(5) choose any kernel of this subgraph;

(6) colour the vertices of the kernel by a colour c;

(7) repeat steps 3–6 with colours c not previously used until the colouring is complete.

Galvin's proof implicitly says that after a finite number of steps (not more than s steps, where s is the size of the board) the colouring is complete and we obtain a good choice.

Notice that in this case every Latin square combined with any sequence of colours gives us a good choice according to the colouring algorithm. However, not all good choices are obtainable by this algorithm. It can even happen that two distinct Latin squares or two distinct sequences of colours can give us the same good choice (see Section 6).

3 Conjecture on the number of good choices

In this section we formulate a conjecture on the number of good choices on an arbitrary board which we find quite important with respect to a good understanding of the Dinitz Problem.

Let B_n be a board of order n. We shall denote $\sigma(B_n)$ the number of all distinct good choices on B_n. Galvin has shown that $\sigma(B_n) \geq 1$ for any board B_n.

By a *homogeneous* board of order n we shall mean a board of $n \times n$ cells with the same set $\{1, 2, \ldots, n\}$ of numbers in each cell. Hence the size of the homogeneous board is equal to its order.

Let $\ell(n)$ be the number of all Latin squares of order n. It is clear that $\ell(n)$ is the number of good choices on a homogeneous board of order n. So, $\sigma(B_n) = \ell(n)$ if the board B_n is homogeneous.

The following conjecture says that $\ell(n)$ is the optimal lower bound for the number of good choices on an *arbitrary* board of order n.

Conjecture. $\sigma(B_n) \geq \ell(n)$ for any board B_n.

The following table lists the values of $\ell(n)$, which are so far known for $1 \leq n \leq 11$ [4]. Thus, for given n, these are our conjectured optimal lower bounds for the number of good choices on an arbitrary board of order n.

n	$\ell(n)$
1	1
2	2
3	12
4	576
5	161280
6	812851200
7	61479419904000
8	108776032459082956800
9	5524751496156892842531225600
10	9982437658213039871725064756920320000
11	776966836171770144107444346734230682311065600000

4 Graph Kernel

In the next definition we introduce a new concept regarding graph kernels of induced subgraphs of a square graph.

Definition 4.1. Let $S_n[A]$ be a subgraph of a square graph S_n induced by some set A of vertices. We say that the graph $S_n[A]$ is k-**kerneled** if for some Latin squares $L_1, L_2, \ldots, L_m$ the digraphs $\vec{S_n}^{L_1}[A]$, $\vec{S_n}^{L_2}[A], \ldots$, $\vec{S_n}^{L_m}[A]$ have together at least k distinct kernels.

Lemma 4.2. Let $S_n[A]$ be a discrete graph. Then it is 1-kerneled and it is not k-kerneled for any $k > 1$.

Proof. It is easy to see that the only kernel in the discrete graph is the whole vertex set. $\square$

In Figure 3 (on the left) we draw the same digraph as in Figure 2 and present the Latin square corresponding to its edge orientations. The kernel of this directed graph is encircled. Now we focus on the vertex u. If we orient all edges towards u and make a new digraph, then u must be in the kernel of this new digraph, because of the second condition of the graph kernel. We want to find some Latin square such that it will correspond to the orientation of this new graph. But it is easy, because we only need the entry in the cell corresponding to the vertex u to be lower than the entries in the cells corresponding to the vertices connected with u. So the entry in the cell corresponding to the vertex u will be 1. One of the possible Latin squares is shown on the right side of the Figure 3. The graph next to it is the graph with the new edge orientations and with a new kernel encircled.

The method described above can be simply generalised. It suffices to have some edge with end vertices whose degrees do not exceed 2, because in that case it is possible to orient all edges into one of its end vertices. By the first kernel condition, two end vertices of one edge cannot both belong to one kernel. We take the vertex which is not there and construct a new orientation of the graph, where this taken vertex will already be in some kernel.

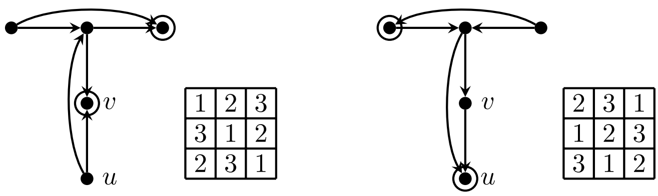


Figure 3. Two kernels of the same graph with changed edge orientations

Lemma 4.3. If there exists an edge $uv \in S_3[A]$ such that $d(u) \leq 2$ and $d(v) \leq 2$, then $S_3[A]$ is 2-kerneled.

Proof. From Lemma 2.7 we know that $\vec{S_3}^{L_1}[A]$ has a kernel, given by some Latin square L_1. We denote this kernel K_1. We need to show that there exists a kernel $K_2 \neq K_1$. We will do it the way that we change the orientations of edges in $\vec{S_3}^{L_1}[A]$ and obtain the new kernel K_2 of $\vec{S_3}^{L_2}[A]$, which will correspond to some Latin square L_2. Since a kernel is a set of independent vertices and $uv \in S_3[A]$, then $u \notin K_1$ or $v \notin K_1$. Without loss of generality we can assume that $u \notin K_1$. As $d(u) \leq 2$, we can orient all edges in $S_3^{L_2}[A]$ in such a way that $d^-(u) = d(u)$ and $d^+(u) = 0$. Note that a Latin square L_2 which will give such orientations always exists. Now $d^+(u) = 0$ implies that $u \in K_2$, because from the Definition 2.5 all vertices which are not in the kernel must be initial vertices of some edge with terminal vertex in the kernel (and so their outdegree must be at least 1). Thus $K_1 \neq K_2$ and the proof is complete. $\square$

Definition 4.4. Let $S_n[A]$ be an induced subgraph of S_n and let $1 \leq r \leq n$. Then the r-th **row** R_r of the graph $S_n[A]$ is the set of vertices $\{(r, j) \in S_n[A],\ 1 \leq j \leq n\}$.

For the graph S_n and for every $i \in \{1, 2, \ldots, n\}$ we have $|R_i| = n$. For any induced subgraph $S_n[A]$ we have $|R_i| \leq n$.

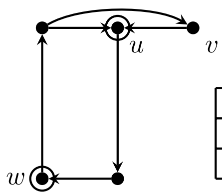
Lemma 4.5. Let $S_3[A]$ be an induced subgraph such that $|R_i| = 3$ and $|R_j| = 0$ for some $1 \leq i, j \leq 3$, $i \neq j$. Then $S_3[A]$ is 2-kerneled.

Proof. Let $1 \leq k \leq 3$, $k \notin \{i, j\}$ (note that such k is unique). If $|R_k| = 0$ then by Lemma 4.3, $S_3[A]$ is 2-kerneled. So let $|R_k| \geq 1$. We can choose any vertex $w \in R_k$ (see Figure 4, in this case $i = 1$, $j = 2$ and $k = 3$). Then there exist vertices $u, v \in R_i$ such that u, w and v, w are independent. Now take any of these two pairs, for example take u, w and construct a Latin square which has an entry 3 in the cells corresponding to the vertices u, w. Since 3 is the biggest number in the Latin square of order 3, all vertices in the same line will be directed into vertices u, w. But this already means that $\{u, w\}$ is a kernel. For the pair v, w it can be showed analogously. Thus $\{u, w\}$ and $\{v, w\}$ are two distinct kernels of $S_3[A]$ and so $S_3[A]$ is 2-kerneled. $\square$

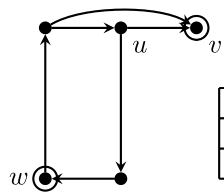
Definition 4.6. A **diagonal** of a graph $S_n[A]$ is any set of n independent vertices.

Lemma 4.7. Every graph $S_n[A]$ containing k different diagonals is k-kerneled.

Proof. To a given diagonal we can take any Latin square of order n which has an entry n in all the cells corresponding to the vertices of the diagonal. Then the edge from all the other vertices will be oriented towards these vertices, so the diagonal will be a kernel. $\square$



1	3	2
2	1	3
3	2	1



1	2	3
2	3	1
3	1	2

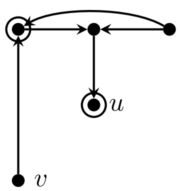
Figure 4. Two kernels of the same graph with an empty row

Clearly, the graph S_n has $n!$ diagonals. Certainly it can happen that some of its induced subgraphs contain no diagonals (for example the graph in Figure 4).

Lemma 4.8. Every graph $S_3[A]$ in which $|R_i| = 3$, for some $i \in \{1, 2, 3\}$, and which contains exactly one pair of independent vertices not contained in R_i, is 2-kerneled.

Proof. Note that such a graph always contains one diagonal, because the two independent vertices can be supplemented by an independent vertex from the i-th row. So by Lemma 4.7, this diagonal is already a kernel. We will show that there always exists a kernel K such that $|K| = 2$ (i.e. different from the first one).

So take any of the two independent vertices u and v. Say we take u. Then we take the vertex from the i-th row, which is independent with u, but not independent with v (there is exactly one such vertex). Up to isomorphism we can assume that we have one of the graphs in Figure 5. Let us first consider the graph on the left side. Its kernel corresponding to the Latin square is encircled. One can notice that if we add a vertex so that there will still be exactly one independent pair of vertices, up to isomorphism we obtain the graph on the right side of Figure 5. Now we can use the same Latin square as before to orient the edges, and the kernel will remain the same. □



2	3	1
1	2	3
3	1	2

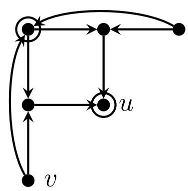


Figure 5. Graphs with one full row and one independent pair of vertices

Theorem 4.9. Every nondiscrete subgraph of S_3 is 2-kerneled.

Proof. Let $S_3[A]$ be a nondiscrete subgraph of S_3. If maximal degree of its vertices $\Delta(S_3[A]) \leq 2$, then $S_3[A]$ has two different kernels by Lemma 4.3. So now let there exist a vertex v with $d(v) > 2$. It is easy to see that then for the line R_i, where $v \in R_i$, we have $|R_i| = 3$ and $|R_j| \geq 1$ for some $1 \leq j \leq 3$, $i \neq j$. Without loss of generality we can assume that $|R_1| = 3$ and $|R_2| \geq 1$. Now we have these possibilities:
1. $|R_3| = 0$ — then the statement holds by Lemma 4.5.
2. $|R_2| = 1$ and $|R_3| = 1$ — if the two vertices in R_2 and R_3 are independent, the statement holds by Lemma 4.8, otherwise by Lemma 4.3.
3. $|R_2| = 2$ and $|R_3| = 1$ — if there is exactly one independent pair of vertices, the statement holds by Lemma 4.8. Otherwise there are two or more independent pairs in which case each of them can be completed to a diagonal with a vertex in R_1, so the

statement holds by Lemma 4.7.

4. $|R_2| \geq 2$ and $|R_3| \geq 2$ — the statement holds by Lemma 4.7.

The proof is complete. □

5 From Kernels to Good Choices

Theorem 5.1. Let c be a colour of a board B of order n and let A be the set of all vertices of the graph S_n corresponding to those cells which contain the colour c. If the graph $S_n[A]$ is k-kerneled, then the board B has at least k distinct good choices.

Proof. Let $S_n[A]$ be a k-kerneled induced subgraph of the graph S_n, i.e. there exist distinct kernels $K_1, K_2, \ldots, K_k$ of the digraphs $\vec{S_n}^{L_i}[A]$ with orientations given by some Latin squares $L_1, L_2, \ldots, L_k$, respectively. According to the colouring algorithm from Galvin's proof (see Subsection 2.2), we take the Latin square L_1. The Latin square gives us an orientation of the graph S_n and in the first step we choose a colour c (in the final good choice the colour c will remain precisely in those cells corresponding to the vertices of the kernel K_1). After this, we continue with an arbitrary sequence of colours (see Subsection 2.3) until we obtain a good choice. We denote the obtained good choice by D_1. Similarly, when we choose in the first step the Latin square L_2 and the same colour c, we obtain a good choice that we can denote D_2. We do the same for all the other kernels so that we have good choices $D_1, D_2, \ldots, D_k$. Note that the sets of cells with a colour c are distinct in all these good choices, since all the kernels were distinct. Thus we obtained k distinct good choices and the proof is complete. □

We recall that Galvin [2] has shown that $\sigma(B_n) \geq 1$ for any board B_n of order n.

Before Galvin, in 1991 Hrnčiar [3] showed that $\sigma(B_3) \geq 1$ for any board of order 3. We conjecture in Section 3 that $\sigma(B_3) \geq 12$. Our following result is improving the lower bound for $\sigma(B_3)$ and so can be understood as the first little step towards proving the conjecture.

Theorem 5.2. $\sigma(B_3) \geq 2$ for any board of order 3.

Proof. Let B_3 be a board of order 3 and let S_3 be its assigned square graph. We will distinguish two cases: (1) every colour of the board is only once in the same row or column, (2) there is a colour c such that it is at least twice in the same row or column.

(1) Let every colour be only once in the same row or column. We take any cell of the board, denote it by E. It contains 3 colours, say a, b, c. Note that in this case the induced subgraph of S_3 generated by any colour is discrete and so its kernel is the whole subgraph. Now according to the colouring algorithm we can take any Latin square and in the first step we take colour a. As the whole subgraph belongs to the kernel, we colour the cell E by the colour a. Then we continue with an arbitrary colour sequence to obtain a good choice. Similarly, when we take a colour b in the first step, we obtain a good choice with b in the cell E. So we constructed two distinct good choices.

(2) Let c be a colour of B_3 such that it is at least twice in the same row or column. Then the subgraph generated by this colour is nondiscrete, and, by Theorem 4.9, it is 2-kerneled. Now by Theorem 5.1, it has two distinct good choices. □

In the last part of this section we introduce some concepts corresponding to a board and present our final result.

Definition 5.3. Let B be a board of order n and let a **diagonal of a board** B be a set of any n cells such that none of them lie in the same row or column. Then any set of n colours from distinct cells of a diagonal will be called the **choice on a diagonal**.

Let B be a board and let D be a choice on some diagonal. We denote every colour in D as $d_{(a,b)}$ to express that this colour was chosen from the cell (a, b). Now for every cell (i, j) of B and for every colour $d_{(k,l)}$ of D we delete this colour from the cell (i, j) if $i = k$ or $j = l$, i.e. we delete every colour of the choice on a diagonal from all cells which lie in the same row or column.

Definition 5.4. We shall call the choice D **tame** if a reduced board obtained by the process above has at least $n - 1$ remaining colours in every cell.

For example, the choice $\{1, 2, 5\}$ on the diagonal $(1, 1), (2, 2), (3, 3)$ in the board on the left below is tame, because the reduced board on the right has at least 2 colours remaining in every cell.

$\{\mathbf{1}, 2, 3\}$	$\{3, 4, 5\}$	$\{2, 3, 5\}$
$\{4, 5, 6\}$	$\{\mathbf{2}, 3, 4\}$	$\{1, 4, 6\}$
$\{2, 3, 5\}$	$\{1, 2, 6\}$	$\{4, \mathbf{5}, 6\}$

$\{2, 3\}$	$\{3, 4, 5\}$	$\{2, 3\}$
$\{4, 5, 6\}$	$\{3, 4\}$	$\{1, 4, 6\}$
$\{2, 3\}$	$\{1, 6\}$	$\{4, 6\}$

As we shall see, the existence of tame choices guarantees the existence of good choices.

Theorem 5.5. Let B be a board. If there exist k tame choices on some diagonal of B, then $\sigma(B) \geq k$.

Proof. Denote this diagonal by D. We take any Latin square L such that it has an entry n in all cells corresponding to the diagonal D, where n is the order of L. We consider the square digraph $\vec{S_n}^L$ with edge orientations given by the Latin square L. The outdegree of every vertex will then be $n - 1$. Now we can delete all the vertices corresponding to D from $\vec{S_n}^L$. Since in every row there was an edge oriented from every vertex to the diagonal, now after we deleted it, the outdegree of every vertex will be $n - 2$.

For every tame choice we can now also delete all colours of a choice from all lists of colours for every vertex. By the definition, every vertex will still have at least $n - 1$ colours. So $|C(v)| \geq n - 1$ for every v. Now by Lemma 2.6, the subgraph with deleted diagonals can be list coloured with colours from $C(v)$ for every v. The deleted vertices can be coloured with colours of a choice on a diagonal and we obtain a good choice.

We can do the same for all tame choices and we obtain k distinct good choices on B. $\square$

6 Misguided Conjectures and Counterexamples

In this section we present various conjectures arising in the process of our investigation and the counterexamples to these conjectures which we later found using (in almost all cases) self-developed computer programs. The aim of this section is to give a helpful hand to those who would follow similar steps as we did and come up with possibly the same conjectures in the process of their investigation.

Misguided conjecture 6.1. Let B be a board of order n. Then there exists a diagonal of B which has at least n tame choices.

Counterexample 6.1. The following board has exactly one tame choice in every diagonal:

$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$
$\{1, 2, 4\}$	$\{1, 2, 4\}$	$\{1, 2, 4\}$
$\{1, 3, 4\}$	$\{1, 3, 4\}$	$\{2, 3, 4\}$

Misguided conjecture 6.2. Let B be a board of order n. Then every diagonal of B has at least one tame choice.

Counterexample 6.2. The following board has no tame choices on the diagonal (1,1), (2,2), (3,3):

$\{1,2,3\}$	$\{1,3,4\}$	$\{3,4,6\}$
$\{2,3,4\}$	$\{3,4,5\}$	$\{2,3,4\}$
$\{1,2,4\}$	$\{2,5,6\}$	$\{2,4,6\}$

Colours of a board can be in general any natural numbers, but note that if a board has size s then we can replace all numbers greater than s with numbers smaller than or equal to s. So we can always obtain the board with colours $\{1,2,\dots,s\}$. We shall call this process a *board normalisation* and the board obtained this way we shall call a *normalised board*.

Misguided conjecture 6.3. Let B be a normalised board of size s. Let L_1, L_2 be distinct Latin squares and let S be a sequence of colours from $\{1,2,\dots,s\}$. Let D_1, D_2 be the good choices obtained from the Latin squares L_1, L_2, respectively, combined with the colour sequence S. Then D_1 and D_2 are distinct.

Counterexample 6.3. Consider the following board:

$\{1,2,3\}$	$\{2,3,4\}$	$\{3,4,5\}$
$\{2,3,5\}$	$\{1,3,4\}$	$\{1,3,4\}$
$\{1,4,5\}$	$\{1,2,3\}$	$\{1,3,5\}$

Now if we take the following two Latin squares

1	2	3
3	1	2
2	3	1

1	3	2
2	1	3
3	2	1

and we use the sequence of colours $1,2,3,4,5$, we will obtain the same good choice:

1	2	4
2	3	1
4	1	3

Misguided conjecture 6.4. Let B be a normalised board of size s. Let L be a Latin square and let S_1, S_2 be distinct sequences of colours from $\{1,2,\dots,s\}$. Let D_1, D_2 be the good choices obtained from the Latin square L combined with the colour sequences S_1, S_2, respectively. Then D_1 and D_2 are distinct.

Counterexample 6.4. Take the same board as in Counterexample 6.3, the Latin square

1	2	3
2	3	1
3	1	2

and the sequences $2, 3, 4, 1, 5$ and $4, 3, 2, 5, 1$. Then for both sequences we obtain the same good choice:

2	3	4
3	4	1
4	2	3

Misguided conjecture 6.5. Let B be a normalised board of size s. Let L_1, L_2 be distinct Latin squares and let S_1, S_2 be distinct sequences of colours from $\{1, 2, \ldots, s\}$. Let D_1, D_2 be the good choices obtained from the Latin squares L_1, L_2 combined with the colour sequences S_1, S_2, respectively. Then D_1 and D_2 are distinct.

Counterexample 6.5. Consider the same board as in Counterexample 6.3, the Latin squares

2	1	3
1	3	2
3	2	1

3	1	2
1	2	3
3	2	1

and for these Latin squares take sequences of colours $1, 2, 4, 5, 3$ and $4, 2, 5, 1, 3$, respectively. In both cases we obtain the same good choice:

1	2	4
2	4	1
4	1	5

Misguided conjecture 6.6. Every good choice on a board can be obtained via the colouring algorithm (from Subsection 2.3) using some Latin square and some sequence of colours.

Counterexample 6.6. We can take the same board as in Counterexample 6.3. Then the following good choice can not be obtained via the colouring algorithm for any Latin square and sequence of colours:

3	4	5
2	1	3
4	2	1

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