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An application of generalized Bessel functions on certain analytic functions

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Abstract

The purpose of the present paper is to investigate some characterization for generalized Bessel functions of first kind to be in various subclasses of analytic functions. We also consider an integral operator related to the generalized Bessel function.

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1 Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Further, we denote by S the subclass of A consisting of functions of the form 1.1 which are also univalent in U and T be the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \tag{1.2}$$

Let $T(\lambda, \alpha)$ be the subclass of T consisting of functions which satisfy the condition

$$Re\left\{\frac{zf'(z)}{\lambda z f'(z) + (1-\lambda) f(z)}\right\} > \alpha, \tag{1.3}$$

for some $\alpha (0 \leq \alpha < 1)$, $\lambda (0 \leq \lambda < 1)$ and for all $z \in U$.

Also, we let $C(\lambda, \alpha)$ denote the subclass of T consisting of functions which satisfy the condition

$$Re\left\{\frac{f'(z)+zf''(z)}{f'(z)+\lambda zf''(z)}\right\} > \alpha, \tag{1.4}$$

for some $\alpha(0 \le \alpha < 1)$, $\lambda(0 \le \lambda < 1)$ and for all $z \in U$.

From 1.3 and 1.4 it is easy to verify that

$$f(z) \in C(\lambda, \alpha) \Leftrightarrow zf'(z) \in T(\lambda, \alpha).$$

The classes $T(\lambda, \alpha)$ and $C(\lambda, \alpha)$ were extensively studied by Altintas and Owa [1] and certain conditions for hypergeometric functions for these classes were studied by Mostafa [11].

It is worthy to note that $T(0, \alpha) \equiv T^*(\alpha)$, the class of starlike functions of order $\alpha(0 \le \alpha < 1)$ and $C(0, \alpha) \equiv C(\alpha)$, the class of convex functions of order $\alpha(0 \le \alpha < 1)$ (see [13]).

We recall that the generalized Bessel function of the first kind $w = w_{p,b,c}$ is defined as the particular solution of the second-order linear homogenous differential equation

$$z^2\omega''(z) + bz\omega'(z) + \left[cz^2 - p^2 + (1-b)p\right]\omega(z) = 0, \tag{1.5}$$

where $b, p, c \in C$, which is a natural generalization of Bessel's equation. This function has the familiar representation

$$\omega(z) = \omega_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n!\Gamma(p+n+\frac{b+1}{2})}\left(\frac{z}{2}\right)^{2n+p}, \quad z \in C. \tag{1.6}$$

The differential equation 1.5 permits the study of Bessel function, modified Bessel function, spherical Bessel function and modified spherical Bessel functions all together. Solutions of 1.5 are referred to as the generalized Bessel function of order p. The particular solution given by 1.6 is called the generalized Bessel function of the first kind of order p. Although the series defined above is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in U. It is worth mentioning that, in particular, when $b = c = 1$, we reobtain the Bessel function $\omega_{p,1,1} = J_p$, and for $c = -1, b = 1$ the function $\omega_{p,1,-1}$ becomes the modified Bessel function I_p. Now, consider the function $u_{p,b,c}$ defined by the transformation

$$u_{p,b,c}(z) = 2^p\Gamma\left(p + \frac{b+1}{2}\right) z^{-p/2}\omega_{p,b,c}(z^{1/2}).$$

By using the well-known Pochhammer (or Appell) symbol, defined in terms of the Euler Gamma function for $a \neq 0, -1, -2, ...$ by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0 \\ a(a+1)...(a+n-1) & \text{if } n = 1, 2, 3, ..., \end{cases},$$

we obtain for the function $u_{p,b,c}$ the following representation

$$u_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c/4)^n}{\left(p + \frac{(b+1)}{2}\right)_n} \frac{z^n}{n!}, \tag{1.7}$$

where $p + (b+1)/2 \neq 0, -1, -2, ...$ This function is analytic on C and satisfies the second-order linear differential equation

$$4z^2u''(z) + 2(2p+b+1)zu'(z) + czu(z) = 0.$$

The convolution (or Hadamard product) of two series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Now, we considered a linear operator $I(k,c) : A \to A$ defined by

$$\begin{aligned} I(k,c)f &= z u_{p,b,c}(z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} a_n z^n, \end{aligned}$$

where $k = p + \frac{b+1}{2}$. The generalized Bessel function is a recent topic of study in Geometric Function Theory (e.g. see the work of [2], [3], [4], [5] and [10]). Motivated by results on connections between various subclasses of analytic univalent functions by using hypergeometric functions (see [6], [7], [9], [12], [14], [15]) and by work of Baricz [2]-[5], we obtain sufficient condition for function $z(2 - u_p(z))$ belonging to the classes $T(\lambda, \alpha)$, $C(\lambda, \alpha)$ and connections between $R^{\tau}(A, B)$ and $C(\lambda, \alpha)$. Finally, we give a condition for an integral operator $G(k, c, z)$ belonging to the class $C(\lambda, \alpha)$.

For convenience throughout in the sequel, we use the following notations:

$$u_{p,b,c} = u_p, \quad k = p + \frac{b+1}{2}.$$

2 Main Results

To establish our main results, we shall require the following lemmas due to Dixit and Pal [8], Altintas and Owa [1] and Baricz [4].

Lemma 1. *([8]) If $f \in R^{\tau}(A, B)$ is of the form 1.1 then*

$$|a_n| \leq \frac{(A-B)|\tau|}{n}, \quad (n \in N \setminus \{1\}). \tag{2.1}$$

The bounds given in 2.1 is sharp.

Lemma 2. *([1]) A function $f(z)$ defined by 1.2 is in the class $T(\lambda, \alpha)$, if and only if*

$$\sum_{n=2}^{\infty} [n - \lambda\alpha n - \alpha + \lambda\alpha] \, |a_n| \leq 1 - \alpha.$$

Lemma 3. *([1]) A function $f(z)$ defined by 1.2 is in the class $C(\lambda, \alpha)$, if and only if*

$$\sum_{n=2}^{\infty} n \, [n - \lambda\alpha n - \alpha + \lambda\alpha] \, |a_n| \leq 1 - \alpha.$$

Lemma 4. *([4]) If $b, p, c \in C$ and $k \neq 0, -1, -2, \ldots$ then the function u_p satisfies the recursive relation $4k u_p'(z) = -c u_{p+1}(z)$ for all $z \in C$.*

Theorem 5. *If $c < 0$, $k > 0 (k \neq 0, -1, -2, \ldots)$, then $z(2 - u_p(z))$ is in $T(\lambda, \alpha)$ if and only if*

$$(1 - \alpha\lambda) u_p'(1) + (1 - \alpha) u_p(1) \leq 2(1 - \alpha), \tag{2.2}$$

Proof. Since

$$z\left(2-u_p(z)\right)=z-\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}z^n,$$

according to Lemma 2, we must show that

$$\sum_{n=2}^{\infty}\left[n(1-\lambda\alpha)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\leq 1-\alpha.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left[n(1-\alpha\lambda)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\
&=\sum_{n=0}^{\infty}\left[(n+2)(1-\alpha\lambda)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!}\\
&=(1-\alpha\lambda)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}n!}+(1-\alpha)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!}\\
&=(1-\alpha\lambda)u_p'(1)+(1-\alpha)\left[u_p(1)-1\right].
\end{aligned}$$

But this last expression is bounded above by $1-\alpha$ if and only if 2.2 holds. Thus the proof of Theorem 5 is established. □

Remark 6. In particular when $c=-1$ and $b=1$, the condition 2.2 becomes

$$2^{p-2}\Gamma(p+1)\left[(1-\alpha\lambda)I_{p+1}(1)+2(1-\alpha)I_p(1)\right]\leq 1-\alpha, \tag{2.3}$$

which is a necessary and sufficient condition for $z\left(2-\zeta_p(z^{1/2})\right)$ to be in $T(\lambda,\alpha)$, where

$$\zeta_p(z^{1/2})=2^p\Gamma(p+1)z^{-p/2}I_p(z^{1/2}). \tag{2.4}$$

Theorem 7. *If $c<0$, $k>0(k\neq 0,-1,-2,...)$, then $z\left(2-u_p(z)\right)$ is in $C(\lambda,\alpha)$ if and only if*

$$(1-\alpha\lambda)u_p''(1)+(3-2\alpha\lambda-\alpha)u_p'(1)+(1-\alpha)u_p(1)\leq 2(1-\alpha), \tag{2.5}$$

Proof. Since

$$z\left(2-u_p(z)\right)=z-\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}z^n,$$

according to Lemma 3, we must show that

$$\sum_{n=2}^{\infty}n\left[n(1-\lambda\alpha)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\leq 1-\alpha.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} n\left[n(1-\alpha\lambda)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\
=&\sum_{n=2}^{\infty}\left\{(1-\alpha\lambda)(n-1)(n-2)+(3-2\alpha\lambda-\alpha)(n-1)+(1-\alpha)\right\}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\
=&(1-\alpha\lambda)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-3)!}+(3-2\alpha\lambda-\alpha)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-2)!}+(1-\alpha)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\
=&(1-\alpha\lambda)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}(n-1)!}+(3-2\alpha\lambda-\alpha)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}n!}+(1-\alpha)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!}\\
=&(1-\alpha\lambda)\frac{(-c/4)^2}{k(k+1)}\sum_{n=0}^{\infty}\frac{(-c/4)^{n-1}}{(k+2)_{n-1}(n-1)!}+(3-2\alpha\lambda-\alpha)\frac{(-c/4)}{k}\sum_{n=0}^{\infty}\frac{(-c/4)^{n}}{(k+1)_{n}n!}+(1-\alpha)\left\{u_p(1)-1\right\}\\
=&(1-\alpha\lambda)\frac{(-c/4)^2}{k(k+1)}u_{p+2}(1)+(3-2\alpha\lambda-\alpha)\frac{(-c/4)}{k}u_{p+1}(1)+(1-\alpha)\left\{u_p(1)-1\right\}\\
=&(1-\alpha\lambda)u_p''(1)+(3-2\alpha\lambda-\alpha)u_p'(1)+(1-\alpha)\left\{u_p(1)-1\right\}
\end{aligned}$$

But this last expression is bounded above by $1-\alpha$ if and only if 2.5 holds. This completes the proof of Theorem 7. □

Theorem 8. *Let $c<0$, $k>0(k\neq 0,-1,-2,...)$. If $f\in R^{\tau}(A,B)$ and the inequality*

$$(A-B)|\tau|\left[(1-\alpha\lambda)u_p'(1)+(1-\alpha)\left\{u_p(1)-1\right\}\right]\leq 1-\alpha, \tag{2.6}$$

is satisfied then $I(k,c)f\in C(\lambda,\alpha)$.

Proof. By Lemma 3, it suffices to show that

$$P_1=\sum_{n=2}^{\infty} n\left[n-\lambda\alpha n-\alpha+\lambda\alpha\right]|a_n|\leq 1-\alpha.$$

Since $f\in R^{\tau}(A,B)$ then by Lemma 1 we have

$$|a_n|\leq\frac{(A-B)|\tau|}{n}.$$

Hence

$$\begin{aligned}
P_1\leq&(A-B)|\tau|\sum_{n=2}^{\infty}\left[n(1-\alpha\lambda)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\
=&(A-B)|\tau|\sum_{n=0}^{\infty}\left[(n+2)(1-\alpha\lambda)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!}\\
=&(A-B)|\tau|\left[(1-\alpha\lambda)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}n!}+(1-\alpha)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!}\right]\\
=&(A-B)|\tau|\left[(1-\alpha\lambda)\frac{(-c/4)}{k}\sum_{n=0}^{\infty}\frac{(-c/4)^{n}}{(k+1)_{n}n!}+(1-\alpha)\left\{u_p(1)-1\right\}\right]\\
=&(A-B)|\tau|\left[(1-\alpha\lambda)\frac{(-c/4)}{k}u_{p+1}(1)+(1-\alpha)\left\{u_p(1)-1\right\}\right]\\
=&(A-B)|\tau|\left[(1-\alpha\lambda)u_p'(1)+(1-\alpha)\left\{u_p(1)-1\right\}\right].
\end{aligned}$$

But this last expression is bounded above by $1-\alpha$ if and only if 2.6 holds. $\square$

3 An Integral Operator

In the following theorem, we obtain similar results in connection with a particular integral operator $G(k,c,z)$ as follows

$$G(k,c,z)=\int_0^z (2-u_p(t))\,dt \tag{3.1}$$

Theorem 9. *If $c<0$, $k>0(k\neq 0,-1,-2,...)$, then $G(k,c,z)$ defined by 3.1 is in $C(\lambda,\alpha)$ if and only if*

$$(1-\alpha\lambda)u_p'(1)+(1-\alpha)\left[u_p(1)-1\right]\leq(1-\alpha). \tag{3.2}$$

Proof. Since

$$\begin{aligned}G(k,c,z)&=z-\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\frac{z^n}{n}\\&=z-\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}}\frac{z^n}{n!}\end{aligned}$$

by Lemma 3, we need only to show that

$$\sum_{n=2}^{\infty}n\left[n(1-\lambda\alpha)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}n!}\leq 1-\alpha.$$

Now

$$\begin{aligned}&\sum_{n=2}^{\infty}n\left[n(1-\lambda\alpha)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}n!}\\=&\sum_{n=2}^{\infty}\left[n(1-\lambda\alpha)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\=&\sum_{n=0}^{\infty}\left[(n+2)(1-\lambda\alpha)-\alpha(1-\lambda)\right]\frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!}\\=&(1-\alpha\lambda)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}n!}+(1-\alpha)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!}\\=&(1-\alpha\lambda)u_p'(1)+(1-\alpha)\left[u_p(1)-1,\right]\end{aligned}$$

which is bounded above by $1-\alpha$, if and only if 3.2 holds. $\square$

Remark 10. If we put $c=-1$ and $b=1$ in Theorem 7-9 we obtain analogues results of 2.3.

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