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$\mathcal{I}$-derivative

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Abstract

This paper deals with a derivative of a real function based on the notion of $\mathcal{I}$-convergence.

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1 Introduction

In the connection to some known results about functions preserving $\mathcal{I}$-convergence and $\mathcal{I}$-continuity (see [1, 2, 3, 4]) we introduce the $\mathcal{I}$-derivative of a real function, i.e. the derivative based on the notion of $\mathcal{I}$-convergence.

$\mathcal{I}$-convergence was introduced in [1] as a generalization of statistical convergence (see [5, 6]).

In this paper we will elucidate the relationship of $\mathcal{I}$-derivative to usual derivative with respect to the choice of ideals used in the definition of $\mathcal{I}$-derivative.

Definition 1. (see [7, p. 6]) A non-void family $\mathcal{I}$ of subsets of a given set X is called an *ideal* on X if it is hereditary and additive, i.e.

(1) $A \in \mathcal{I}$ and $B \subset A \Rightarrow B \in \mathcal{I}$,

(2) $A \in \mathcal{I}$ and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$.

An ideal $\mathcal{I}$ is called a *proper* ideal if $X \notin \mathcal{I}$.

A proper ideal $\mathcal{I}$ is said to be *admissible* (see [1]) if $\mathcal{I}$ contains every singleton.

The dual notion to the notion of an ideal is the notion of a filter.

Definition 2. (see [7, p. 6]) A non-void family $\mathcal{F}$ of subsets of a given set X is called a *filter* on X if

(1) $A \in \mathcal{F}$ and $A \subset B \Rightarrow B \in \mathcal{F}$,

(2) $A \in \mathcal{F}$ and $B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$.

A filter $\mathcal{F}$ is *proper* if $\emptyset \notin \mathcal{F}$.

Obviously, for each ideal $\mathcal{I}$ the system

$$\mathcal{F}(\mathcal{I}) = \{X \setminus A : A \in \mathcal{I}\}$$

is a filter on X.

Definition 3. (see [1]) Let $\mathcal{I}$ be a proper ideal on the set $\mathbb{N}$. A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-*convergent* to $\xi \in \mathbb{R}$ ($\mathcal{I}$-$\lim x_n = \xi$) if and only if for each $\varepsilon > 0$ the set

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - \xi| \geq \varepsilon\}$$

belongs to $\mathcal{I}$. The element ξ is called $\mathcal{I}$-limit of the sequence $(x_n)_{n\in\mathbb{N}}$.

In what follows we recall some basic properties of $\mathcal{I}$-convergence and of the notions $\mathcal{I}$-limit inferior and $\mathcal{I}$-limit superior(see [8, 9]).

Theorem 4. *(see [9]) Let $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ be sequences of real numbers such that $\mathcal{I}$-$\lim x_n = \xi$, $\mathcal{I}$-$\lim y_n = \eta$. Then*

(a) $\mathcal{I}$-$\lim(x_n \cdot y_n) = \xi \cdot \eta$,

(b) $\mathcal{I}$-$\lim(x_n + y_n) = \xi + \eta$.

Let $t \in \mathbb{R}$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Put

$$M_t = \{n : x_n > t\}, \qquad M^t = \{n : x_n < t\}.$$

Definition 5. (see [9])

(a) If there is a $t \in \mathbb{R}$ such that $M_t \notin \mathcal{I}$, we put

$$\mathcal{I}\text{-}\limsup x_n = \sup\{t \in \mathbb{R} : M_t \notin \mathcal{I}\}.$$

If $M_t \in \mathcal{I}$ for each $t \in \mathbb{R}$, then $\mathcal{I}\text{-}\limsup x_n = -\infty$.

(b) If there is a $t \in \mathbb{R}$ such that $M^t \notin \mathcal{I}$, we put

$$\mathcal{I}\text{-}\liminf x_n = \inf\{t \in \mathbb{R} : M^t \notin \mathcal{I}\}.$$

If $M^t \in \mathcal{I}$ for each $t \in \mathbb{R}$, then $\mathcal{I}\text{-}\liminf x_n = +\infty$.

Theorem 6. *(see [9]) The inequality*

$$\mathcal{I}\text{-}\liminf x_n \leq \mathcal{I}\text{-}\limsup x_n$$

holds for every sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers.

Theorem 7. *(see [9]) The sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is $\mathcal{I}$-convergent if and only if*

$$\mathcal{I}\text{-}\liminf x_n = \mathcal{I}\text{-}\limsup x_n.$$

If this equality holds, then

$$\mathcal{I}\text{-}\lim x_n = \mathcal{I}\text{-}\liminf x_n = \mathcal{I}\text{-}\limsup x_n.$$

We use various types of convergences in the article. We also often use sequences of real numbers with restrictions on their members. In order to keep the text comprehensible we introduce following notations.

Let $\mathbb{R}^{\mathbb{N}}$ denote the set of all sequences of real numbers. Let $x = (x_n)_{n\in\mathbb{N}}$. We put

$$\mathcal{S} = \{x \in \mathbb{R}^{\mathbb{N}} : x_n \neq 0, n \in \mathbb{N}\},$$

$$\mathcal{S}_{\mathcal{I}} = \{x : \mathcal{I}\text{-lim}\, x_n = 0, x_n \neq 0, n \in \mathbb{N}\}$$

and

$$\mathcal{S}_{\mathcal{I}}^{+} = \{x \in \mathcal{S}_{\mathcal{I}} : x_n > 0, n \in \mathbb{N}\},\ \mathcal{S}_{\mathcal{I}}^{-} = \{x \in \mathcal{S}_{\mathcal{I}} : x_n < 0, n \in \mathbb{N}\}.$$

Note that $\mathcal{S}_{\mathcal{I}_f}$, where $\mathcal{I}_f$ is the Fréchet ideal, contains only sequences convergent in the usual sense.

2 $\mathcal{I}$-derivative

Definition 8. Let $\mathcal{I}$ be an admissible ideal on the set $\mathbb{N}$. A function $f : \mathbb{R} \to \mathbb{R}$ has an *$\mathcal{I}$-derivative* $d \in \mathbb{R}$ at a point x_0, i.e. $\mathcal{I}$-$f'(x_0) = d$, if and only if

$$\mathcal{I}\text{-lim} \frac{f(x_0 + x_n) - f(x_0)}{x_n} = d \tag{2.1}$$

holds for each sequence $(x_n)_{n\in\mathbb{N}} \in \mathcal{S}_{\mathcal{I}}$.

In [2, Theorem 1] it is showed that if a function $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{I}$-continuous at a point x_0, i.e.

$$\mathcal{I}\text{-lim}\, x_n = x_0 \Rightarrow \mathcal{I}\text{-lim}\, f(x_n) = f(x_0) \tag{2.2}$$

holds for each sequence $(x_n)_{n\in\mathbb{N}}$, then it is continuous at the point x_0.

Proposition 9. *Let a function $f : \mathbb{R} \to \mathbb{R}$ have an $\mathcal{I}$-derivative $d \in \mathbb{R}$ at a point x_0 where $\mathcal{I}$ is an admissible ideal on the set $\mathbb{N}$. Then f is continuous at the point x_0.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \in \mathcal{S}_{\mathcal{I}}$. Obviously

$$f(x_0 + x_n) - f(x_0) = \frac{f(x_0 + x_n) - f(x_0)}{x_n} \cdot x_n$$

holds for each $n \in \mathbb{N}$. Thus

$$\mathcal{I}\text{-lim}(f(x_0 + x_n) - f(x_0)) = \mathcal{I}\text{-lim} \frac{f(x_0 + x_n) - f(x_0)}{x_n} \cdot x_n.$$

According to the assumption and Theorem 4 we have

$$\mathcal{I}\text{-lim}\, f(x_0 + x_n) = f(x_0)$$

what means that (2.2) holds for each sequence of real numbers and so f is continuous at the point x_0. □

For $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ put

$$D^{+}f(x_0) = \limsup_{h\to 0^{+}} \frac{f(x_0 + h) - f(x_0)}{h},$$

$$D_{+}f(x_0) = \liminf_{h\to 0^{+}} \frac{f(x_0 + h) - f(x_0)}{h},$$

$$D^- f(x_0) = \limsup_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h},$$

$$D_- f(x_0) = \liminf_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h}.$$

The numbers $D^+f(x_0), D_+f(x_0), D^-f(x_0), D_-f(x_0)$ are called Dini's derivatives (see e.g. [10, p. 27]) and it is well known that all Dini's derivatives at the point x_0 are equal to $d \in \mathbb{R}$ if and only if $f'(x_0) = d$.

For each $x \in \mathcal{S}_\mathcal{I}^+$ we put

$$D_x^+ f(x_0) = \mathcal{I}\text{-}\limsup \frac{f(x_0+x_n)-f(x_0)}{x_n},$$

$$D_+^x f(x_0) = \mathcal{I}\text{-}\liminf \frac{f(x_0+x_n)-f(x_0)}{x_n},$$

and for each $y \in \mathcal{S}_\mathcal{I}^-$

$$D_y^- f(x_0) = \mathcal{I}\text{-}\limsup \frac{f(x_0+y_n)-f(x_0)}{y_n},$$

$$D_-^y f(x_0) = \mathcal{I}\text{-}\liminf \frac{f(x_0+y_n)-f(x_0)}{y_n}.$$

Lemma 10. *Let $f:\mathbb{R}\to\mathbb{R}$ and $x_0\in\mathbb{R}$. Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers such that $\mathcal{I}$-$\lim x_n = x_0$, $x_n \neq x_0$ for each $n\in\mathbb{N}$ where $\mathcal{I}$ is an admissible ideal on the set $\mathbb{N}$. Then*

$$\liminf_{x\to x_0} f(x) \leq \mathcal{I}\text{-}\liminf f(x_n), \tag{2.3}$$

$$\mathcal{I}\text{-}\limsup f(x_n) \leq \limsup_{x\to x_0} f(x). \tag{2.4}$$

Proof. We will prove (2.3), the proof of (2.4) is quite similar.

Let $\lambda > 0$. Put $O_\lambda = \{f(x) : 0 < |x-x_0| < \lambda\}$. Because $(x_n)_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent, the set $\{n : |x_n - x_0| \geq \lambda\}$ belongs to $\mathcal{I}$ what implies $\{n : f(x_n) \in \mathbb{R}\setminus O_\lambda\} \in \mathcal{I}$. Let $s = \inf O_\lambda$. Case $s = -\infty$ is trivial so suppose that $s\in\mathbb{R}$. It is obvious that $\{f(x_n) : f(x_n) < s\} \subset \mathbb{R}\setminus O_\lambda$ and so $\{n : f(x_n) < s\}$ belongs to $\mathcal{I}$ what implies $s \notin \{t\in\mathbb{R} : \{n : f(x_n) < t\} \notin \mathcal{I}\}$. From the arbitrariness of the choice of λ we have

$$\sup_{\lambda>0}\{\inf O_\lambda\} \leq \inf\{t\in\mathbb{R} : \{n : f(x_n) < t\} \notin \mathcal{I}\}$$

and the statement holds. □

Theorem 11. *Let $\mathcal{I}$ be an admissible ideal on the set $\mathbb{N}$. A function $f:\mathbb{R}\to\mathbb{R}$ has a derivative $d\in\mathbb{R}$ at a point x_0 if and only if $\mathcal{I}$-$f'(x_0) = d$.*

Proof. Let $f'(x_0) = d$. According to Theorem 6, Theorem 7 and Lemma 10 for each $x \in \mathcal{S}_\mathcal{I}^+$ we have

$$D_+ f(x_0) \leq D_+^x f(x_0) \leq D_x^+ f(x_0) \leq D^+ f(x_0)$$

and for each $y \in \mathcal{S}_\mathcal{I}^-$

$$D_- f(x_0) \leq D_-^y f(x_0) \leq D_y^- f(x_0) \leq D^- f(x_0).$$

Therefore (2.1) holds for each sequence from $\mathcal{S}_{\mathcal{I}}^+ \cup \mathcal{S}_{\mathcal{I}}^-$. It is sufficient to show, that (2.1) holds for each $z \in \mathcal{S}_{\mathcal{I}} \setminus (\mathcal{S}_{\mathcal{I}}^+ \cup \mathcal{S}_{\mathcal{I}}^-)$. By contradiction. Let there exists $z \in \mathcal{S}_{\mathcal{I}} \setminus (\mathcal{S}_{\mathcal{I}}^+ \cup \mathcal{S}_{\mathcal{I}}^-)$ such that (2.1) does not hold. For $\eta > 0$ we put

$$K_\eta = \left\{ n : \frac{f(x_0 + z_n) - f(x_0)}{z_n} \notin B(d, \eta) \right\},$$

where $B(d, \eta)$ is open ball with center d and radius η. There is η_0 such that $K_{\eta_0} \notin \mathcal{I}$. Hence at least one of sets

$$A = \{n : n \in K_{\eta_0}, z_n < 0\}, B = \{n : n \in K_{\eta_0}, z_n > 0\}$$

does not belong to $\mathcal{I}$ and it is infinite. Let it be the set B. Define the sequence $s = (s_n)_{n\in\mathbb{N}}$ as follows. For $n \in B$ put $s_n = z_n$ and $s_n = \frac{1}{n}$ if $n \in \mathbb{N} \setminus B$. Obviously $s \in \mathcal{S}_{\mathcal{I}}^+ \cup \mathcal{S}_{\mathcal{I}}^-$, a contradiction. If the set A does not belong to $\mathcal{I}$ for $n \in A$ we put $s_n = z_n$ and $s_n = -\frac{1}{n}$ if $n \in \mathbb{N} \setminus A$.

Suppose now, that f does not have the derivative at the point x_0. This implication follows immediately from the fact, that if one of Dini's derivatives is equal to $s \in \mathbb{R}$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ convergent to x_0 such that the sequence $(y_n)_{n\in\mathbb{N}}$ of numbers

$$y_n = \frac{f(x_0 + x_n) - f(x_0)}{x_n}$$

converges to s or if $s = +\infty(-\infty)$ there is a sequence $(x_n)_{n\in\mathbb{N}}$ convergent to x_0 such that the sequence $(y_n)_{n\in\mathbb{N}}$ is increasing (decreasing) with limit $+\infty(-\infty)$. Hence for each admissible ideal $\mathcal{I}$ the function f does not have $\mathcal{I}$-derivative at the point x_0. $\square$

In Definition 8 we have used the same ideal $\mathcal{I}$ for $\mathcal{I}$-convergence in $\mathcal{S}_{\mathcal{I}}$ and in (2.1). It is quite natural to ask what (if any) difference in our results will be reached by using various ideals.

Definition 12. Let $\mathcal{I}_1$, $\mathcal{I}_2$ be admissible ideals on the set $\mathbb{N}$. A function $f : \mathbb{R} \to \mathbb{R}$ has a $(\mathcal{I}_1, \mathcal{I}_2)$-*derivative* $d \in \mathbb{R}$ at a point x_0, i.e. $(\mathcal{I}_1, \mathcal{I}_2)$-$f'(x_0) = d$, if and only if

$$\mathcal{I}_1\text{-}\lim x_n = 0 \Rightarrow \mathcal{I}_2\text{-}\lim \frac{f(x_0 + x_n) - f(x_0)}{x_n} = d \tag{2.5}$$

holds for each sequence $(x_n)_{n\in\mathbb{N}} \in \mathcal{S}$.

Remark 13. Studying the proof of Theorem 11 we find that in case $\mathcal{I}_1 \subset \mathcal{I}_2$ we get the same results for $(\mathcal{I}_1, \mathcal{I}_2)$-derivative as well. The difference is reached if $\mathcal{I}_1 \setminus \mathcal{I}_2 \neq \emptyset$. The following theorem says that in this case $(\mathcal{I}_1, \mathcal{I}_2)$-derivative is no longer only local property of a real function at a point.

Theorem 14. *Let $\mathcal{I}_1$, $\mathcal{I}_2$ be admissible ideals on the set $\mathbb{N}$ such that $\mathcal{I}_1 \setminus \mathcal{I}_2 \neq \emptyset$. Then a function $f : \mathbb{R} \to \mathbb{R}$ has a $(\mathcal{I}_1, \mathcal{I}_2)$-derivative $d \in \mathbb{R}$ at a point x_0 if and only if f is a linear function.*

Proof. The case if f is a linear function is trivial.

Suppose that f is not linear and $(\mathcal{I}_1, \mathcal{I}_2)$-$f'(x_0) = d$. So there exists $z \in \mathbb{R} \setminus \{0\}$ such that

$$\frac{f(x_0 + z) - f(x_0)}{z} = d' \neq d. \tag{2.6}$$

Because $\mathcal{I}_1$, $\mathcal{I}_2$ are both admissible, there is an infinite set $A \in \mathcal{I}_1 \setminus \mathcal{I}_2$. Define sequence $(x_n)_{n\in\mathbb{N}}$ as follows. If $n \in A$ put $x_n = z$ else $x_n = \frac{1}{n}$. Obviously $\mathcal{I}_1$-$\lim x_n = 0$. Let $\eta > 0$ be such that $d' \notin B(d, \eta)$. Then

$$\left\{n : \frac{f(x_0 + x_n) - f(x_0)}{x_n} \notin B(d, \eta)\right\} \notin \mathcal{I}_2.$$

That is a contradiction because for the sequence $(x_n)_{n\in\mathbb{N}}$ the implication (2.5) does not hold. □

Simultaneously with $\mathcal{I}$-convergence another closely related kind of convergence called $\mathcal{I}^*$-convergence was introduced and investigated in [1, 9, 3], later generalized as $\mathcal{I}^{\mathcal{K}}$-convergence in [11].

Let $\mathcal{I}$ be an ideal on a set S and X be a topological space. A function $f : S \to X$ is said to be $\mathcal{I}$-convergent to $x \in X$ if

$$f^{-1}(U) = \{s \in S : f(s) \in U\} \in \mathcal{F}(\mathcal{I})$$

holds for each neighborhood of x.

Definition 15. (see [11]) Let $\mathcal{I}$, $\mathcal{K}$ be ideals on a set S. Let X be a topological space and $x \in X$. A function $f : S \to X$ is said to be $\mathcal{I}^{\mathcal{K}}$*-convergent* to x if there exists a set $M \in \mathcal{F}(\mathcal{I})$ such that the function $g : S \to X$ given by

$$g(s) = \begin{cases} f(s), & \text{if } s \in M \\ x, & \text{otherwise} \end{cases}$$

is $\mathcal{K}$-convergent to x.

Proposition 16. *Let $\mathcal{I}$ be an arbitrary and $\mathcal{K}$ an admissible ideal on the set $\mathbb{N}$. Let $(y_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, $\lim y_n = x$. Then a real sequence $(x_n)_{n\in\mathbb{N}}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to x if and only if there is a set $M \in \mathcal{F}(\mathcal{I})$ such that the sequence $(z_n)_{n\in\mathbb{N}}$, $z_n = x_n$ for $n \in M$ otherwise $z_n = y_n$, is $\mathcal{K}$-convergent to x.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be $\mathcal{I}^{\mathcal{K}}$-convergent to x, $\eta > 0$. Let M be the corresponding set belonging to $\mathcal{F}(\mathcal{I})$. Then $U = \{n : z_n \notin B(x, \eta) \wedge n \in M\}$ belongs to $\mathcal{K}$. The set $V = \{n : z_n \notin B(x, \eta) \wedge n \notin M\}$ is always finite or empty and since $\mathcal{K}$ is admissible the union $U \cup V$ belongs to $\mathcal{K}$ as well.

The converse implication can be proved using similar consideration. □

In [11] it is showed that $\mathcal{I}^{\mathcal{K}}$-convergence implies $\mathcal{I}$-convergence if $\mathcal{K} \subset \mathcal{I}$. The converse implication holds (assuming that X is a first countable topological space) if $\mathcal{I}$ has additive property with respect to $\mathcal{K}$, or more briefly that the condition $AP(\mathcal{I}, \mathcal{K})$ holds. The condition $AP(\mathcal{I}, \mathcal{K})$ holds, if for every sequence of mutually disjoint sets $(A_n)_{n\in\mathbb{N}}$ belonging to $\mathcal{I}$ there is a sequence $(B_n)_{n\in\mathbb{N}}$ of sets belonging to $\mathcal{I}$ such that $A_n \triangle B_n \in \mathcal{K}$ for $n \in \mathbb{N}$ and $B = \bigcup_{n\in\mathbb{N}} B_n \in \mathcal{I}$.

Definition 17. Let $\mathcal{I}$, $\mathcal{K}$ be admissible ideals on the set $\mathbb{N}$.
A function $f : \mathbb{R} \to \mathbb{R}$ has an $\mathcal{I}^{\mathcal{K}}$*-derivative* $d \in \mathbb{R}$ at a point x_0, i.e. $\mathcal{I}^{\mathcal{K}}$-$f'(x_0) = d$, if and only if

$$\mathcal{I}^{\mathcal{K}}\text{-}\lim x_n = 0 \Rightarrow \mathcal{I}^{\mathcal{K}}\text{-}\lim \frac{f(x_0 + x_n) - f(x_n)}{x_n} = d$$

holds for each sequence $(x_n)_{n\in\mathbb{N}} \in \mathcal{S}$.

Theorem 18. *Let $\mathcal{I}$, $\mathcal{K}$ be admissible ideals on the set $\mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$.*

(a) If $f'(x_0) = d$, then $\mathcal{I}^{\mathcal{K}}$-$f'(x_0) = d$.

(b) If $\mathcal{I}$ fulfils condition $AP(\mathcal{I}, \mathcal{K})$, $\mathcal{K} \subset \mathcal{I}$ and $\mathcal{I}^{\mathcal{K}}$-$f'(x_0) = d$, then $f'(x_0) = d$.

Proof. (a) Let $(x_n)_{n\in\mathbb{N}} \in \mathcal{S}_{\mathcal{I}^{\mathcal{K}}}$, $(y_n)_{n\in\mathbb{N}} \in \mathcal{S}_{\mathcal{I}_f}$. Thus there is a set $M \in \mathcal{F}(I)$ such that the sequence $(z_n)_{n\in\mathbb{N}}$, $z_n = x_n$ if $n \in M$ else $z_n = y_n$, is $\mathcal{K}$-convergent to the point 0. According to assumption, Proposition 16 and Theorem 11 we have

$$\mathcal{I}^{\mathcal{K}}\text{-}\lim \frac{f(x_0 + x_n) - f(x_n)}{x_n} = d.$$

(b) Let $(x_n)_{n\in\mathbb{N}} \in \mathcal{S}_{\mathcal{I}}$. Because $\mathcal{I}$ fulfils condition $AP(\mathcal{I}, \mathcal{K})$ and $\mathcal{K} \subset \mathcal{I}$ the implications

$$\mathcal{I}\text{-}\lim x_n = 0 \Rightarrow \mathcal{I}^{\mathcal{K}}\text{-}\lim x_n = 0,$$

$$\mathcal{I}^{\mathcal{K}}\text{-}\lim \frac{f(x_0 + x_n) - f(x_n)}{x_n} = d \Rightarrow \mathcal{I}\text{-}\lim \frac{f(x_0 + x_n) - f(x_n)}{x_n} = d$$

hold. Therefore $\mathcal{I}$-$f'(x_0) = d$ what is equivalent (Theorem 11) with $f'(x_0) = d$. □

According to Definition 12 we can introduce next definition.

Definition 19. Let $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{K}_1$, $\mathcal{K}_2$ be admissible ideals on the set $\mathbb{N}$. A function $f : \mathbb{R} \to \mathbb{R}$ has a $(\mathcal{I}_1^{\mathcal{K}_1}, \mathcal{I}_2^{\mathcal{K}_2})$-*derivative* $d \in \mathbb{R}$ at a point x_0, i.e. $(\mathcal{I}_1^{\mathcal{K}_1}, \mathcal{I}_2^{\mathcal{K}_2})$-$f'(x_0) = d$, if and only if

$$\mathcal{I}_1^{\mathcal{K}_1}\text{-}\lim x_n = 0 \Rightarrow \mathcal{I}_2^{\mathcal{K}_2}\text{-}\lim \frac{f(x_0 + x_n) - f(x_n)}{x_n} = d$$

holds for each sequence $(x_n)_{n\in\mathbb{N}} \in \mathcal{S}$.

Theorem 20. *Let $f : \mathbb{R} \to \mathbb{R}$ and $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{K}_1$, $\mathcal{K}_2$ be admissible ideals on the set $\mathbb{N}$. Let $\mathcal{I}_1$, $\mathcal{I}_2$ possess conditions $AP(\mathcal{I}_1, \mathcal{K}_1)$, $AP(\mathcal{I}_2, \mathcal{K}_2)$ and $\mathcal{K}_1 \subset \mathcal{I}_1$, $\mathcal{K}_2 \subset \mathcal{I}_2$.*

(a) If $\mathcal{I}_1 \subset \mathcal{I}_2$ then $(\mathcal{I}_1^{\mathcal{K}_1}, \mathcal{I}_2^{\mathcal{K}_2})$-$f'(x_0) = d$ if and only if $f'(x_0) = d$.

(b) If $\mathcal{I}_1 \setminus \mathcal{I}_2 \neq \emptyset$ then $(\mathcal{I}_1^{\mathcal{K}_1}, \mathcal{I}_2^{\mathcal{K}_2})$-$f'(x_0) = d$ if and only if f is a linear function.

Proof. According to assumption the following equivalences

$$\mathcal{I}_1\text{-}\lim x_n = 0 \Leftrightarrow \mathcal{I}_1^{\mathcal{K}_1}\text{-}\lim x_n = 0,$$

$$\mathcal{I}_2\text{-}\lim \frac{f(x_0 + x_n) - f(x_n)}{x_n} = d \Leftrightarrow \mathcal{I}_2^{\mathcal{K}_2}\text{-}\lim \frac{f(x_0 + x_n) - f(x_n)}{x_n} = d$$

hold. The statement of Theorem 20 follows from Remark 13 and Theorem 14. □

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