Bounded linear functionals on the $n$-normed space of $p$-summable sequences

Harmanus Batkunde  
Department of Mathematics, Bandung Institute of Technology, Bandung 40132, Indonesia  
batkunde@yahoo.com

Hendra Gunawan $^*$  
Department of Mathematics, Bandung Institute of Technology, Bandung 40132, Indonesia  
hgunawan@math.itb.ac.id

Yosafat E.P. Pangalela  
Department of Mathematics, Bandung Institute of Technology, Bandung 40132, Indonesia  
matrix.yepp@gmail.com

Abstract  
Let $(X, \|\cdot,\cdots,\cdot\|)$ be a real $n$-normed space, as introduced by S. Gähler in 1969. We shall be interested in bounded linear functionals on $X$, using the $n$-norm as our main tool. We study the duality properties and show that the space $X'$ of bounded linear functionals on $X$ also forms an $n$-normed space. We shall present more results on bounded multilinear $n$-functionals on the space of $p$-summable sequences being equipped with an $n$-norm. Open problems are also posed.

Received 10 October 2012  
Accepted in final form March 6, 2013  
Communicated with Vladimír Janiš.

Keywords $p$-summable sequences, $n$-normed spaces, bounded linear functionals.


1 Introduction

Let $n$ be a nonnegative integer and $X$ be a real vector space of dimension $d \geq n$. A real-valued function $\|\cdot,\cdots,\cdot\|$ on $X^n$ satisfying the following four properties:

N.1 $\|x_1,\ldots,x_n\| = 0$ if and only if $x_1,\ldots,x_n$ are linearly dependent,

N.2 $\|x_1,\ldots,x_n\|$ is invariant under permutation,

N.3 $\|\alpha x_1,\ldots,x_n\| = |\alpha| \|x_1,\ldots,x_n\|$ for any $\alpha \in \mathbb{R}$,

N.4 $\|x_1 + x'_1,x_2,\ldots,x_n\| \leq \|x_1,x_2,\ldots,x_n\| + \|x'_1,x_2,\ldots,x_n\|$, 

is called an $n$-norm on $X$, and the pair $(X, \|\cdot,\cdots,\cdot\|)$ is called an $n$-normed space.

In an $n$-normed space $(X, \|\cdot,\cdots,\cdot\|)$, one may observe that $\|x_1,\ldots,x_n\| \geq 0$ and

$$\|x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, x_2,\ldots,x_n\| = \|x_1, x_2,\ldots,x_n\|$$ \hspace{1cm} (1.1)

for every $x_1,\ldots,x_n \in X$ and $\alpha_2,\ldots,\alpha_n \in \mathbb{R}$.

$^*$corresponding author
If \((X, \|\cdot\|)\) is a normed space and \(X'\) is its dual (consisting of bounded linear functionals on \(X\)), the following function defines an \(n\)-norm on \(X\):

\[
\|x_1, \ldots, x_n\|_G := \sup_{f_i \in X', \|f_i\| \leq 1} \left| \begin{array}{ccc}
    f_1(x_1) & \cdots & f_n(x_1) \\
    \vdots & \ddots & \vdots \\
    f_1(x_n) & \cdots & f_n(x_n)
  \end{array} \right|.
\]

(1.2)

Note that the determinant on the right hand side may be negative for certain \(f_i\)'s, but in such a case we may replace one of the \(f_i\)'s by its negative, so that the supremum of these determinants is always nonnegative.

For another example, if \((X, \langle \cdot, \cdot \rangle)\) is an inner product space, we can define the standard \(n\)-norm on \(X\) by

\[
\|x_1, \ldots, x_n\|_S := \left| \begin{array}{ccc}
    \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_n \rangle \\
    \vdots & \ddots & \vdots \\
    \langle x_n, x_1 \rangle & \cdots & \langle x_n, x_n \rangle
  \end{array} \right|^{1/2}.
\]

(1.3)

The determinant above is known as Gram's determinant, whose value is always nonnegative. Geometrically, the value of \(\|x_1, \ldots, x_n\|_S\) represents the volume of the \(n\)-dimensional parallelepiped spanned by \(x_1, \ldots, x_n\) (see [5]).

The concept of \(n\)-normed spaces was initially introduced by Gähler [1, 2, 3, 4] in the 1960’s. Recent results and related topics may be found in [8, 9, 10, 7, 11].

In this paper, we shall be interested in studying bounded linear functionals on \(X\), using the \(n\)-norm as our main tool. We prove an analog of the Riesz-Fréchet Theorem and show that the dual space \(X'\), consisting of all bounded linear functionals on \(X\), also forms an \(n\)-normed space. We shall present more results when \(X\) is the space of \(p\)-summable sequences being equipped with an \(n\)-norm. In addition, some open problems will be posed.

2 Bounded Linear Functionals

Let \((X, \|\cdot\|, \ldots, \|\cdot\|)\) be a real \(n\)-normed space and \(f : X \rightarrow \mathbb{R}\) be a linear functional on \(X\). We may define bounded linear functionals on \(X\) by using the \(n\)-norm in several ways as follows.

2.1 Bounded linear functionals (of 1st index)

Fix a linearly independent set \(Y := \{y_1, \ldots, y_n\}\) in \(X\). We say that \(f\) is bounded with respect to \(Y\) if and only if there exists \(K > 0\) such that

\[
|f(x)| \leq K \sum \|x, y_{i_2}, \ldots, y_{i_n}\|
\]

(2.1)

for all \(x \in X\), where the sum is taken over \(\{i_2, \ldots, i_n\} \subseteq \{1, \ldots, n\}\) with \(i_2 < \cdots < i_n\). [One might ask why we do not just take a linearly independent set \(\{y_2, \ldots, y_n\}\) in \(X\) and put \(|f(x)| \leq K \|x, y_2, \ldots, y_n\|\) for all \(x \in X\). The drawback with this is that for a nonzero vector \(x\) in the linear span of \(\{y_2, \ldots, y_n\}\), we have \(\|x, y_2, \ldots, y_n\| = 0\) while \(f(x) \neq 0\). This problem is overcome by taking a set of \(n\) linearly independent vectors and form the sum as in (2.1). Indeed, one might observe that the sum is equal to 0 if and only if \(x = 0\).]

For simplicity, we shall say ‘bounded’ instead of ‘bounded with respect to \(Y\)’. Clearly the set \(X'_1\) of all linear functionals which are bounded on \(X\) forms a vector space. Now, for \(f \in X'_1\), we define

\[
\|f\|_1 := \inf \{K > 0 : (2.1) \text{ holds}\}.
\]

(2.2)
It is easy to see that
\[ \|f\|_1 = \sup \{|f(x)| : \sum \|x, y_i, \ldots, y_n\| \leq 1\} \]
Moreover, the formula (2.2) defines a norm on \( X_1' \).

To give an example, we invoke the notion of \( n \)-inner product spaces \[\square\]. Assume that \( X \) is of dimension \( d \geq n + 1 \). A real-valued function \( \langle \cdot, \cdot, \ldots, \cdot \rangle \) on \( X^{n+1} \) satisfying the following properties:
I.1 \( \langle x_1, x_1|x_2, \ldots, x_n \rangle \geq 0 \) and it is equal to 0 if and only if \( x_1, \ldots, x_n \) are linearly dependent,
I.2 \( \langle x_i, x_i|x_i, \ldots, x_n \rangle = \langle x_1, x_1|x_2, \ldots, x_n \rangle \) for any permutation \( \{i_1, \ldots, i_n\} \) of \( \{1, \ldots, n\} \),
I.3 \( \langle x, y|x_2, \ldots, x_n \rangle = \langle y, x|x_2, \ldots, x_n \rangle \),
I.4 \( \langle \alpha x, y|x_2, \ldots, x_n \rangle = \alpha \langle x, y|x_2, \ldots, x_n \rangle \) for any \( \alpha \in \mathbb{R} \),
I.5 \( \langle x + x', y|x_2, \ldots, x_n \rangle = \langle x, y|x_2, \ldots, x_n \rangle + \langle x', y|x_2, \ldots, x_n \rangle \),

is called an \( n \)-inner product on \( X \), and the pair \( (X, \langle \cdot, \cdot, \ldots, \cdot \rangle) \) is called an \( n \)-inner product space.

Note that if \( (X, \langle \cdot, \cdot, \ldots, \cdot \rangle) \) is an \( n \)-inner product space, then we can define an \( n \)-norm \( \|\cdot, \ldots, \cdot\| \) on \( X \) by
\[ \|x_1, x_2, \ldots, x_n\| := \langle x_1, x_1|x_2, \ldots, x_n \rangle^{1/2}. \]
Here we have the Cauchy-Schwarz inequality:
\[ |\langle x, y|x_2, \ldots, x_n \rangle| \leq \|x_1, x_2, \ldots, x_n\| \|y, x_2, \ldots, x_n\|. \]

Now we give an example of bounded linear functionals on \( X \). Let \( (X, \langle \cdot, \cdot, \ldots, \cdot \rangle) \) be an \( n \)-inner product space, and \( \|\cdot, \ldots, \cdot\| := \langle \cdot, \cdot, \ldots, \cdot \rangle^{1/2} \) be the induced \( n \)-norm on \( X \).

With respect to the set \( Y = \{y_1, \ldots, y_n\} \), define \( f : X \rightarrow \mathbb{R} \) by
\[ f(x) := \sum \langle x, y_{i_1}|y_{i_2}, \ldots, y_{i_n}\rangle, \tag{2.3} \]
where the sum is taken over \( \{i_2, \ldots, i_n\} \subset \{1, \ldots, n\} \) with \( i_2 < \cdots < i_n \) and \( i_1 \in \{1, \ldots, n\} \setminus \{i_2, \ldots, i_n\} \). Clearly \( f \) is linear. Furthermore, we have:

**Fact 1.** The linear functional \( f \) defined by (2.3) is bounded with \( \|f\|_1 = \|y_1, \ldots, y_n\| \).

**Proof.** We observe that for every \( x \in X \), we have
\[ |f(x)| \leq \sum |\langle x, y_{i_1}|y_{i_2}, \ldots, y_{i_n}\rangle| \]
\[ \leq \sum \|x, y_{i_2}, \ldots, y_{i_n}\| \|y_{i_1}, y_{i_2}, \ldots, y_{i_n}\| \]
\[ = \|y_1, \ldots, y_n\| \sum \|x, y_{i_2}, \ldots, y_{i_n}\| \]
where the sum is taken over \( \{i_2, \ldots, i_n\} \subset \{1, \ldots, n\} \) with \( i_2 < \cdots < i_n \). Thus \( f \) is bounded with \( \|f\|_1 \leq \|y_1, \ldots, y_n\| \).

To show that \( \|f\|_1 = \|y_1, \ldots, y_n\| \), just take \( x := \|y_1, \ldots, y_n\|^{-1} y_1 \). Then we see that
\[ \sum \|x, y_{i_2}, \ldots, y_{i_n}\| = 1 \]
and
\[ |f(x)| = \|y_1, \ldots, y_n\|^{-1} f(y_1) \]
\[ = \|y_1, \ldots, y_n\|^{-1} \sum \langle y_1, y_{i_1}|y_{i_2}, \ldots, y_{i_n}\rangle \]
\[ = \|y_1, \ldots, y_n\|^{-1} \langle y_1, y_{i_2}, \ldots, y_{i_n}\rangle \]
\[ = \|y_1, \ldots, y_n\|^{-1} \|y_1, \ldots, y_n\|^2 \]
\[ = \|y_1\|. \]
[Note that when $i_1 \neq 1$ and $\{i_2, \ldots, i_n\} = \{1, \ldots, n\} \setminus \{i_1\}$, we have
\[ \langle y_1, y_i, y_{i_2}, \ldots, y_{i_n} \rangle \leq \|y_1, y_{i_2}, \ldots, y_{i_n}\| \|y_{i_1}, y_{i_2}, \ldots, y_{i_n}\| = 0 \]
because one of $y_{i_2}, \ldots, y_{i_n}$ must be equal to $y_1$.]

\[ \Box \]

## 2.2 Bounded linear functionals of $p$-th index

Fix a linearly independent set $Y := \{y_1, \ldots, y_n\}$ in $X$ and $1 \leq p \leq \infty$. We say that $f$ is *bounded of $p$-th index* (with respect to $Y$) if and only if there exists $K > 0$ such that
\[ |f(x)| \leq K \left( \sum \|x, y_{i_2}, \ldots, y_{i_n}\|^p \right)^{1/p} \tag{2.4} \]
where the sum is taken over $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$ with $i_2 < \cdots < i_n$. If $p = \infty$, then the sum is the maximum of all possible values of $\|x, y_{i_2}, \ldots, y_{i_n}\|$.

As in the case where $p = 1$, the set $X'_p$ of all linear functionals which are bounded of $p$-index on $X$ forms a vector space. Now, for $f \in X'_p$, we define
\[ \|f\|_p := \inf \{K > 0 : (2.4) \text{ holds}\}. \tag{2.5} \]

One then has
\[ \|f\|_p = \sup \{|f(x)| : \sum \|x, y_{i_2}, \ldots, y_{i_n}\|^p \leq 1 \}. \]

Moreover, the formula (2.5) defines a norm on $X'_p$.

**Fact 2.** The linear functional $f$ defined by (2.3) is bounded of $p$-th index with $\|f\|_p = n^{1/p'} \|y_1, \ldots, y_n\|$, where $p'$ is the dual exponent of $p$ (that is, $\frac{1}{p} + \frac{1}{p'} = 1$).

**Proof.** For every $x \in X$, it follows from Hölder’s inequality that
\[ |f(x)| \leq \sum \|x, y_{i_2}, \ldots, y_{i_n}\| \|y_1, \ldots, y_n\| \leq n^{1/p'} \|y_1, \ldots, y_n\| \left( \sum \|x, y_{i_2}, \ldots, y_{i_n}\|^p \right)^{1/p}, \]
whence $\|f\|_p \leq n^{1/p'} \|y_1, \ldots, y_n\|$.

To obtain the equality, take $x := n^{-1/p} \|y_1, \ldots, y_n\|^{-1} (y_1 + \cdots + y_n)$. Then, using (1.1), one may verify that $\sum \|x, y_{i_2}, \ldots, y_{i_n}\|^p = 1$. Moreover, we have
\[
\begin{align*}
  f(x) &= n^{-1/p} \|y_1, \ldots, y_n\|^{-1} \sum \langle y_1 + \cdots + y_n, y_{i_2}, \ldots, y_{i_n} \rangle \\
        &= n^{-1/p} \|y_1, \ldots, y_n\|^{-1} \sum \langle y_{i_1}, y_1, y_{i_2}, \ldots, y_{i_n} \rangle \\
        &= n^{-1/p} \|y_1, \ldots, y_n\|^{-1} \cdot n \|y_1, \ldots, y_n\|^2 \\
        &= n^{1/p'} \|y_1, \ldots, y_n\|.
\end{align*}
\]

This convinces us that $\|f\|_p = n^{1/p'} \|y_1, \ldots, y_n\|$. \[ \Box \]

The following theorem tells us that $X'_1$ and $X'_p$ are identical as a set.

**Theorem 3.** Let $f$ be a linear functional on $X$. If $f$ is bounded of $1$st index, then $f$ is bounded of $p$-th index; and vice versa. In other words, $X'_1 = X'_p$.

**Proof.** Suppose that $f$ is bounded of $p$-index (with respect to $Y = \{y_1, \ldots, y_n\}$). If $x$ satisfies $\sum \|x, y_{i_2}, \ldots, y_{i_n}\| \leq 1$, then each term of the sum is less than 1, i.e., $\|x, y_{i_2}, \ldots, y_{i_n}\| \leq 1$. Hence $\|x, y_{i_2}, \ldots, y_{i_n}\|^p \leq \|x, y_{i_2}, \ldots, y_{i_n}\|$, and so
\[ \sum \|x, y_{i_2}, \ldots, y_{i_n}\|^p \leq \sum \|x, y_{i_2}, \ldots, y_{i_n}\| \leq 1. \]
Consequently, \(|f(x)| \leq \|f\|_p\), and thus \(f\) is bounded of 1st index with \(\|f\|_1 \leq \|f\|_p\).

Conversely, suppose that \(f\) is bounded of 1st index. If \(x\) satisfies \(\sum \|x, y_{i_2}, \ldots, y_{i_n}\| \leq 1\), then \(\sum \|x, y_{i_2}, \ldots, y_{i_n}\| \leq n^{1/p'}\), where \(p'\) is the dual exponent of \(p\). Hence

\[
\sum \| \frac{x}{n^{1/p'}} , y_{i_2}, \ldots, y_{i_n} \| \leq 1,
\]

and so \(|f(\frac{x}{n^{1/p'}})| \leq \|f\|_1\) or \(|f(x)| \leq n^{1/p'} \|f\|_1\). We therefore conclude that \(f\) is bounded of \(p\)-th index with \(\|f\|_p \leq n^{1/p} \|f\|_1\).

**Remark 4.** Unless we need to specify the index explicitly, we may simply use the word ‘bounded’ instead of ‘bounded of \(p\)-th index.’ We also denote by \(X'\) the set of all bounded linear functionals on \(X\) and call it the dual space of \(X\) (with respect to \(Y\)). Theorem 3 states further that, on \(X'\), the norms \(\| \cdot \|_p\) are all equivalent to \(\| \cdot \|_1\), with

\[
\|f\|_1 \leq \|f\|_p \leq n^{1/p'} \|f\|_1,
\]

for every \(f \in X'\).

**2.3 Duality properties for \(p = 2\)**

Let us now discuss another example of bounded linear functionals on the \(n\)-inner product space \(X\), using the linearly independent set \(Y = \{y_1, \ldots, y_n\}\). Let \(y \neq y_i\) for \(i = 1, \ldots, n\). Define \(f_y : X \rightarrow \mathbb{R}\) by

\[
f_y(x) := \sum \langle x, y|y_{i_2}, \ldots, y_{i_n} \rangle,
\]

where the sum is taken over \(\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}\) with \(i_2 < \cdots < i_n\). Then \(f_y\) is linear. Moreover, we have:

**Fact 5.** The linear functional \(f_y\) defined by (2.6) is bounded of 2nd index with \(\|f_y\|_2 = (\sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2)^{1/2}\).

**Proof.** For every \(x \in X\), it follows from Cauchy-Schwarz inequalities that

\[
|f_y(x)| \leq \sum |\langle x, y|y_{i_2}, \ldots, y_{i_n} \rangle |
\leq \sum \|x, y_{i_2}, \ldots, y_{i_n}\| \|y, y_{i_2}, \ldots, y_{i_n}\|
\leq (\sum \|x, y_{i_2}, \ldots, y_{i_n}\|^2)^{1/2} (\sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2)^{1/2},
\]

whence \(\|f_y\|_2 \leq \left(\sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2\right)^{1/2}\).

Now, if we take \(x := \left(\sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2\right)^{-1/2} y\), we get

\[
f_y(x) = \left(\sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2\right)^{-1/2} f_y(y)
= \left(\sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2\right)^{-1/2} \sum \langle y, y|y_{i_2}, \ldots, y_{i_n} \rangle
= \left(\sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2\right)^{-1/2} \sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2
= \left(\sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2\right)^{1/2}.
\]

We must therefore have \(\|f_y\|_2 = \left(\sum \|y, y_{i_2}, \ldots, y_{i_n}\|^2\right)^{1/2}\).

It is desirable to have an analog of the Riesz-Fréchet Theorem for linear functionals which are bounded of 2nd index on an \(n\)-inner product space. For that, we import the following theorem from [9].
Theorem 6. Let \((X, \langle \cdot, \cdot, \ldots, \cdot \rangle)\) be an \(n\)-inner product space and \(\| \cdot, \ldots, \cdot \| = \langle \cdot, \cdot, \ldots, \cdot \rangle^{1/2}\) be the induced \(n\)-norm on \(X\). With respect to the linearly independent set \(Y = \{y_1, \ldots, y_n\}\), the mapping \(\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{R}\) given by

\[
\langle x, y \rangle := \sum \langle x, y \rangle y_{i_2}, \ldots, y_{i_n} \tag{2.7}
\]

defines an inner product on \(X\), and its induced norm \(\| \cdot \|_2 : X \rightarrow \mathbb{R}\) is given by

\[
\| x \|_2 := \left( \sum \| x, y_{i_2}, \ldots, y_{i_n} \|^2 \right)^{1/2}. \tag{2.8}
\]

Corollary 7. If \((X, \langle \cdot, \cdot, \ldots, \cdot \rangle)\) is complete with respect to the norm \(\| \cdot \|_2\) in (2.8), then for every linear functional \(f\) which is bounded of 2nd index on \(X\) there exists a unique \(y \in X\) such that

\[
f(x) = \langle x, y \rangle, \quad x \in X,
\]

where \(\langle \cdot, \cdot \rangle\) is the inner product in (2.7). Moreover, we have \(\| y \|_2 = \| f \|_2\).

Theorem 8. Let \((X, \| \cdot, \ldots, \cdot \|)\) be an \(n\)-normed space, \(X'\) be the dual space of \(X\) (with respect to \(Y\)), and \(\| \cdot \|_2\) be the derived norm on \(X\) given by

\[
\| x \|_2 := \left( \sum \| x, y_{i_2}, \ldots, y_{i_n} \|^2 \right)^{1/2}.
\]

Then, the function \(\| \cdot, \ldots, \cdot \|' : (X')^n \rightarrow \mathbb{R}\) given by

\[
\| f_1, \ldots, f_n \|' := \sup_{x_i \in X, \| x_i \|_2 \leq 1} \begin{vmatrix} f_1(x_1) & \ldots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \ldots & f_n(x_n) \end{vmatrix}
\]

defines an \(n\)-norm on \(X'\).

Proof. Similar to the proof of Fact 2 in [6].

3 Bounded Multilinear \(n\)-Functionals on \(\ell^p\)

In this section, we shall focus on the space of \(p\)-summable sequences of real numbers, denoted by \(\ell^p = \ell^p_n(\mathbb{R})\), where \(1 \leq p < \infty\). Recall that a sequence \(u := \{u_k\}_{k=1}^\infty\) (of real numbers) belongs \(\ell^p\) space if \(\| u \|_p := \left( \sum_{k=1}^\infty |u_k|^p \right)^{1/p} < \infty\). It is known that the dual space of \(\ell^p\) is \(\ell^{p'}\) where \(\frac{1}{p} + \frac{1}{p'} = 1\).

3.1 Several \(n\)-norms on \(\ell^p\)

Using the formula (1.2), \(\ell^p\) may be equipped with the following \(n\)-norm:

\[
\| x_1, \ldots, x_n \|_p^G := \sup_{y_i \in \ell^{p'}, \| y_i \|_{p'} \leq 1} \left| \sum_{k=1}^\infty x_{1k} y_{1k} \cdots \sum_{k=1}^\infty x_{1k} y_{nk} \right|, \tag{3.1}
\]

where \(p'\) denotes the dual exponent of \(p\). But there is another formula of \(n\)-norm that we can define on \(\ell^p\), namely

\[
\| x_1, \ldots, x_n \|_p^H := \left[ \frac{1}{n!} \sum_{k_1=1}^\infty \cdots \sum_{k_n=1}^\infty \left| \begin{array}{ccc} x_{1k_1} & \cdots & x_{1k_n} \\ \vdots & \ddots & \vdots \\ x_{nk_1} & \cdots & x_{nk_n} \end{array} \right| \right]^{1/p}, \tag{3.2}
\]
where \( x_i = \{x_{ik}\}_{k=1}^\infty, i = 1, \ldots, n \). As shown in [12], the two \( n \)-norms are equivalent:

\[
(n!)^{(1/p) - 1} \|x_1, \ldots, x_n\|_p^H \leq \|x_1, \ldots, x_n\|_p^G \leq (n!)^{1/p} \|x_1, \ldots, x_n\|_p^H.
\]

On \( \ell^2 \), both \( n \)-norms coincide with the standard \( n \)-norm given by (1.3) [6].

Next, one may observe that, by taking the sums and like terms out of the determinant and knowing that there are \( n! \) possible ways to do so (see [7]), the determinant on the right hand side of (3.1) can be rewritten as

\[
\frac{1}{n!} \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \begin{vmatrix} x_{1k_1} & \cdots & x_{1k_n} \\ \vdots & \ddots & \vdots \\ x_{nk_1} & \cdots & x_{nk_n} \end{vmatrix} \begin{vmatrix} y_{1k_1} & \cdots & y_{1k_n} \\ \vdots & \ddots & \vdots \\ y_{nk_1} & \cdots & y_{nk_n} \end{vmatrix}.
\]

By Hölder’s inequality, we find that this sum is dominated by

\[
\|x_1, \ldots, x_n\|_p^H \|y_1, \ldots, y_n\|_p^H.
\]

This inspires us to define another \( n \)-norm on \( \ell^p \), namely

\[
\|x_1, \ldots, x_n\|_p^I := \sup_{y_i \in \ell^{p'}}, \|y_1, \ldots, y_n\|_p^i \leq 1 \begin{vmatrix} \sum_{k=1}^{\infty} x_{1k}y_{1k} & \cdots & \sum_{k=1}^{\infty} x_{1k}y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{\infty} x_{nk}y_{1k} & \cdots & \sum_{k=1}^{\infty} x_{nk}y_{nk} \end{vmatrix}.
\]

**Theorem 9.** The three \( n \)-norms on \( \ell^p \), namely \( \|\cdot, \ldots, \cdot\|_p^I, \|\cdot, \ldots, \cdot\|_p^H, \) and \( \|\cdot, \ldots, \cdot\|_p^G \), are equivalent.

**Proof.** By the observation above, we have \( \|x_1, \ldots, x_n\|_p^I \leq \|x_1, \ldots, x_n\|_p^H \). By Theorem 2.3 of [12], we have \( \|x_1, \ldots, x_n\|_p^H \leq (n!)^{1/p} \|x_1, \ldots, x_n\|_p^G \). Now, using the inequality

\[
\|y_1, \ldots, y_n\|_p^I \leq (n!)^{1/p} \|y_1, \ldots, y_n\|_p^H \|
\]

(see Fact 3.1 of [7]), we see that if \( \|y_i\|_{p'} \leq 1 \) for \( i = 1, \ldots, n \), then \( \|y_1, \ldots, y_n\|_p^H \leq (n!)^{1/p} \). Hence we obtain

\[
\|x_1, \ldots, x_n\|_p^G \leq (n!)^{1/p} \|x_1, \ldots, x_n\|_p^I.
\]

The chain of these inequalities shows that the three \( n \)-norms are equivalent.

### 3.2 Multilinear \( n \)-functionals on \( \ell^p \)

By a multilinear \( n \)-functional on a real vector space \( X \) we mean a mapping \( F : X^n \to \mathbb{R} \) which is linear in each variable. A multilinear \( n \)-functional \( F \) is bounded on an \( n \)-normed space \( (X, \|\cdot, \ldots, \cdot\|) \) if and only if there exists \( K > 0 \) such that

\[
|F(x_1, \ldots, x_n)| \leq K \|x_1, \ldots, x_n\|
\]

for every \( x_1, \ldots, x_n \in X \). Note that for a bounded multilinear \( n \)-functional \( F \) on an \( n \)-normed space \( (X, \|\cdot, \ldots, \cdot\|) \), we have \( F(x_1, \ldots, x_n) = 0 \) when \( x_1, \ldots, x_n \) are linearly dependent. Moreover, we have the following proposition.

**Proposition 10.** If \( F \) is a bounded multilinear \( n \)-functional on an \( n \)-normed space \( (X, \|\cdot, \ldots, \cdot\|) \), then \( F \) is antisymmetric, that is

\[
F(x_1, \ldots, x_n) = \text{sgn}(\sigma) F(x_{\sigma(1)}, \ldots, x_{\sigma(n)})
\]

for any \( x_1, \ldots, x_n \in X \) and any permutation \( \sigma \) of \( (1, \ldots, n) \). [Here \( \text{sgn}(\sigma) = 1 \) if \( \sigma \) is an even permutation and \( \text{sgn}(\sigma) = -1 \) if \( \sigma \) is an odd permutation.]
Problem 2. We give the proof for the case where \( n = 2 \) and leave the other case to the reader. Here, \( F \) is antisymmetric if and only if \( F(x_1, x_2) = -F(x_2, x_1) \) for every \( x_1, x_2 \in X \). To see this, we observe that

\[
F(x_1 + x_2, x_1 + x_2) = F(x_1, x_1) + F(x_1, x_2) + F(x_2, x_1) + F(x_2, x_2).
\]

But \( F(x, x) = 0 \) for every \( x \in X \), and so we are done. \( \square \)

We note that the set \( X^* \) of all bounded multilinear \( n \)-functionals on \( (X, \|\cdot, \ldots, \cdot\|) \) forms a vector space. Next, for a bounded multilinear \( n \)-functional \( F \), we may define

\[
\|F\| := \inf\{K > 0 : \text{(3.4) holds}\},
\]

or equivalently

\[
\|F\| := \sup\{|F(x_1, \ldots, x_n)| : \|x_1, \ldots, x_n\| \leq 1\}.
\]

This formula defines a norm on \( X^* \).

We shall now discuss some multilinear \( n \)-functionals on \( \ell^p \) (where \( 1 \leq p < \infty \)). Let \( Y := \{y_1, \ldots, y_n\} \) in \( \ell^{p'} \), where \( p' \) is the dual exponent of \( p \). We define

\[
F_Y(x_1, \ldots, x_n) := \frac{1}{n!} \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \begin{vmatrix}
\begin{array}{cccc}
x_{1k_1} & \cdots & x_{1k_n} \\
\vdots & \ddots & \vdots \\
x_{nk_1} & \cdots & x_{nk_n}
\end{array}
\end{vmatrix}
\begin{vmatrix}
\begin{array}{cccc}
y_{1k_1} & \cdots & y_{1k_n} \\
\vdots & \ddots & \vdots \\
y_{nk_1} & \cdots & y_{nk_n}
\end{array}
\end{vmatrix}, \tag{3.5}
\]

for \( x_1, \ldots, x_n \in \ell^p \). Clearly \( F_Y \) is linear in each variable. Further, we have

\[
|F_Y(x_1, \ldots, x_n)| \leq \|x_1, \ldots, x_n\|_{\ell^p}^H \|y_1, \ldots, y_n\|_{\ell^{p'}}^H,
\]

and so \( F_Y \) is bounded on \( (\ell^p, \|\cdot, \ldots, \cdot\|_p, \|\cdot\|_p^H) \) with \( \|F_Y\| \leq \|y_1, \ldots, y_n\|_{\ell^{p'}}^H \).

For \( p = 2 \), we have the following fact.

Fact 11. Consider the \( n \)-normed space \( (\ell^2, \|\cdot, \ldots, \cdot\|_2, \|\cdot\|_2^H) \). For fixed linearly independent \( Y := \{y_1, \ldots, y_n\} \) in \( \ell^2 \), let \( F_Y \) be the multilinear \( n \)-functional defined as in (3.5). Then \( F_Y \) is bounded on \( (\ell^2, \|\cdot, \ldots, \cdot\|_2^H) \) with

\[
\|F_Y\| = \|y_1, \ldots, y_n\|_2^H.
\]

Proof. From the inequality

\[
|F_Y(x_1, \ldots, x_n)| \leq \|x_1, \ldots, x_n\|_2^H \|y_1, \ldots, y_n\|_2^H,
\]

we see that \( F_Y \) is bounded with \( \|F_Y\| \leq \|y_1, \ldots, y_n\|_2^H \). Next, if we take

\[
x_i := \frac{y_i}{\sqrt[4]{\|y_1, \ldots, y_n\|_2^H}}, \quad i = 1, \ldots, n,
\]

then \( \|x_1, \ldots, x_n\|_2^H = 1 \) and \( F_Y(x_1, \ldots, x_n) = \|y_1, \ldots, y_n\|_2^H \). Hence we conclude that \( \|F_Y\| = \|y_1, \ldots, y_n\|_2^H \). \( \square \)

Regarding the \( n \)-functional \( F_Y \) on \( (\ell^p, \|\cdot, \ldots, \cdot\|_p, \|\cdot\|_p^H) \), we have an open problem.

Problem 1. Compute the exact norm of \( F_Y \) in (3.5), especially for \( p \neq 2 \).

Problem 2. Can every bounded multilinear \( n \)-functional on \( \ell^p \) be identified by \( (y_1, \ldots, y_n) \) where \( y_i \in \ell^{p'}, i = 1, \ldots, n \)?
Note that the multilinear $n$-functional $F_Y$ may be reformulated as
$$F_Y(x_1, \ldots, x_n) = \left| \sum_{k=1}^{\infty} x_{1k} y_{1k} \cdots \sum_{k=1}^{\infty} x_{nk} y_{nk} \right|.$$ From this expression, we get the following result.

**Fact 12.** Let $e_k := (0, \ldots, 0, 1, 0, \ldots)$ where the $k$-th term is the only term with value 1. Then, for $k_1, \ldots, k_n \in \mathbb{N}$, we have
$$F_Y(e_{k_1}, \ldots, e_{k_n}) = \left| \begin{array}{ccc} y_{1k_1} & \cdots & y_{1k_n} \\ \vdots & \ddots & \vdots \\ y_{nk_1} & \cdots & y_{nk_n} \end{array} \right|.$$ Accordingly, the multiindex sequence $\{F_Y(e_{k_1}, \ldots, e_{k_n})\}_{k_1, \ldots, k_n}$ is $p'$-summable, in the sense that
$$\left[ \frac{1}{n!} \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \left| \begin{array}{ccc} y_{1k_1} & \cdots & y_{1k_n} \\ \vdots & \ddots & \vdots \\ y_{nk_1} & \cdots & y_{nk_n} \end{array} \right|^{p'} \right]^{\frac{1}{p'}} < \infty.$$ **Proof.** The first part is straightforward, while the second part follows from the fact that $y_1, \ldots, y_n \in \ell^{p'}$ and that the sum is actually equal to $\|y_1, \ldots, y_n\|_{H^{p'}}$. \hfill \qed

The following problem is still open.

**Problem 3.** Let $F$ be a bounded multilinear $n$-functional on $\ell^p$. Must the multiindex sequence $\{F(e_{k_1}, \ldots, e_{k_n})\}_{k_1, \ldots, k_n}$ be $p'$-summable?

In general, the converse of Fact 11 holds, as follows. (We leave the proof to the reader.)

**Proposition 13.** Let $c := \{c_{k_1\ldots k_n}\}_{k_1, \ldots, k_n}$ be a multiindex sequence which is antisymmetric and $p'$-summable. Then, the $n$-functional $F_c$ given by
$$F_c(x_1, \ldots, x_n) := \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} x_{1k_1} \cdots x_{nk_n} c_{k_1\ldots k_n}, \quad (3.6)$$
where $x_i := (x_{ik})_{k=1}^{\infty} \in \ell^p$ ($i = 1, \ldots, n$), is linear in each variable, and is bounded on $(\ell^p, \| \cdot \|, \ldots, \| \cdot \|_{H^p})$ with
$$\|F_c\| \leq \left[ \frac{1}{n!} \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} |c_{k_1\ldots k_n}|^{p'} \right]^{1/p'}. \quad (3.6)$$**Remark 14.** Similar to Problem 1, we do not know the exact norm of the $n$-functional $F_c$ in (3.6).

**Acknowledgement**

This research is supported by ITB Research and Innovation Program 2012. The main ideas of the results were presented by the second author in International Conference of Honam Mathematical Society, which was held in Jeju, South Korea, on June 15-17, 2012. The participation in the conference was supported by ITB I-MHERE International Conference Grant 2012. We thank the anonymous referee for his/her useful comments on the earlier version of this paper.
References


