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Acta Universitatis Matthiae Belii, ser. Mathematics
Institute of Mathematics, Faculty of Natural Sciences
Matej Bel University
Tajovského 40
974 01 Banská Bystrica
Slovakia
E-mail: actamath@savbb.sk

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Boundedness of the solution set for a fourth-order nonlinear differential equation with multiple deviating arguments

Hilmi Ergören
Department of Mathematics, Faculty of Science, Yuzuncu Yil University, 65080, Van, Turkey
hergoren@yahoo.com

Cemil Tunç
Department of Mathematics, Faculty of Science, Yuzuncu Yil University, 65080, Van, Turkey
cemtunc@yahoo.com

Abstract

This paper deals with a fourth-order non-linear differential equation with multiple deviating arguments. Some sufficient conditions are set up for all solutions and their derivatives to be bounded. Our results are new and complement to previously known results.

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1 Introduction

We consider the nonlinear differential equation of fourth order with multiple deviating arguments

$$x^{(4)}(t) + f_1(t, x(t))x^{(3)}(t) + f_2(t, x(t))x^{(2)}(t) + f_3(t, x(t))x^{(1)}(t)$$
$$+g_0(t, x(t)) + \sum_{i=1}^{n} g_i(t, x(t - \tau_i(t)) = p(t) \tag{1.1}$$

where f_1, f_2, f_3 and $g_i (i = 0, 1, 2, ..., n)$ are continuous functions on $R^+ \times R$, $\tau_i(t) \geq 0$ $(i = 1, 2, ..., n)$ and $p(t)$ are bounded continuous functions on $R^+ = [0, +\infty)$.

Define $y(t) = \dfrac{dx(t)}{dt} + d_1 x(t)$, $z(t) = \dfrac{dy(t)}{dt} + d_2 y(t)$ and $w(t) = \dfrac{dz(t)}{dt} + d_3 z(t)$ where d_1, d_2 and d_3 are some constants. Then, we can transform Eq. (1.1) into the system, as follows,

$$
\begin{aligned}
\frac{dx(t)}{dt} &= -d_1x(t) + y(t)\\
\frac{dy(t)}{dt} &= -d_2y(t) + z(t)\\
\frac{dz(t)}{dt} &= -d_3z(t) + w(t)\\
\frac{dw(t)}{dt} &= -(f_1(t, x(t)) - d_1 - d_2 - d_3)w(t)\\
&+ (-(d_1 + d_2 - f_1(t, x(t)))(d_1 + d_2 + d_3) + (d_1d_2 - f_2(t, x(t))) - d_3^2)z(t)\\
&+ ((d_1 + d_2 - f_1(t, x(t)))(d_1^2 + d_1d_2 + d_2^2) - (d_1d_2 - f_2(t, x(t)))(d_1 + d_2)\\
&- f_3(t, x(t)))y(t)\\
&+ ((f_1(t, x(t)) - d_1 - d_2)d_1^3 + (d_1d_2 - f_2(t, x(t)))d_1^2 + f_3(t, x(t))d_1)x(t)\\
&- g_0(t, x(t)) - \sum_{i=1}^{n} g_i(t, x(t - \tau_i(t)) + p(t). \qquad (1.2)
\end{aligned}
$$

In applied science, some practical problems are associated with higher-order nonlinear differential equations, such as nonlinear oscillations ([1]–[4]), electronic theory [5], biological model and other models ([6], [7]). Just as above, in the past few decades, the study of qualitative behaviors for higher order differential equations has been paid attention to by many scholars. And, many results relative to the stability and boundedness of solutions of higher order differential equations with delays or without delays have been obtained in view of various methods, especially, Liapunov's method (see [8]–[23] and references therein). On the other hand, some researchers have obtained their results for higher order differential equations with several deviating arguments without using Liapunov's method and Liapunov functional (see [24]–[27]). However, to the best of our knowledge, no authors have considered the boundedness of solutions of fourth order differential equations with multiple deviating arguments in non-Liapunov sense. By the way, we interpret that forming the forthcoming conditions in non-Liapunov sense for our results is easier and more useful than determining a Liapunov functional for higher order differential equations with delays. Thus, it is worthwhile to continue to investigate the boundedness of solutions of Eq. (1.1) in this case.

The main objective of this paper is to study the uniformly boundedness of solutions of (1.2). We will establish some sufficient conditions satisfying the solutions of (1.2) to be uniformly bounded. Our results are new and complement to previously known results.

2 Definition and Assumptions

We assume that $h = \max\limits_{1\le i\le n}\left\{\sup\limits_{t\in R}\tau_i(t)\right\} \ge 0$. Let $C([-h, 0], R)$ denote the space of continuous functions $\phi : [-h, 0] \to R$ with the supremum norm. It is known from ([28]–[30]) that for $g_i(i = 0, 1, 2, ..., n), \phi, f_1, f_2, f_3, p$ and $\tau_i(t)(i = 1, 2, ..., n)$ continuous, given a continuous initial function $\phi \in C([-h, 0], R)$ and a vector $(y_0, z_0, w_0) \in R^3$, there exists a solution of (1.2) on an interval $[0, T)$ satisfying the initial condition and satisfying (1.2) on $[0, T)$. If the solution remains bounded, then $T = +\infty$. We denote such a solution by $(x(t), y(t), z(t), w(t)) = (x(t, \phi, y_0, z_0, w_0), y(t, \phi, y_0, z_0, w_0), z(t, \phi, y_0, z_0, w_0), w(t, \phi, y_0, z_0, w_0))$ where $y(s) = y(0)$, $z(s) = z(0)$ and $w(s) = w(0)$ for all $s \in [-h, 0]$. Then, it follows that $(x(t), y(t), z(t), w(t))$ can be defined on $[-h, +\infty)$.

Definition. Solutions of (1.2) are called uniformly bounded (UB) if for each $B_1 > 0$ there is a $B_2 > 0$ such that $(\phi, y_0, z_0, w_0) \in C([-h, 0], R) \times R^3$ and $\|\phi\| + \|y_0\| + \|z_0\| + \|w_0\| \leq B_1$ imply that $|x(t, \phi, y_0, z_0, w_0)| + |y(t, \phi, y_0, z_0, w_0)| + |z(t, \phi, y_0, z_0, w_0)| + |w(t, \phi, y_0, z_0, w_0)| \leq B_2$ for all $t \in R^+$.

In this work, we also assume that the following conditions hold:

There exist constants $K > 0$, $d_1 > 1$, $d_2 > 1$, $d_3 > 1$, $d_4 > 0$ and nonnegative constants L_i and q_i $(i = 0, 1, 2, ..., n)$ such that

i) $\left|(f_1(t, x(t)) - d_1 - d_2)d_1^3 + (d_1 d_2 - f_2(t, x(t)))d_1^2 + f_3(t, x(t))d_1)u - g_0(t, u)\right| \leq L_0 |u|$, for all $u \in R$ and $t \geq K$,

ii) $|g_1(t, u)| \leq L_1 |u| + q_1$, $|g_2(t, u)| \leq L_2 |u| + q_2, ..., |g_n(t, u)| \leq L_n |u| + q_n$ for all $u \in R$ and $t \geq K$,

iii) $d_4 = \inf\limits_{t \geq K} (f_1(t, x(t)) - d_1 - d_2 - d_3)$

$$-(\sup_{t\geq K} \left|(d_1 + d_2 - f_1(t, x(t)))(d_1^2 + d_1 d_2 + d_2^2) - (d_1 d_2 - f_2(t, x(t)))(d_1 + d_2) - f_3(t, x(t))\right|$$

$$+\sup_{t\geq K} \left|-(d_1 + d_2 - f_1(t, x(t)))(d_1 + d_2 + d_3) + (d_1 d_2 - f_2(t, x(t))) - d_3^2\right|) > \sum_{i=0}^{n} L_i.$$

3 Main Results

Theorem 1. *Suppose (i)-(iii) hold. Then solutions of (1.2) are uniformly bounded.*

Proof. Let $(x(t), y(t), z(t), w(t))$ be a solution of system (1.2) with initial conditions $x(s) = \phi(s)$, $y(0) = y_0$, $z(0) = z_0$ and $w(0) = w_0$ for all $s \in [-h, 0]$ where $\phi \in C([-h, 0], R)$ and $(y_0, z_0, w_0) \in R^3$.

Calculating the upper right derivatives of $|x(s)|, |y(s)|, |z(s)|$ and $|w(s)|$ along (1.2), in view of (*i*)-(*iii*), we have

$$\begin{aligned} D^+(|x(s)|)_{s=t} &= sgn(x(t))\{-d_1 x(t) + y(t)\} \\ &\leq -d_1 |x(t)| + |y(t)|, \end{aligned} \tag{3.1}$$

$$\begin{aligned} D^+(|y(s)|)_{s=t} &= sgn(y(t))\{-d_2 y(t) + z(t)\} \\ &\leq -d_2 |y(t)| + |z(t)|, \end{aligned} \tag{3.2}$$

$$\begin{aligned} D^+(|z(s)|)_{s=t} &= sgn(z(t))\{-d_3 z(t) + w(t)\} \\ &\leq -d_3 |z(t)| + |w(t)|, \end{aligned} \tag{3.3}$$

and

$$
\begin{aligned}
&D^+(|w(s)|)_{s=t}\\
&= sgn(w(t))\{-(f_1(t,x(t))-d_1-d_2-d_3)w(t)\\
&+(-(d_1+d_2-f_1(t,x(t)))(d_1+d_2+d_3)+(d_1d_2-f_2(t,x(t)))-d_3^2)z(t)\\
&+((d_1+d_2-f_1(t,x(t)))(d_1^2+d_1d_2+d_2^2)-(d_1d_2-f_2(t,x(t)))(d_1+d_2)\\
&-f_3(t,x(t)))y(t)\\
&+((f_1(t,x(t))-d_1-d_2)d_1^3+(d_1d_2-f_2(t,x(t)))d_1^2+f_3(t,x(t))d_1)x(t))\\
&-g_0(t,x(t))-\sum_{i=1}^{n}g_i(t,x(t-\tau_i(t))+p(t)\}\\
&\le \{(-\inf_{t\ge K}\ (f_1(t,x(t))-d_1-d_2-d_3))\,|w(t)|\\
&+\sup_{t\ge K}\left|-(d_1+d_2-f_1(t,x(t)))(d_1+d_2+d_3)+(d_1d_2-f_2(t,x(t)))-d_3^2\right|\,|z(t)|\\
&+\sup_{t\ge K}\big|(d_1+d_2-f_1(t,x(t)))(d_1^2+d_1d_2+d_2^2\big)\\
&-(d_1d_2-f_2(t,x(t)))(d_1+d_2)-f_3(t,x(t))|\,|y(t)|\\
&+L_0\,|x(t)|+\sum_{i=1}^{n}L_i\,|x(t-\tau_i(t))|\}+\sum_{i=0}^{n}q_i+|p(t)|\,. \qquad (3.4)
\end{aligned}
$$

Let

$$M(t)=\max_{-h\le s\le t}\{\max\{|x(s)|,|y(s)|,|z(s)|,|w(s)|\}\}.$$

It is clear that $\max\{|x(t)|,|y(t)|,|z(t)|,|w(t)|\}\le M(t)$ and $M(t)$ is non-decreasing. Now, we consider the following two cases:

Case I): If there exists a sufficiently large constant $K_1>K$ such that

$$M(t)>\max\{|x(t)|,|y(t)|,|z(t)|,|w(t)|\} \qquad (3.5)$$

for all $t\ge K_1$, then we claim that

$$M(t)\equiv M(K_1) \qquad (3.6)$$

is a constant for all $t\ge K_1$.

By contrapositive, assume (3.6) does not hold, then, there exists $t_1\ge K_1$ such that $M(t_1)>M(K_1)$.

Here $\max\{|x(t)|,|y(t)|,|z(t)|,|w(t)|\}\le M(K_1)$ for all $-h\le t\le K_1$ and there exists $\beta\in(K_1,t_1)$ such that $\max\{|x(\beta)|,|y(\beta)|,|z(\beta)|,|w(\beta)|\}=M(t_1)\ge M(\beta)$ which contradicts (3.5). This implies that (3.6 holds. It follows that there exists $t_2\ge K_1$ such that $\max\{|x(t)|,|y(t)|,|z(t)|,|w(t)|\}<M(t)=M(K_1)$ for all $t\ge t_2$.

Case II): There is a point $t_0\ge K_1$ such that

$$M(t_0)=\max\{|x(t_0)|,|y(t_0)|,|z(t_0)|,|w(t_0)|\}.$$

Let $\eta=\min\left\{d_1-1,d_2-1,d_3-1,\ d_4-\sum_{i=0}^{n}L_i\right\}>0$ and $\theta=\sum_{i=0}^{n}q_i+\sup_{t\in R^+}|p(t)|+1$ where $t\ge K_1$. Then, if $M(t_0)=\max\{|x(t_0)|,|y(t_0)|,|z(t_0)|,|w(t_0)|\}=|x(t_0)|$, then, in view of (3.1), we obtain

$$\begin{aligned} 0 &\leq D^+(|x(s)|)_{s=t_0} \leq -d_1 |x(t)| + |y(t)| \\ &\leq (1-d_1)M(t_0) \\ &< -\eta M(t_0) + \theta, \end{aligned} \tag{3.7}$$

if $M(t_0) = \max\{|x(t_0)|, |y(t_0)|, |z(t_0)|, |w(t_0)|\} = |y(t_0)|$, then, in view of (3.2), we have

$$\begin{aligned} 0 &\leq D^+(|y(s)|)_{s=t_0} \leq -d_2 |y(t)| + |z(t)| \\ &\leq (1-d_2)M(t_0) \\ &< -\eta M(t_0) + \theta, \end{aligned} \tag{3.8}$$

if $M(t_0) = \max\{|x(t_0)|, |y(t_0)|, |z(t_0)|, |w(t_0)|\} = |z(t_0)|$, then, in view of (3.3), we get

$$\begin{aligned} 0 &\leq D^+(|z(s)|)_{s=t_0} \leq -d_3 |z(t)| + |w(t)| \\ &\leq (1-d_3)M(t_0) \\ &< -\eta M(t_0) + \theta, \end{aligned} \tag{3.9}$$

if $M(t_0) = \max\{|x(t_0)|, |y(t_0)|, |z(t_0)|, |w(t_0)|\} = |w(t_0)|$, then, in view of (3.4), we get

$$\begin{aligned} 0 \leq & D^+(|w(s)|)_{s=t_0} \\ & \leq \{(-\inf_{t\geq K}(f_1(t) - d_1 - d_2 - d_3))\,|w(t)| \\ & + \sup_{t\geq K}\left|-(d_1+d_2-f_1(t,x(t)))(d_1+d_2+d_3) + (d_1d_2 - f_2(t,x(t))) - d_3^2\right| |z(t)| \\ & + \sup_{t\geq K}\left|(d_1+d_2-f_1(t,x(t)))(d_1^2+d_1d_2+d_2^2)\right. \\ & \left. - (d_1d_2 - f_2(t,x(t)))(d_1+d_2) - f_3(t,x(t))\right| |y(t_0)| \\ & + L_0 |x(t_0)| + \sum_{i=1}^{n} L_i |x(t_0 - \tau_i(t_0))|\} + \sum_{i=0}^{n} q_i + |p(t)| \\ & \leq (\sum_{i=0}^{n} L_i - d_4)M(t_0) + \sum_{i=0}^{n} q_i + |p(t)| \\ & < -\eta M(t_0) + \theta. \end{aligned} \tag{3.10}$$

In addition, if $M(t_0) \geq \dfrac{\theta}{\eta}$, (3.7), (3.8), (3.9) and (3.10) imply that $M(t)$ is strictly decreasing in a small neighborhood $(t_0, t_0 + \delta_0)$. This contradicts that $M(t)$ is non-decreasing. Therefore, $M(t_0) < \dfrac{\theta}{\eta}$ and

$$\max\{|x(t_0)|, |y(t_0)|, |z(t_0)|, |w(t_0)|\} < \frac{\theta}{\eta}. \tag{3.11}$$

For $\forall t > t_0$, by the same approach used in the proof of (3.11), we have

$$\max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\} < \frac{\theta}{\eta}, \text{ if } M(t) = \max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\}.$$

On the other hand, if $M(t) > \max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\}, t > t_0$, then, we can choose $t_0 \leq t_3 < t$ such that $M(t_3) = \max\{|x(t_3)|, |y(t_3)|, |z(t_3)|, |w(t_3)|\} < \dfrac{\theta}{\eta}$

and $M(s) > \max\{|x(s)|, |y(s)|, |z(s)|, |w(s)|\}$ for all $s \in (t_3, t]$. Using a similar argument as in the proof of Case (I), we can show that $M(s) \equiv M(t_3)$ is a constant, for all $s \in (t_3, t]$, which implies $\max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\} < M(t) = M(t_3) = \max\{|x(t_3)|, |y(t_3)|, |z(t_3)|, |w(t_3)|\} < \dfrac{\theta}{\eta}$.

To sum up, the solutions of (1.2) are uniformly bounded. The proof is complete. □

4 An Example

Consider the following fourth-order non-linear differential equation

$$\begin{aligned} x^{(4)}(t) + (18 - \frac{1}{1+t+x^2(t)})x^{(3)}(t) + (78 - \frac{4}{1+t+x^2(t)})x^{(2)}(t) \\ +(127 - \frac{3}{1+t+x^2(t)})x^{(1)}(t) + (69 + \frac{3}{1+t+x^2(t)})x(t) + \sin x(t - |\sin t|) \\ + \cos x(t - 2|\sin t|) + \sin t \sin x(t - e^{|\sin t|}) + \cos t \cos x(t - e^{2|\sin t|}) = \frac{1}{1+t^2} \end{aligned} \tag{4.1}$$

Setting $y(t) = \dfrac{dx(t)}{dt} + 2x(t)$, $z(t) = \dfrac{dy(t)}{dt} + 2y(t)$ and $w(t) = \dfrac{dz(t)}{dt} + 2z(t)$ we can transform (4.1) into the following system

$$\begin{aligned} \frac{dx(t)}{dt} &= -2x(t) + y(t) \\ \frac{dy(t)}{dt} &= -2y(t) + z(t) \\ \frac{dz(t)}{dt} &= -2z(t) + w(t) \end{aligned}$$

$$\begin{aligned} \frac{dw(t)}{dt} = -(10 - \frac{1}{1+t+x^2(t)})w(t) + (2 - \frac{2}{1+t+x^2(t)})z(t) + (1 - \frac{1}{1+t+x^2(t)})y(t) \\ (1 - \frac{1}{1+t+x^2(t)})x(t) - \sin x(t - |\sin t|) - \cos x(t - 2|\sin t|) \\ - \sin t \sin x(t - e^{|\sin t|}) - \cos t \cos x(t - e^{2|\sin t|}) + \frac{1}{1+t^2}. \end{aligned} \tag{4.2}$$

Then we can satisfy the assumptions $(i - iii)$:

(i) $\left|1 - \frac{1}{1+t+u^2}\right| \leq L_0|u| + q_0$ for all $t, u \in R$,

(ii) $|g_1(t,u)| = |\sin u| \leq L_1|u| + q_1, |g_2(t,u)| = |\cos u| \leq L_2|u| + q_2, |g_3(t,u)| = |\sin t \sin u| \leq L_3|u| + q_3, |g_4(t,u)| = |\cos t \cos u| \leq L_4|u| + q_4$ for all $t, u \in R$,

(iii) $d_4 = \inf\limits_{t \geq K}(10 - \frac{1}{1+t+x^2(t)}) - (\sup\limits_{t \geq K}\left|2 - \frac{2}{1+t+x^2(t)}\right| + \sup\limits_{t \geq K}\left|1 - \frac{1}{1+t+x^2(t)}\right|) > \sum\limits_{i=0}^{4} L_i$ by taking suitable L_i and q_i such as $L_0 = L_1 = L_2 = L_3 = L_4 = 1$ for appropriate q_i $(i = 0, 1, 2, 3, 4)$.Hence,all solutions of the system(4.2) are uniformly bounded.

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Fuzzy Soft Convex Subalgebras on Residuated Lattices

Shokoofeh Ghorbani
Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran
sh.ghorbani@uk.ac.ir

Abstract
In this paper, the notions of fuzzy soft subalgebra and fuzzy soft convex subalgebra of a residuated lattice are introduced and some related properties are investigated. Then, we define fuzzy soft congruence on a residuated lattice and obtain the relation between fuzzy soft convex subalgebras and fuzzy soft congruence relations on residuated lattices. The concept of soft homomorphism is defined and some related results are obtained.

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1 Introduction

M. Ward and R. P. Dilworth [12] introduced the concept of residuated lattice in the 1930's. Their investigation stemmed from attempts to generalize properties of the lattice of ideals of a ring. Residuated lattices provide an algebraic semantics for logics without contraction also known as resource sensitive logic. For the previous study of these algebras see [1, 9, 10, 11].

Molodtsov [4] introduced the concept of soft sets in 1999 as a new mathematical tool for dealing with uncertainties. He established the fundamental results of the new theory and successfully applied the soft theory into several directions, such as game theory, theory of probability, smoothness of functions, etc. Maji et al. [5] described first practical application of soft sets in decision making problems which is based on the notion of knowledge reduction in rough set theory. Maji et al. [6] defined and studied several basic notions of soft set theory and several operations on the theory of soft sets.

The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [13]. The notion of fuzzy soft sets, as a generalization of the standard soft sets, is introduced in [7], and Roy et al. [8] presented an application of fuzzy soft sets in a decision making problem.

Fuzzy equivalence relations were introduced by Zadeh [14] as a generalization of the concept of an equivalence relation. They have been studied as a way to measure the degree of similarity between the objects of a given universe of discourse. It has been shown that they are useful in different contexts such as fuzzy control, approximate reasoning, fuzzy cluster analysis, etc.

In this paper, we deal with the algebraic structure of residuated lattice by applying the notion of fuzzy soft sets and study fuzzy soft congruence relations.

In Section 2, some basic definitions and results are mentioned. In Section 3, we introduce the notions of fuzzy soft (convex) subalgebras of a residuated lattice and fuzzy soft congruence relation and investigate some related properties. Then we study the relation between them. Finally, we define soft homomorphisms of residuated lattices and study them.

2 Preliminaries

We recall some definitions and theorems which will be needed in this paper.

Definition 1 ([1, 12])**.** A *residuated lattice* is an algebraic structure $(L, \wedge, \vee, \rightarrow, *, e)$ such that

(1) $(L, \wedge, \vee)$ is a lattice,

(2) $(L, *, e)$ is a commutative monoid where e is a unit element,

(3) $x * y \leq z$ iff $x \leq y \rightarrow z$, for all $x, y, z \in L$.

In the rest of this paper, we denote the residuated lattice $(L, \wedge, \vee, *, \rightarrow, e)$ by L. If e is the greatest element of L, then L is called an integral residuated lattice.

A *fuzzy set* of a non-empty set X is a mapping $\mu : X \rightarrow [0, 1]$. For each $\alpha \in [0, 1]$, the set $\mu_\alpha = \{x \in X : \mu(x) \geq \alpha\}$ is called *α-level subset* of μ.

Definition 2 ([3])**.** A fuzzy subset S of a residuated lattice L is called a *fuzzy subalgebra* of L, if

(1) $S(e) \geq S(x)$,

(2) $S(y \rightarrow x) = \min\{S(x), S(y)\}$,

(3) $S(x * y) = \min\{S(x), S(y)\}$,

(4) $S(x \wedge y) = \min\{S(x), S(y)\}$,

(5) $S(x \vee y) = \min\{S(x), S(y)\}$,

for all $x, y \in L$.

Definition 3 ([3])**.** A fuzzy subalgebra S of a residuated lattice L is said to be a *fuzzy convex subalgebra* of L if $a \in S_\alpha$, $b \in S_\beta$ and $a \leq c \leq b$, then there exists γ between α and β such that $c \in S_\gamma$.

Definition 4 ([14])**.** Let X, Y be to sets and $\mathfrak{F}(X \times Y)$ be the set of all fuzzy subset of $X \times Y$. Then a fuzzy subset $R \in \mathfrak{F}(X \times Y)$ is called a *fuzzy binary relation* from X to Y and $R(x, y)$ is called the *degree of relation* between x and y, where $(x, y) \in X \times Y$. If $X = Y$, then R is called a *fuzzy relation* on X.

Definition 5 ([14])**.** A *fuzzy equivalence relation* R on a non-empty set X is a fuzzy relation on X satisfying the following conditions:

(R1) $R(x, x) = \sup\{R(y, z) : y, z \in X\}$ (reflexive),

(R2) $R(x, y) = R(y, x)$ (symmetric),

(R3) $R(x,z) = \min\{R(x,y), R(y,z)\}$, for all $x, y, z \in X$ (transitive).

Definition 6 ([3])**.** A fuzzy equivalence relation θ on a residuated lattice L is called a *fuzzy congruence relation* on L if

(C1) $\theta(y \to x, w \to z) = \min\{\theta(x,z), \theta(y,w)\}$,

(C2) $\theta(x * y, z * w) = \min\{\theta(x,z), \theta(y,w)\}$,

(C3) $\theta(x \wedge y, z \wedge w) = \min\{\theta(x,z), \theta(y,w)\}$,

(C4) $\theta(x \vee y, z \vee w) = \min\{\theta(x,z), \theta(y,w)\}$,

for all $x, y, z, w \in L$.

Definition 7 ([4])**.** Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the power set of U and $A \subseteq E$. A pair (F, A) is called a *soft set* over U, where $F : A \to P(U)$ is a map. In other words, a soft set over U is a parametrized family of subsets of the universe U.

Definition 8 ([7])**.** Let U be an initial universe set and E be a set of parameters. Let $\mathfrak{F}(U)$ denote the set of all fuzzy sets in U. Then (F, A) is called a *fuzzy soft set* over U where $A \subseteq E$ and $F : A \to \mathfrak{F}(U)$ is a map.

In general, for every $u \in A$, $F[u]$ is a fuzzy set in U and it is called *fuzzy value set* of parameter u. If for every $u \in A$, $F[u]$ is a fuzzy subset of U, then (F, A) is degenerated to be the fuzzy soft set. Thus, from the above definition, it is clear that fuzzy soft sets are a generalization of soft sets.

Definition 9. Let X and Y be two non-empty subsets of some universal set U and E be a set of parameters. A pair (R, E) is called a *soft relation* where $R : E \to \mathfrak{F}(X \times Y)$ is a map.

3 Fuzzy soft convex subalgebras

In what follows, let E be a set of parameters unless otherwise specified and L be a residuated lattice.

Definition 10. Let (F, A) be a fuzzy soft set over a residuated lattice L where A is a subset of E. If there exists $u \in A$ such that $F[u]$ is a fuzzy subalgebra of L, we say that (F, A) is a *fuzzy soft subalgebra based on a parameter* u over L. If (F, A) is a fuzzy soft subalgebra based on a parameter u over L for all $u \in A$, we say that (F, A) is a *fuzzy soft subalgebra* over L.

Example 11. Suppose that there are five players in the universe, that is

$$U = \{a, b, c, e, d\}.$$

Let $\Cap$, $\Cup$, $\circledast$, $\twoheadrightarrow$ be four soft game machines for two players to play accordingly in such a way, we have the following results:

$a \Cap x = a$ and $d \Cap x = x$ for all $x \in U$,

$$b \Cap x = \begin{cases} a & if \quad x = a \\ b & if \quad x \in \{b, c, e, d\} \end{cases} \quad c \Cap x = \begin{cases} x & if \quad x \in \{a, b\} \\ c & if \quad x \in \{c, e, d\} \end{cases}$$

$$e \Cap x = \begin{cases} x & if \quad x \in \{a,b,c\} \\ e & if \quad x \in \{e,d\} \end{cases}$$

$a \Cup x = x$ and $d \Cup x = d$ for all $x \in U$,

$$b \Cup x = \begin{cases} b & if \quad x \in \{a,b\} \\ x & if \quad x \in \{c,e,d\} \end{cases} \quad c \Cup x = \begin{cases} c & if \quad x \in \{a,b,c\} \\ x & if \quad x \in \{e,d\} \end{cases}$$

$$e \Cup x = \begin{cases} e & if \quad x \in \{a,b,c,e\} \\ d & if \quad x = d \end{cases}$$

$a \circledast x = a$ and $e \circledast x = x$ for all $x \in U$,

$$c \circledast x = \begin{cases} a & if \quad x = a \\ b & if \quad x = b \\ c & if \quad x \in \{c,e\} \\ d & if \quad x = d \end{cases} \quad d \circledast x = \begin{cases} a & if \quad x = a \\ b & if \quad x = b \\ d & otherwise \end{cases}$$

$$b \circledast x = \begin{cases} a & if \quad x = a \\ b & otherwise \end{cases}$$

$a \twoheadrightarrow x = d$ and $e \twoheadrightarrow x = x$ for all $x \in U$,

$$c \twoheadrightarrow x = \begin{cases} a & if \quad x = a \\ b & if \quad x = b \\ e & if \quad x \in \{b,e\} \\ d & if \quad x = d \end{cases} \quad d \twoheadrightarrow x = \begin{cases} a & if \quad x = a \\ d & if \quad x = d \\ b & otherwise \end{cases}$$

$$b \twoheadrightarrow x = \begin{cases} a & if \quad x = a \\ d & otherwise \end{cases}$$

Then $(U, \Cap, \Cup, \circledast, \twoheadrightarrow, e)$ is a residuated lattice. Consider a set of parameters:

$$E = \{Clever, Agile\}.$$

(1) Let (F,E) be a fuzzy soft set over U. Then $F[Clever]$ and $F[Agile]$ are fuzzy sets in U. Define them as follows:

F	a	b	c	e	d
Clever	0.4	0.8	0.5	0.9	0.8
Agile	0.2	0.3	0.8	0.6	0.7

Then (F,E) is a fuzzy soft subalgebra based on a parameter “Clever” over U but it is not a fuzzy soft subalgebra based on a parameter “Agile” over U. Hence (F,E) is not a fuzzy soft subalgebra over U.

(2) Let (G,E) be a fuzzy soft set over U. Then $G[Clever]$ and $G[Agile]$ are fuzzy sets in U. Define them as follows:

G	a	b	c	e	d
Clever	0.5	0.7	0.5	1	0.7
Agile	0.1	0.3	0.2	0.4	0.3

Then $G[Clever]$ and $G[Agile]$ are fuzzy soft subalgebras based on parameters “Clever” and “Agile” over U, respectively. Hence (G,E) is a fuzzy soft subalgebra over U.

Definition 12. Let (F, A) be a fuzzy soft subalgebra over a residuated lattice L where A is a subset of E. If there exists $u \in A$ such that $F[u]$ is a fuzzy convex subalgebra of L, we say that (F, A) is a *fuzzy soft convex subalgebra based on a parameter* u over L. If (F, A) is a fuzzy soft convex subalgebra based on a parameter u over L for all $u \in A$, we say that (F, A) is a *fuzzy soft convex subalgebra* over L.

Example 13. Suppose that there are five players in the universe, that is

$$U = \{a, b, c, e, d\}.$$

Let $\Cap$, $\Cup$, $\circledast$, $\twoheadrightarrow$ be four soft game machines for two players to play accordingly in such a way, we have the following results:

$a \Cap x = a$, $d \Cap x = x$, for all $x \in U$,

$$b \Cap x = \begin{cases} a & if \quad x \in \{a, c\} \\ b & \quad otherwise \end{cases} \quad c \Cap x = \begin{cases} a & if \quad x \in \{a, b\} \\ c & \quad otherwise \end{cases}$$

$$e \Cap x = \begin{cases} x & if \quad x \in \{e, d\} \\ b & \quad otherwise \end{cases}$$

$a \Cup x = x$, $d \Cup x = d$ for all $x \in U$,

$$b \Cup x = \begin{cases} b & if \quad x \in \{a, b\} \\ e & if \quad x \in \{c, e\} \\ d & if \quad x = d \end{cases} \quad c \Cup x = \begin{cases} c & if \quad x \in \{a, c\} \\ e & if \quad x \in \{e, b\} \\ d & if \quad x = d \end{cases}$$

$$e \Cup x = \begin{cases} d & if \quad x = d \\ x & \quad otherwise \end{cases}$$

$a \circledast x = a$ and $e \circledast x = x$ for all $x \in U$,

$$b \circledast x = \begin{cases} a & if \quad x \in \{a, c\} \\ b & if \quad otherwise \end{cases} \quad c \circledast x = \begin{cases} a & if \quad x \in \{a, b\} \\ c & if \quad otherwise \end{cases}$$

$$d \circledast x = \begin{cases} d & if \quad x \in \{e, d\} \\ x & if \quad otherwise \end{cases}$$

$a \twoheadrightarrow x = d$ and $e \twoheadrightarrow x = x$ for all $x \in U$,

$$b \twoheadrightarrow x = \begin{cases} c & if \quad x \in \{a, c\} \\ d & if \quad otherwise \end{cases} \quad c \twoheadrightarrow x = \begin{cases} b & if \quad x \in \{a, b\} \\ d & if \quad otherwise \end{cases}$$

$$d \twoheadrightarrow x = \begin{cases} a & if \quad x = a \\ b & if \quad x = b \\ c & if \quad x = c \\ d & \quad otherwise \end{cases}$$

Then $(U, \Cap, \Cup, \circledast, \twoheadrightarrow, e)$ is a residuated lattice. Consider a set of parameters:

$$E = \{Clever, Smart\}.$$

(1) Let (F, E) be a fuzzy soft set over U. Then $F[Clever]$ and $F[Smart]$ are fuzzy sets in U. Define them as follows:

F	a	b	c	e	d
Clever	0.5	0.5	0.6	0.7	0.7
Smart	0.3	0.5	0.1	0.8	0.7

Then (F, E) is a fuzzy soft convex subalgebra on the parameter "Clever" over U but it is not a fuzzy soft convex subalgebra based on the parameter "Smart" over U. Hence (F, E) is not a fuzzy soft convex subalgebra over U.

(2) Consider fuzzy soft subalgebra (G, E) over U where $G[Clever]$ and $G[Agile]$ are fuzzy sets in U:

G	a	b	c	e	d
Clever	0.3	0.3	0.65	0.82	0.7
Smart	0.11	0.223	0.11	0.4	0.34

Then $G[Clever]$ and $G[Smart]$ are fuzzy soft convex subalgebras of residuated lattice based on parameters "Clever"and "Smart" over U, respectively. Hence (G, E) is a fuzzy soft convex subalgebra over U.

Remark 14. We notice that each fuzzy soft subalgebra may not be a fuzzy soft convex subalgebra of a residuated lattice. Consider a fuzzy soft subalgebra (G, E) over U in Example 3.2 which is not a fuzzy soft convex subalgebra over U.

Definition 15 ([5])**.** Let (F, A) and (G, B) be two fuzzy soft sets over a common universe U. The *union* of (F, A) and (G, B) is defined to be the fuzzy soft set (H, C) satisfying the following conditions:

(i) $C = A \cup B$,

(ii) for all $u \in C$, $H[u] = \begin{cases} F[u] & if \quad u \in A \backslash B \\ G[u] & if \quad u \in B \backslash A \\ F[u] \cup G[u] & if \quad u \in A \cap B \end{cases}$

where $F[u] \cup G[u]$ is union of fuzzy sets. In this case, we write $(F, A) \sqcup (G, B) = (H, C)$.

Theorem 16. *Let (F, A) and (G, A) be two fuzzy soft convex subalgebras over a residuated lattice L. If A and B are disjoint, then the union $(F, A) \sqcup (G, B)$ is a fuzzy soft convex subalgebra over L.*

Proof. Suppose that $(F, A) \sqcup (G, B) = (H, C)$, where $C = A \cup B$ and for all $u \in C$,

$$H[u] = \begin{cases} F[u] & if \quad u \in A \backslash B \\ G[u] & if \quad u \in B \backslash A \\ F[u] \cup G[u] & if \quad u \in A \cap B \end{cases}$$

By assumption, $A \cap B = \emptyset$. Hence we have either $u \in A \backslash B$ or $u \in B \backslash A$ for all $u \in C$. Consider the following cases:

(1) If $u \in A \backslash B$, then $H[u] = F[u]$ is a fuzzy convex subalgebra over L because (F, A) is a fuzzy soft convex subalgebra over L.

(2) If $u \in B \backslash A$, then $H[u] = G[u]$ is a fuzzy convex subalgebra over L because (G, A) is a fuzzy soft convex subalgebra of over L.

Therefore $(H, C) = (F, A) \sqcup (G, B)$ is a fuzzy soft convex subalgebra over L. $\square$

The following example shows that Theorem 16 is not valid, if A and B are not disjoint.

Example 17. Let $(U, \Cap, \Cup, \circledast, \twoheadrightarrow, e)$ be the residuated lattice in Example 13. Consider two sets of parameters:

$$A = \{Attentive, Brave\},\ B = \{Attentive\}$$

Then A and B are not disjoint.

Let (F, A) be a fuzzy soft set over U. Then $F[Attentive]$, $F[Brave]$ are fuzzy sets in U. Define them as follows:

F	a	b	c	e	d
Smart	0.2	0.4	0.2	0.7	0.5
Attentive	0.1	0.1	0.4	0.6	0.5

Then (F, A) is a fuzzy soft convex subalgebra over U.

Let (G, B) be a fuzzy soft set over U. Then $G[Attentive]$ is a fuzzy set in U. Define it as follows:

G	a	b	c	e	d
Attentive	0.3	0.5	0.3	0.8	0.7

Then (G, B) is a fuzzy soft convex subalgebra over U.

But the union (F, A) and (G, B) is not a fuzzy soft convex subalgebra over U. Suppose that $u = Attentive$. Then

$$(F[u] \cup G[u])(b \circledast c) = (F[u] \cup G[u])(a) = max\{F[u](a), G[u](a)\} = 0.3$$
$$\min\{(F[u] \cup G[u])(b), (F[u] \cup G[u])(c)\} = 0.4$$

but $0.3 \ngeq 0.4$.

Definition 18 ([5]). Let (F, A) and (G, B) be two fuzzy sets over a common universe U. The *extended intersection* of (F, A) and (G, B) is defined to be the fuzzy soft set (H, C) satisfying the following conditions:

(i) $C = A \cup B$

(ii) for all $u \in C$,

$$H[u] = \begin{cases} F[u] & if \quad u \in A \backslash B \\ G[u] & if \quad u \in B \backslash A \\ F[u] \cap G[u] & if \quad u \in A \cap B \end{cases}$$

where $F[u] \cap G[u]$ is intersectin of fuzzy sets. In this case, we write $(F, A) \sqcap_e (G, B) = (H, C)$.

Theorem 19. *Let (F, A) and (G, B) be two fuzzy soft convex subalgebras over a residuated lattice L. Then the extended intersection of (F, A) and (G, B) is a fuzzy soft convex subalgebra over L.*

Proof. **Proof.** Let $(F, A) \sqcap_e (G, B) = (H, C)$ be the extended intersection of (F, A) and (G, B). We have $C = A \cup B$. Suppose that $u \in C$ be arbitrary.

(i) if $u \in A \backslash B$, then $H[u] = F[u]$ is a fuzzy convex subalgebra over L,

(ii) if $u \in B \backslash A$, then $H[u] = G[u]$ is a fuzzy convex subalgebra over L,

(iii) if $A \cap B \neq \emptyset$, then $H[u] = F[u] \cap G[u]$ is a fuzzy convex subalgebra for all $u \in A \cap B$, because the intersection of two fuzzy convex subalgebras in L is an fuzzy convex subalgebra. Therefore (H, C) is a fuzzy soft convex subalgebra over L.

□

Corollary 20. *Let (F, A) and (G, A) be two fuzzy soft convex subalgebras over a residuated lattice L. Then the extended intersection of (F, A) and (G, A) is a fuzzy soft convex subalgebra over L.*

Definition 21 ([5])**.** Let (F, A) and (G, B) be two fuzzy soft sets over a common universe U such that $A \cap B \neq \emptyset$. The *restricted intersection* of (F, A) and (G, B) is defined to be the fuzzy soft set (H, C) satisfying the following conditions:

(i) $C = A \cap B$,

(ii) $H[u] = F[u] \cap G[u]$ for all $u \in C$.

In this case, we write $(F, A) \sqcap (G, B) = (H, C)$.

Corollary 22. *The restricted intersection of two fuzzy soft convex subalgebras over a residuated lattice L is a fuzzy soft convex subalgebra over L.*

Notation The set of all fuzzy soft convex subalgebras over a residuated lattice L is denoted by $FSC(L)$.

Clearly $FSC(L)$ is a lattice, because if $(F, A), (G, B) \in FSC(L)$, then $(F, A) \bigvee (G, B)$ (i.e. the intersection of all fuzzy soft convex subalgebras containing $(F, A), (G, B)$) is the least upper bound of (F, A) and (G, B). Also, $(F, A) \sqcap_e (G, B) \in FSC(L)$ is the greatest lower bound of (F, A) and (G, B). Since we can replace the set (F, A) and (G, B) by an arbitrary family of fuzzy soft convex subalgebras, so the lattice $(FSC(L), \sqcap, \bigvee)$ is a complete lattice.

Definition 23. Let (R, A) be a fuzzy soft relation on a residuated lattice L where A be a subset of E.

(1) If $R[u]$ is a fuzzy equivalence relation on L for all $u \in A$, we say that (R, A) is a *fuzzy soft equivalence relation* over L.

(2) A fuzzy soft equivalence relation (θ, A) over L is called *fuzzy soft congruence relation*, if $\theta[u]$ is a fuzzy congruence relation on L for all $u \in A$.

Example 24. Consider the residuated lattice which is defined in Example 11. Suppose that $A = \{Smart\}$ and $u = Smart$. Define

$$\theta[u](a, b) = \theta[u](a, d) = \theta[u](a, c) = \theta[u](a, e) = 0.2,$$
$$\theta[u](d, b) = \theta[u](b, c) = \theta[u](e, d) = \theta[u](b, e) = \theta[u](c, d) = 0.45,$$
$$\theta[u](c, e) = 0.67,$$
$$\theta[u](e, e) = \theta[u](a, a) = \theta[u](b, b) = \theta[u](c, c) = \theta[u](d, d) = 0.87.$$

Then (θ, A) is a fuzzy soft congruence relation on L.

Definition 25. Let (F, A) be a fuzzy soft convex subalgebras over L. Fuzzy soft relation (θ_F, A) on L which is defined by

$$\theta_F[u](x, y) = min\{F[u]((y \to x) \wedge e), F[u]((x \to y) \wedge e)\}$$

is called the *fuzzy soft relation induced* by (F, A).

Proposition 26. *Let (F, A) be a fuzzy soft convex subalgebras over a residuated lattice L. Then the fuzzy soft relation (θ_F, A) induced by (F, A) is a fuzzy soft congruence relation on L.*

Proof. By assumption (F, A) is a fuzzy soft congruence relation on L. Thus $F[u]$ is a fuzzy convex subalgebra of L for all $u \in A$. By Theorem 3.19 in [3],

$$min\{F[u]((y \to x) \wedge e), F[u]((x \to y) \wedge e)\}$$

is a fuzzy congruence relation on L for all $u \in A$. We get that $\theta_F[u]$ is a fuzzy congruence relation on L for all $u \in A$. Therefore (θ_F, A) is a fuzzy soft congruence relation on L. □

Definition 27. Let (θ, A) be a fuzzy soft congruence relation on a residuated lattice L. Then the fuzzy subset (F_θ, A) which is defined by

$$F_\theta[u](x) = \theta[u](x, e)$$

is called the *fuzzy soft subset induced* by (θ, A).

Proposition 28. *Let (θ, A) be a fuzzy soft congruence relation on a residuated lattice L. Then the fuzzy soft subset (F_θ, A) induced by (θ, A) is a fuzzy soft convex subalgebra over L.*

Proof. Since (θ, A) is a fuzzy soft congruence relation on L, then $\theta[u]$ is a fuzzy congruence relation on L for all $u \in A$. By Theorem 3.22 in [3], $\theta[u](x, e)$ is a fuzzy convex subalgebra of L for all $u \in A$. Hence (F_θ, A) is a fuzzy soft convex subalgebra over L. □

Theorem 29. *There is a bijection between the set of all fuzzy soft convex subalgebras over a residuated lattice L and the set of all fuzzy soft congruence relations on L.*

Proof. It follows from Proposition 26 and Proposition 28. □

Definition 30. Let (θ, A) be a fuzzy soft congruence relation on a residuated lattice L, $u \in A$ and $x \in L$. Define the fuzzy set $[x]_{\theta[u]}$ by $[x]_{\theta[u]}(y) = \theta[u](x, y)$. The fuzzy set $[x]_{\theta[u]}$ is called a *fuzzy soft congruence class* of x by $\theta[u]$ in L.

Proposition 31. *Let (θ, A) be a soft congruence relation over a residuated lattice L and $u \in A$. Define $L/\theta[u] = \{[x]_{\theta[u]} : x \in L\}$. Then $(L/\theta[u], \wedge, \vee, *, \to, [e]_{\theta[u]})$ is a residuated lattice where*

$$_{\theta[u]} \wedge [y]_{\theta[u]} = [x \wedge y]_{\theta[u]}$$
$$[x]_{\theta[u]} \vee [y]_{\theta[u]} = [x \vee y]_{\theta[u]}$$
$$[x]_{\theta[u]} * [y]_{\theta[u]} = [x * y]_{\theta[u]}$$
$$[x]_{\theta[u]} \to [y]_{\theta[u]} = [x \to y]_{\theta[u]}$$

for all $x, y \in L$.

Definition 32. Let (F, A) and (G, B) be two fuzzy soft subalgebras over residuated lattices L and M respectively and let $f : L \to M$ and $g : A \to B$ be two functions. Then a pair (f, g) is called a *soft homomorphism*, if the following conditions hold:

(1) f is a residuated lattice homomorphism from L to M,

(2) $f(F[u]) = G[g(u)]$ for all $u \in A$.

Then we say that (F, A) is a *soft homomorphic* to (G, B).

Proposition 33. *Let L and M be two residuated lattices, (F, A) be a fuzzy soft subalgebra over L and let $f : L \to M$ be a surjective homomorphism of residuated lattices. Then*

(1) $(f(F), A)$ is a fuzzy soft subalgebra over M.

(2) If L is an integral residuated lattice, then $(f(F), A)$ is a fuzzy soft convex subalgebra over M.

Proof. (1) Since (F, A) is a fuzzy soft subalgebra over L, then $F[u]$ is a fuzzy subalgebra in L for all $u \in L$. Hence $f(F) : A \to \mathfrak{F}(M)$ is a mapping given by

$$f(F)[u](y) = \begin{cases} \sup_{y=f(x)} F[u](x) & if \quad f^{-1}(G) \neq \emptyset \\ 0 & if \quad f^{-1}(G) = \emptyset \end{cases}$$

for all $u \in A$. We will show that $f(F)[u]$ is a fuzzy subalgebra of M for all $u \in A$.

Suppose that $\star \in \{\wedge, \vee, *, \to\}$. Let $x, y \in M$. Since f is onto, there exist $a, b \in L$ such that $f(a) = x$ and $f(y) = b$. Since f is a homomorphism, $x \star y = f(a) \star f(b) = f(a \star b)$. Hence $a \star b \in f^{-1}(x \star y)$. We have

$$\begin{aligned} f(F)[u](x \star y) &= \sup\{f(F)[u](z) : z \in f^{-1}(x \star y)\} \\ &\geq \sup\{f(F)[u](a \star b) : a \in f^{-1}(x), b \in f^{-1}(y)\} \\ &\geq \sup\{\min\{f(F)[u](a), f(F)[u](b)\} : a \in f^{-1}(x) \text{ and } b \in f^{-1}(y)\} \\ &= \min\{\sup\{f(F)[u](a) : a \in f^{-1}(x)\}, \sup\{f(F)[u](b) : b \in f^{-1}(y)\}\} \\ &= \min\{f(F)[u](x), f(F)[u](y)\}. \end{aligned}$$

Hence $f(F)[u]$ is a fuzzy subalgebra of M, for all $u \in A$.

(2) Now, suppose that $a \in f(F)[u]_\alpha$, $b \in f(F)[u]_\beta$ and $a \leq c \leq b$. We will show that $f(F)[u](a) \leq f(F)[u](c)$.

Suppose that $f(F)[u](a) > f(F)[u](c)$. Then there exists $a_0 \in L$ such that $f(a_0) = a$ and $F[u](a_0) > \sup\{F[u](y) : y \in f^{-1}(c)\}$. We have $f(a_0) = a \leq c = f(y)$ for all $y \in f^{-1}(c)$. Let $y \in f^{-1}(c)$ be arbitrary. Since f is a homomorphism, then $c = a \vee c = f(a_0) \vee f(y) = f(a_0 \vee y)$, that is $a_0 \vee y \in f^{-1}(c)$. We get that $F[u](a_0 \vee y) < F[u](a_0)$. Since L is an integral commutative residuated lattice, then $F[u](a_0 \vee y) \geq \max\{F[u](a_0), F[u](y)\} = F[u](a_0)$ which is a contradiction. Hence $f(F)[u](a) \leq f(F)[u](c)$.

Similarly, we can prove that $f(F)[u](c) \leq f(F)[u](b)$. Hence we have $f(F)[u](a) \leq f(F)[u](c) \leq f(F)[u](b)$. Put $\gamma_1 = f(F)[u](c)$ and $\gamma = \min\{\gamma_1, \beta\}$. We have $\alpha \leq \gamma \leq \beta$ and $c \in f(F)[u]_\gamma$. Therefore $f(F)[u]$ is a fuzzy convex subalgebra of M, for all $u \in A$. So $(f(F), A)$ is a fuzzy soft convex subalgebra over M. □

Proposition 34. *Let L and M be two residuated lattices, (G, A) be a fuzzy soft subalgebra over residuated lattice M and let $f : L \to M$ be a homomorphism of residuated lattices. Then*

(1) $(f^{-1}(G), A)$ is a fuzzy soft subalgebra over L.

(2) If M is an integral residuated lattice, then $(f^{-1}(G), A)$ is a fuzzy soft convex subalgebra over L.

Proof. (1) By assumption (G, A) is a fuzzy soft subalgebra over M, then $G[u]$ is a fuzzy subalgebra in M for all $u \in M$. Since f is a homomorphism, then $f^{-1}(G[u])$ is a fuzzy set in L where $f^{-1}(G[u])(x) = G[u](f(x))$. We have $f^{-1}(G) : A \to \mathfrak{F}(L)$. Let $a, b \in L$ and $\star \in \{\wedge, \vee, *, \to\}$. We have

$$\begin{aligned} f^{-1}(G[u])(a \star b) = G[u](f(a) * f(b)) &\geq \min\{G[u](f(a)), G[u](f(b))\} \\ &= \min\{f^{-1}(G[u])(a), f^{-1}(G[u])(b)\}. \end{aligned}$$

Then $(f^{-1}(G), A)$ is a fuzzy soft subalgebra over L.

(2) Now, suppose that $a \in f^{-1}(G[u])_\alpha$, $b \in f^{-1}(G[u])_\beta$ and $a \leq c \leq b$. Since f is a homomorphism, $f(a) \leq f(c) \leq f(b)$. By assumption M is an integral commutative residuated lattice, then we get that $G[u](f(a)) \leq G[u](f(c)) \leq G[u](f(b))$, that is $f^{-1}(G[u])(a) \leq f^{-1}(G[u])(c) \leq f^{-1}(G[u])(b)$. Put $\gamma_1 = f^{-1}(G[u])(c)$ and $\gamma = \min\{\gamma_1, \beta\}$. We have $\alpha \leq \gamma \leq \beta$ and $c \in f^{-1}(G[u])_\gamma$. Therefore $f^{-1}(G[u])$ is a fuzzy convex subalgebra of L, for all $u \in A$. Hence $(f^{-1}(G), A)$ is a fuzzy soft convex subalebra over L. □

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On the Convergence of Three Step Iterative Process for Three Asymptotically Nonexpansive Multi-maps in Banach Spaces

Pankaj Kumar Jhade
Department of Mathematics, NRI Institute of Information Science & Technology, Bhopal, India
pmathsjhade@gmail.com

Amarjeet Singh Saluja
Department of Mathematics, J H Government Postgraduate College, Betul, India
dssaluja@rediffmail.com

Abstract
In this paper, we introduced a new three-step iterative scheme with errors for finding common fixed points of three asymptotically nonexpansive multi-maps in Banach spaces and prove a strong convergence theorem of the purposed algorithm under some control conditions. Our results improved and extended many known results existing in the literature.

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1 Introduction

Let D be a nonempty convex subset of a Banach space E. The set D is called *proximinal* if for each $x \in E$, there exists an element $y \in D$ such that $\|x - y\| = d(x, D)$, where $d(x, D) = \inf\{\|x - z\| : z \in D\}$. Let $CB(D), K(D)$ and $P(D)$ denote the families of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of D, respectively. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$$

for $A, B \in CB(D)$.

A single-valued map $T : D \to D$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. A multi- valued map $T : D \to CB(D)$ is said to be *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called fixed point of $T : D \to D$ (respectively,$T : D \to CB(D)$) if $p = Tp$ (respectively, $p \in Tp$). The set of fixed points of T is denoted by $F(T)$.

The mapping $T : D \to CB(D)$ is called
(i) *asymptotically nonexpansive* if there exists a sequence $r_n \geq 1, \lim_{n\to\infty} r_n = 1$ and

$H(T^n x, T^n y) \leq r_n \|x - y\|$ for all $x, y \in D$ and $n \in \mathbb{N}$;
(ii) *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that $H(T^n x, T^n y) \leq L\|x - y\|$ for all $x, y \in D$ and $n \in \mathbb{N}$;

(iii) *hemi compact* if, for any sequence $\{x_n\}$ in D such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in D$. We note that if D is compact, then every multi-valued mapping $T : D \to CB(D)$ is hemicompact.

In 1953, Mann [8] introduced the following iteration scheme, starting from $x_0 \in D$, to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H:

$$x_{n+1} = \alpha_n T x_n + (1 - \alpha_n) x_n \quad \text{for all} \quad n \in \mathbb{N} \tag{1.1}$$

where $\{\alpha_n\}$ be a sequence in [0,1] satisfies certain conditions. However, we note that Mann's iteration process (1.1) has only weak convergence, in general; for instance, see([1],[4],[11]).
The Ishikawa [5] iteration scheme, starting from $x_0 \in D$, is the sequence $\{x_n\}$ defined by

$$\begin{aligned} y_n &= \beta_n T x_n + (1 - \beta_n) x_n \ \text{ for all } \ n \in \mathbb{N} \\ x_{n+1} &= \alpha_n T y_n + (1 - \alpha_n) x_n \ \text{ for all } \ n \in \mathbb{N} \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in [0,1] satisfies certain conditions.
Iterative techniques for approximating fixed points of nonexpansive single-valued mappings have been studied by various authors (see; e.g. [5], [11], [14], [18]) using the Mann iteration or the Ishikawa iteration scheme. For details on the subject, we refer the reader to Berinde [2].

Sastry and Babu [12] defined the Mann and Ishikawa iteration schemes for multi-valued mappings.
Let $T : D \to P(D)$ be a multi-valued map and fix $p \in F(T)$.

(A) The sequence of Mann iterates is defined by $x_0 \in D$,

$$x_{n+1} = \alpha_n y_n + (1 - \alpha_n) x_n \quad \text{for all} \ \ n \in \mathbb{N}$$

where $\{\alpha_n\}$ be a sequence in [0,1] and $y_n \in Tx_n$ such that $\|y_n - p\| = d(p, Tx_n)$.

(B) The sequence of Ishikawa iterates is defined by $x_0 \in D$,

$$y_n = \beta_n z_n + (1 - \beta_n) x_n \quad \text{for all} \quad n \in \mathbb{N}$$

where $\{\beta_n\} \in [0, 1]$ and $z_n \in Tx_n$ such that $\|z_n - p\| = d(p, Tx_n)$, and

$$x_{n+1} = \alpha_n z_n^{'} + (1 - \alpha_n) x_n \quad \text{for all} \quad n \in \mathbb{N}$$

where $\{\alpha_n\} \in [0, 1]$ and $z_n^{'} \in Ty_n$ such that $\|z_n^{'} - p\| = d(p, Ty_n)$.

Sastry and Babu [12] proved that the Mann and Ishikawa iteration schemes for a multi-valued map T with a fixed point p converges to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from p. More precisely, they proved the following result for nonexpansive multi-valued map with compact domain.

Theorem 1. *([12], Theorem 5):Let E be a Hilbert space, D be a nonempty compact convex subset of E, and $T : D \to P(D)$ be a multi-valued map with a fixed point $p \in F(T)$. Assume that (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$ and (iii) $\sum \alpha_n\beta_n = \infty$. Then the Ishikawa iterates $\{x_n\}$ defined by (B) converges to a fixed point of T.*

Panyanak [10] extend the above result to uniformly convex Banach spaces but the domain of T remains compact.

Theorem 2. *([10], Theorem 3.1):Let E be a uniformly convex Banach space, D be a nonempty compact convex subset of E, and $T : D \to P(D)$ be a multi-valued map with a fixed point $p \in F(T)$. Assume that (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$ and (iii) $\sum \alpha_n\beta_n = \infty$. Then the Ishikawa iterates $\{x_n\}$ defined by (B) converges to a fixed point of T.*

Later, Song and Wang [17] noted that there was a gap in the proof of Theorem 1 (see [12], Theorem 5)and Theorem 2 (see [10], Theorem 3.1). Because the iteration x_n depends on a fixed $p \in F(T)$ as well as T. If $q \in F(T)$ and $q \neq p$, then the iteration x_n defined by q is different from the one defined by p. Therefore, one cannot derive the monotonicity of sequence $\{\|x_n - q\|\}$ from the monotonicity of $\{\|x_n - p\|\}$. So the conclusion of Theorem 1 and Theorem 2 are ambiguous. They further solved/revised the gap and also gave the affirmative answer the question using the following Ishikawa iteration scheme.

(C): Let D be a nonempty convex subset of E, $\alpha_n, \beta_n \in [0, 1]$ and $\gamma_n \in (0, \infty)$ such that $\lim_{n\to\infty} \gamma_n = 0$. Choose $x_0 \in D$. Then

$$\begin{aligned} y_n &= \beta_n z_n + (1 - \beta_n)x_n \quad \text{for all} \quad n \in \mathbb{N} \\ x_{n+1} &= \alpha_n z_n^{'} + (1 - \alpha_n)x_n \quad \text{for all} \quad n \in \mathbb{N} \end{aligned}$$

where $\|z_n - z_n^{'}\| \leq H(Tx_n, Ty_n) + \gamma_n$ and $\|z_{n+1} - z_n^{'}\| \leq H(Tx_{n+1}, Ty_n) + \gamma_n$ for $z_n \in Tx_n$ and $z_n^{'} \in Ty_n$.
Song and Wang [17] proved the following results. In the result, the domain of T is still compact, which is a strong condition.

Theorem 3. *([17], Theorem 1): Let E be a uniformly convex Banach space, D a nonempty compact convex subset of E and $T : D \to CB(D)$ a nonexpansive multi-valued map with $F(T) \neq \phi$ satisfying $TP = \{p\}$ for any $p \in F(T)$. Assume that (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$ and (iii) $\sum \alpha_n\beta_n = \infty$. Then the Ishikawa iterates $\{x_n\}$ defined by (C) converges to a fixed point of T.*

Recall that a multi-valued map $T : D \to CB(D)$ is said to satisfy Condition (I) [14] if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in D$.

Theorem 4. *([17], Theorem 2): Let E be a uniformly convex Banach space, D a nonempty compact convex subset of E and $T : D \to CB(D)$ a nonexpansive multi-valued map with $F(T) \neq \phi$ satisfying $Tp = \{p\}$ for any $p \in F(T)$. Assume that T satisfies Condition (I) and $0 \leq \alpha_n, \beta_n \in [a, b] \subset (0, 1)$. Then the Ishikawa iterates $\{x_n\}$ defined by (C) converges to a fixed point of T.*

In 2009, Shahzad and Zegeye [15] extended and improved the results of Panyanak [10], Sastry and Babu [12] and Song and Wang [17] to quasi-nonexpansive multi-valued maps. They also relaxed compactness of the domain of T and constructed an iteration scheme which removes the restriction of T namely $Tp = \{p\}$ for any $p \in F(T)$. The results provided an affirmative answer to Panyanak [10] question in a more general setting. They introduced a new iteration as follows:

Let D be a nonempty convex subset of a Banach space E and $\alpha_n, \alpha_n^{'} \in [0,1]$. The sequence of Ishikawa iterates is defined by $x_0 \in D$,

$$\begin{aligned} y_n &= \alpha_n^{'} z_n^{'} + (1-\alpha_n^{'})x_n \quad \text{for all} \quad n \in \mathbb{N} \\ x_{n+1} &= \alpha_n z_n + (1-\alpha_n)x_n \quad \text{for all} \quad n \in \mathbb{N} \end{aligned}$$

where T is a quasi-nonexpansive multi-valued map, $z_n^{'} \in Tx_n$ and $z_n \in Ty_n$.

Since 2003, the iterative schemes with error for a single-valued map in Banach spaces have been studied by many authors, see ([3], [6], [7], [9]). Motivated and inspired by Shahzad and Zegeye [15], we purpose a new three-step iterative scheme for three multi-valued asymptotically nonexpansive maps in Banach spaces and prove strong convergence theorems of the purposed iteration.

2 Preliminaries

In this paper, we use the following iteration scheme.
Let D be a nonempty convex subset of a Banach space E. $\alpha_n, \beta_n, \gamma_n, \alpha_n^{'}, \beta_n^{'}, \gamma_n^{'}, \alpha_n^{''}, \beta_n^{''}, \gamma_n^{''}$ $\in [0.1]$ and $\{u_n\}, \{v_n\}, \{c_n\}$ are bounded sequences in D. Let T_1, T_2, T_3 be three asymptotically nonexpansive multi-valued maps from D into $CB(D)$. Let $\{x_n\}$ be the sequence defined by $x_0 \in D$,

$$\begin{aligned} z_n &= \alpha_n^{''} w_n + \beta_n^{''} x_n + \gamma_n^{''} u_n \quad \text{for all} \quad n \in \mathbb{N} \\ y_n &= \alpha_n^{'} w_n^{'} + \beta_n^{'} x_n + \gamma_n^{'} v_n \quad \text{for all} \quad n \in \mathbb{N} \\ x_{n+1} &= \alpha_n z_n^{'} + \beta_n x_n + \gamma_n c_n \quad \text{for all} \quad n \in \mathbb{N} \end{aligned} \tag{2.1}$$

where $z_n^{'} \in T_1^n y_n, w_n^{'} \in T_2^n z_n$ and $w_n \in T_3^n x_n$

To prove our main results, we shall make use the following definition and lemmas in the sequel.

Definition 5. The mappings $T_1, T_2, T_3 : D \to CB(D)$ with $F := \bigcap_{i=1}^{3} F(T_i) \neq \phi$ are said to satisfy *Condition (II)* if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0,\infty)$ such that

$$\max\{d(x,T_1x), d(x,T_2x), d(x,T_3x)\} \geq f(d(x,F(T)))$$

for all $x \in D$.

Note that when $T_2 = T_3 = I$, the identity map or $T_1 = T_2 = T_3$ Condition (II) reduces to Condition (I) of Senter and Dotson [14]. Our Condition (II) also contains Condition (A') of Khan and Fakhar-ud-din [3].

Lemma 6. *[18]: Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be the sequences of nonnegative numbers satisfying the inequality*

$$\alpha_{n+1} \leq (1+\beta_n)\alpha_n + r_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ *and* $\sum_{n=1}^{\infty} r_n < \infty$, *then* $\lim_{n\to\infty} \alpha_n$ *exists. In particular,* $\{\alpha_n\}_{n=1}^{\infty}$ *has a subsequence which converges to zero, then* $\lim_{n\to\infty} \alpha_n = 0$

Lemma 7. *[13]: Suppose that E is a uniformly convex Banach space and $0 < p \le t_n \le q < 1$ for all positive integer n. Also, suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that* $\limsup_{n\to\infty} \|x_n\| \le r,\ \limsup_{n\to\infty} \|y_n\| \le r$ *and* $\lim_{n\to\infty} \|t_n x_n + (1-t_n y_n\| = r$ *hold for some $r \ge 0$. Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

3 Main Results

Before proving our main result we shall prove the following crucial lemmas.

Lemma 8. *Let E be a uniformly convex Banach space, and let D be a nonempty closed and convex subset of E. Let T_1, T_2, T_3 be three asymptotically nonexpansive multi-maps from D into $CB(D)$ with the sequence $\{r_{i_n}\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty} r_{i_n} < \infty$ for all $i = 1, 2, 3$ and $F := \bigcap_{i=1}^{3} F(T_i) \ne \phi$ and $T_i p = \{p\}, (i = 1, 2, 3)$. Let $\{x_n\}$ be the sequence defined by (2.1), where $\alpha_n, \beta_n, \gamma_n, \alpha_n^{'}, \beta_n^{'}, \gamma_n^{'}, \alpha_n^{''}, \beta_n^{''}, \gamma_n^{''}$ be real sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = \alpha_n^{'} + \beta_n^{'} + \gamma_n^{'} = \alpha_n^{''} + \beta_n^{''} + \gamma_n^{''} = 1$ and $\{u_n\}, \{v_n\}, \{c_n\}$ are bounded sequences in D with $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma_n^{'} < \infty, \sum_{n=1}^{\infty} \gamma_n^{''} < \infty$. Then*

(i) $\lim_{n\to\infty} \|x_n - p\|$ *exists for all $p \in F$;*

(ii) *there exists a constant $M > 0$ such that*

$$\|x_{n+m} - p\| \le M\|x_n - p\| + M \sum_{k=n}^{n+m-1} b_k$$

for all $n, m \ge 1$ and $p \in F$, where $M = e^{3\sum_{k=n}^{n+m-1} r_k}$.

Proof. (i) Let $p \in F$ be the common fixed point of $\{T_i\}, (i = 1, 2, 3)$. Since $\{u_n\}, \{v_n\}, \{c_n\}$ are bounded sequences in D , we can put

$$M \ge \max\{\sup_{n\ge1} \|u_n - p\|, \sup_{n\ge1} \|v_n - p\|, \sup_{n\ge1} \|c_n - p\|\}$$

Then M is a finite number for each $n \in N$. For each $n \ge 1$, let $r_n = \max\{r_{i_n} : i = 1, 2, 3\}$. Thus we have $r_n \ge 0$, $\lim_{n\to\infty} r_{i_n} = 0$ and

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n z_n^{'} + \beta_n x_n + \gamma_n c_n - p\| \\
&\le \alpha_n \|z_n^{'} - p\| + \beta_n \|x_n - p\| + \gamma_n \|c_n - p\| \\
&\le \alpha_n d(z_n^{'}, T_1^n p) + \beta_n \|x_n - p\| + \gamma_n \|c_n - p\| \\
&\le \alpha_n H(T_1^n y_n, T_1^n p) + \beta_n \|x_n - p\| + \gamma_n \|c_n - p\| \\
&\le \alpha_n r_n \|y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|c_n - p\|
\end{aligned} \tag{3.1}$$

Similarly, we have

$$\|y_n - p\| \le \alpha_n^{'} r_n \|z_n - p\| + \beta_n^{'} \|x_n - p\| + \gamma_n^{'} \|v_n - p\| \tag{3.2}$$

and

$$\|z_n - p\| \le \alpha_n^{''} r_n \|x_n - p\| + \beta_n^{''} \|x_n - p\| + \gamma_n^{''} \|u_n - p\| \tag{3.3}$$

Substituting (3.3) in (3.2), we get

$$\begin{aligned}\|y_n - p\| &\leq \alpha_n^{'}\alpha_n^{''}r_n^2\|x_n - p\| + \alpha_n^{'}\beta_n^{''}r_n\|x_n - p\| \\ &\quad + \alpha_n^{'}\gamma_n^{''}r_n\|u_n - p\| + \beta_n^{'}\|x_n - p\| + \gamma_n^{'}\|v_n - p\| \\ &= (1 - \beta_n^{'} - \gamma_n^{'})\alpha_n^{''}r_n^2\|x_n - p\| + \beta_n^{'}\|x_n - p\| \\ &\quad + (1 - \beta_n^{'} - \gamma_n^{'})\beta_n^{''}r_n^2\|x_n - p\| + m_n \\ &\leq (1 - \beta_n^{'})\alpha_n^{"}r_n^2\|x_n - p\| + \beta_n^{'}r_n^2\|x_n - p\| \\ &\quad + (1 - \beta_n^{'})\beta_n^{''}r_n^2\|x_n - p\| + m_n \\ &\leq (1 - \beta_n^{'})r_n^2\|x_n - p\| + \beta_n^{'}r_n^2\|x_n - p\| + m_n \\ &= r_n^2\|x_n - p\| + m_n\end{aligned} \tag{3.4}$$

where $m_n = \alpha_n^{'}\gamma_n^{''}r_n\|u_n - p\| + \gamma_n^{'}\|v_n - p\|$

Note that $\sum_{n=1}^{\infty} m_n < \infty$ as $\sum_{n=1}^{\infty} \gamma_n^{'} < \infty$ and $\sum_{n=1}^{\infty} \gamma_n^{''} < \infty$.

Substituting (3.4) in (3.1), we have

$$\begin{aligned}\|x_{n+1} - p\| &\leq \alpha_n r_n^3\|x_n - p\| + \alpha_n r_n m_n \\ &\quad + \beta_n\|x_n - p\| + \gamma_n\|c_n - p\| \\ &\leq (\alpha_n + \beta_n)r_n^3\|x_n - p\| + b_n \\ &= r_n^3\|x_n - p\| + b_n\end{aligned} \tag{3.5}$$

where $b_n = \alpha_n r_n m_n + \gamma_n\|c_n - p\|$.

Since $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} m_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, which implies that $\sum_{n=1}^{\infty} b_n < \infty$.

It follows that from Lemma (6) that $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$.This proved that the first part of the lemma.

(ii) Since $1 + x \leq e^x$ for all $\varepsilon > 0$. Then from (3.5)

$$\begin{aligned}\|x_{n+m} - p\| &\leq r_{n+m-1}^3\|x_{n+m-1} - p\| + b_{n+m-1} \\ &\leq e^{3r_{n+m-1}}\|x_{n+m-1} - p\| + b_{n+m-1} \\ &\leq e^{3r_{n+m-1}}\left[e^{3r_{n+m-2}}\|x_{n+m-2} - p\| + b_{n+m-2}\right] + b_{n+m-1} \\ &\leq e^{3(r_{n+m-1}+r_{n+m-2})}\|x_{n+m-2} - p\| + e^{3r_{n+m-1}}b_{n+m-2} + b_{n+m-1} \\ &\leq e^{3(r_{n+m-1}+r_{n+m-2})}\|x_{n+m-2} - p\| + e^{3r_{n+m-1}}[b_{n+m-2} + b_{n+m-1}] \\ &\;\vdots \\ &\leq e^{3\sum_{k=n}^{n+m-1} r_k}\|x_n - p\| + e^{3\sum_{k=n}^{n+m-1} r_k}\sum_{k=n}^{n+m-1} b_k \\ &\leq M\|x_n - p\| + M\sum_{k=n}^{n+m-1} b_k\end{aligned}$$

where $M = e^{3\sum_{k=n}^{n+m-1} r_k}$.

This completes the proof of the lemma. □

Lemma 9. *Let E be a uniformly convex Banach space, and let D be a nonempty closed and convex subset of E. Let T_1, T_2, T_3 be three asymptotically nonexpansive multi-maps from D into $CB(D)$ with the sequence $\{r_{i_n}\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty} r_{i_n} < \infty$ for all $i = 1, 2, 3$ and $F := \bigcap_{i=1}^{3} F(T_i) \neq \phi$ and $T_i p = \{p\}, (i = 1, 2, 3)$. Let $\{x_n\}$ be the sequence defined by (2.1) with the following restrictions:*

(i) $0 < \alpha \leq \alpha_n, \alpha_n^{'}, \;\; \alpha_n^{''} \leq 1 - \alpha$ *for some* $\alpha \in (0, 1)$ *and for all* $n \geq n_0, \exists \;\; n_0 \in \mathbb{N}$*;*

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n^{'} < \infty$, *and* $\sum_{n=1}^{\infty} \gamma_n^{''} < \infty$.

Then $\lim_{n\to\infty} \|z_n^{'} - x_n\| = \lim_{n\to\infty} \|w_n^{'} - x_n\| = \lim_{n\to\infty} \|w_n - x_n\| = 0$ *for all* $n \in \mathbb{N}$.

Proof. Let $p \in F$. It follows from Lemma (8) that $\lim_{n\to\infty} \|x_n - p\|$ exists for all $n \in \mathbb{N}$. Let $\lim_{n\to\infty} \|x_n - p\| = c$ for some $c \geq 0$. For each $n \geq 1$, let $r_n = \max\{r_{i_n} : i = 1, 2, 3\}$. Taking the *limsup* of (3.4), we obtain

$$\limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = c \tag{3.6}$$

So

$$\begin{aligned} \limsup_{n\to\infty} \|z_n^{'} - p\| &\leq \limsup_{n\to\infty} d(z_n^{'}, T_1^n p) \\ &\leq \limsup_{n\to\infty} H(T_1^n y_n, T_1^n p) \\ &\leq r_n \|y_n - p\| \\ &\leq c \end{aligned} \tag{3.7}$$

Next, we consider

$$\limsup_{n\to\infty} \|z_n^{'} - p + \gamma_n(c_n - x_n)\| \leq \limsup_{n\to\infty} \|z_n^{'} - p\| + \gamma_n \|c_n - x_n\| \tag{3.8}$$

It follows from (3.7) that

$$\limsup_{n\to\infty} \|z_n^{'} - p + \gamma_n(c_n - x_n)\| \leq c \tag{3.9}$$

By the Triangle inequality

$$\limsup_{n\to\infty} \|x_n - p + \gamma_n(c_n - x_n)\| \leq c \tag{3.10}$$

Moreover, we note that

$$\begin{aligned} c &= \lim_{n\to\infty} \|x_{n+1} - p\| \\ &= \lim_{n\to\infty} \|\alpha_n z_n^{'} + \beta_n x_n + \gamma_n(c_n - (1 - \alpha_n)p - \alpha_n p)\| \\ &= \lim_{n\to\infty} \|\alpha_n z_n^{'} - \alpha_n p + \alpha_n \gamma_n c_n - \alpha_n \gamma_n x_n + (1 - \alpha_n) x_n \\ &\quad - (1 - \alpha_n)p - \gamma_n x_n + \gamma_n c_n - \alpha_n \gamma_n c_n + \alpha_n \gamma_n x_n\| \\ &= \lim_{n\to\infty} \|\alpha_n (z_n^{'} - p + \gamma_n(c_n - x_n) \\ &\quad + (1 - \alpha_n)(x_n - p + \gamma_n(c_n - x_n)\| \end{aligned} \tag{3.11}$$

It follows from (3.9), (3.10) and Lemma (7) that

$$\lim_{n\to\infty} \|z_n^{'} - x_n\| = 0$$

Next, we prove that $\lim_{n\to\infty} \|w_n^{'} - x_n\| = 0$

For each $n \geq 1$,

$$\begin{aligned} \|x_n - p\| &\leq \|z_n^{'} - x_n\| + \|z_n^{'} - p\| \\ &\leq \|z_n^{'} - x_n\| + d(z_n^{'}, T_1^n p) \\ &\leq \|z_n^{'} - x_n\| + H(T_1^n y_n, T_1^n p) \\ &\leq \|z_n^{'} - x_n\| + r_n\|y_n - p\| \end{aligned} \tag{3.12}$$

Since $\lim_{n\to\infty} \|z_n^{'} - x_n\| = 0 = \lim_{n\to\infty} r_n$, it follows from (3.6) and (3.12) that

$$c = \lim_{n\to\infty} \|x_n - p\| \liminf_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|y_n - p\| \leq c \tag{3.13}$$

Hence $\lim_{n\to\infty} \|y_n - p\| = c$

Observe that

$$\|z_n - p\| \leq r_n\|x_n - p\| + \gamma_n^{''}\|v_n - p\|$$

By the boundedness of $\{v_k\}$ and $\lim_{n\to\infty} r_n = 0 = \lim_{n\to\infty} \gamma_n^{''}$, we have

$$\limsup_{n\to\infty} \|z_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| \leq c$$

and so

$$\limsup_{n\to\infty} \|w_n^{'} - p\| \leq \limsup_{n\to\infty} r_n\|z_n - p\| \leq c$$

Next, we consider

$$\|w_n^{'} - p + \gamma_n^{'}(u_n - x_n)\| \leq \|w_n^{'} - p\| + \gamma_n^{'}\|u_n - x_n\| \tag{3.14}$$

Taking *limsup* on both sides, we get

$$\limsup_{n\to\infty} \|w_n^{'} - p + \gamma_n^{'}(u_n - x_n)\| \leq c$$

By the Triangle inequality, we see that

$$\limsup_{n\to\infty} \|x_n - p + \gamma_n^{'}(u_n - x_n)\| \leq c$$

Since $\lim_{n\to\infty} \|y_n - p\| = c$, we obtain

$$\begin{aligned} c &= \lim_{n\to\infty} \|x_n - p\| \\ &= \lim_{n\to\infty} \|\alpha_n^{'} w_n^{'} + \beta_n^{'} x_n + \gamma_n^{'} u_n - p\| \\ &= \lim_{n\to\infty} \|\alpha_n^{'}(w_n^{'} - p + \gamma_n^{'}(u_n - x_n)) + (1 - \alpha_n^{'}(x_n - p + \gamma_n^{'}(u_n - x_n))\| \end{aligned} \tag{3.15}$$

By Lemma (7), we obtain $\lim_{n\to\infty} \|w_n^{'} - x_n\| = 0$
Similarly, by using the same argument as in the proof above, we have $\lim_{n\to\infty} \|w_n - x_n\| = 0$, for all $n \in \mathbb{N}$. This completes the proof. □

Theorem 10. *Let E be a uniformly convex Banach space, and let D be a nonempty closed and convex subset of E. Let T_1, T_2, T_3 be three asymptotically nonexpansive multi-maps from D into $CB(D)$ with the sequence $\{r_{i_n}\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty} r_{i_n} < \infty$ for all $i = 1, 2, 3$ and $F := \bigcap_{i=1}^{3} F(T_i) \neq \phi$ and $T_i p = \{p\}, (i = 1, 2, 3)$. Let $\{x_n\}$ be the sequence defined by (2.1) with the following restrictions:*

(i) $0 < \alpha \leq \alpha_n, \alpha_n^{'}, \alpha_n^{''} \leq 1 - \alpha$ for some $\alpha \in (0, 1)$ and for all $n \geq n_0, \exists\ n_0 \in \mathbb{N}$;

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n^{'} < \infty$, and $\sum_{n=1}^{\infty} \gamma_n^{"} < \infty$.

If $\{T_i\}, (i = 1, 2, 3)$ satisfying Condition (II), then the sequence $\{x_n\}$ converges strongly to a common fixed point of F.

Proof. From Lemma (9), we have

$$\lim_{n \to \infty} \|z_n^{'} - x_n\| = \lim_{n \to \infty} \|w_n^{'} - x_n\| = \lim_{n \to \infty} \|w_n - x_n\| = 0 \tag{3.16}$$

Also,

$$d(x_n, T_3^n x_n) \leq \|w_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty$$

Since $\{x_n\}, \{u_n\}$ are bounded, so is $\{u_n - w_n\}$. Now, let $K = \sup_{n \in \mathbb{N}} \|u_n - w_n\|$. By assumption and (3.16), we get

$$\begin{aligned} \|z_n - w_n\| &\leq \|\alpha_n^{''} w_n + \beta_n^{''} x_n + \gamma_n^{''} u_n - w_n\| \\ &\leq \beta_n^{''} \|x_n - w_n\| + \gamma_n^{''} \|u_n - w_n\| \\ &\leq \beta_n^{''} \|x_n - w_n\| + \gamma_n^{''} K \\ &\to 0 \qquad \text{as} \qquad n \to \infty \end{aligned} \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$\|z_n - x_n\| \leq \|z_n - w_n\| + \|w_n - x_n\| \to 0 \qquad \text{as} \qquad n \to \infty \tag{3.18}$$

Again from (3.16) and (3.18), we have

$$\begin{aligned} d(x_n, T_2^n x_n) &\leq d(x_n, T_2^n z_n) + H(T_2^n z_n, T_2^n x_n) \\ &\leq \|x_n - w_n^{'}\| + r_n \|z_n - x_n\| \\ &\to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

Now, since $\{y_n\}, \{v_n\}$ are bounded, so is $\{v_n - w_n^{'}\}$. Now, let $K = \sup_{n \in \mathbb{N}} \|v_n - w_n^{'}\|$. By assumption and (3.16), we get

$$\begin{aligned} \|y_n - w_n^{'}\| &\leq \|\alpha_n^{'} w_n^{'} + \beta_n^{'} x_n + \gamma_n^{'} v_n - w_n^{'}\| \\ &\leq \beta_n^{'} \|x_n - w_n^{'}\| + \gamma_n^{'} \|v_n - w_n^{'}\| \\ &\leq \beta_n^{'} \|x_n - w_n^{'}\| + \gamma_n^{'} K \\ &\to 0 \quad \text{as} \quad n \to \infty \end{aligned} \tag{3.19}$$

It follows from (3.16) and (3.19) that

$$\|y_n - x_n\| \leq \|y_n - w_n^{'}\| + \|w_n^{'} - x_n\| \to 0 \quad \text{as} \quad n \to \infty \tag{3.20}$$

Therefore from (3.20) and (3.16), we get

$$\begin{aligned} d(x_n, T_1^n x_n) &\leq d(x_n, T_1^n y_n) + H(T_1^n y_n, T_1^n x_n) \\ &\leq \|x_n - z_n^{'}\| + r_n\|y_n - x_n\| \\ &\to 0 \qquad \text{as} \qquad n \to \infty \end{aligned}$$

Now since T_1, T_2, T_3 satisfy Condition (II), we have $d(x_n, F) \to 0$. Thus there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $p_j \subset F$ such that

$$\|x_{n_j} - p_j\| \leq 2^{-j} \tag{3.21}$$

Set $M = e^{3\sum_{k=n}^{n+m-1} r_k}$ and write $n_{j+1} = n_j + l$ for some $l \geq 1$. Then we have by (3.5)

$$\begin{aligned} \|x_{n_{j+1}} - p_j\| &= \|x_{n_j+l} - p_j\| \\ &\leq M\|x_{n_j} - p_j\| + M \sum_{k=n_j}^{n_j+l-1} b_k \\ &< \frac{M}{2^j} + M \sum_{k=n_j}^{n_j+l-1} b_k \end{aligned}$$

Next we shall show that $\{p_j\}$ is Cauchy sequence in D.
Note that

$$\begin{aligned} \|p_{j+1} - p_j\| &\leq \|p_{j+1} - x_{n_{j+1}}\| + \|x_{n_{j+1}} - p_j\| \\ &< \frac{1}{2^{j+1}} + \frac{M}{2^j} + M \sum_{k=n_j}^{n_j+l-1} b_k \\ &< \frac{2M+1}{2^{j+1}} + M \sum_{k=n_j}^{n_j+l-1} b_k \end{aligned}$$

This implies that $\{p_j\}$ is Cauchy sequence in D. Assume that $p_j \to p$ as $j \to \infty$.

Since $d(p_j, T_i^n p) \leq H(T_i^n p, T_i^n p_j) \leq r_n\|p - p_j\|$ for all $i = 1, 2, 3$ and $p_j \to p$ as $j \to \infty$.
It follows that $d(p_j, T_i^n p) = 0$ for all $i = 1, 2, 3$ and thus $p \in F$. It implies by (3.21) that $\{x_{n_j}\}$ converges strongly to p. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, it follows that $\{x_n\}$ converges strongly to p. This completes the proof of the theorem. □

The main result of this paper holds under the assumption that $Tp = \{p\}$ for all $p \in F$. This condition was introduced by Shahzad and Zegeye [15]. The following examples give an example of nonexpansive multi-map T which satisfies the property that $Tp = \{p\}$ for all $p \in F := \bigcap_{i=1}^{3} F(T_i)$ and Tx is not a singletone for all $x \notin F$.

Example 11. Consider $D = [0, 1] \times [0, 1]$ with the usual norm. Define $T : D \to CB(D)$ by

$$T(x, y) = \begin{cases} \{(x, 0)\}, & \text{if } x \neq 0,\ y = 0; \\ \{(0, y)\}, & \text{if } x = 0,\ y \neq 0; \\ \{(x, 0), (0, y)\}, & \text{if } x, y \neq 0; \\ \{(0, 0)\}, & \text{if } x, y = 0. \end{cases}$$

Example 12. Consider $D = [0, 1]$ with the usual norm. Define $T : D \to CB(D)$ by

$$Tx = \left[\frac{x+1}{2}, 1\right]$$

Example 13. Consider $D = [0, 1] \times [0, 1]$ with the usual norm. Define $T : D \to CB(D)$ by

$$T(x, y) = \{x\} \times \left[\frac{y+1}{2}, 1\right]$$

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Products of composition and iterated differentation operators from fractional Cauchy transforms to weighted Bloch-type spaces

Ajay K. Sharma
School of Mathematics, Shri Mata Vaishno Devi University, Kakryal, Katra-182320, J&K, India
aksju_76@yahoo.com

Ram Krishan
School of Mathematics, Shri Mata Vaishno Devi University, Kakryal, Katra-182320, J&K, India
ramk123.verma@gmail.com

Abstract

We consider products of composition and iterated differentiation operators from the space of fractional Cauchy transforms to weighted Bloch-type spaces and little weighted Bloch-type spaces. Upper and lower bounds for norm of these operators are computed and compactness is completely characterized.

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1 Introduction and Preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ its boundary, $dA(z)$ the normalized area measure on $\mathbb{D}$ (i.e. $A(\mathbb{D}) = 1$) and H^∞ the space of all bounded holomorphic functions on $\mathbb{D}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. Let $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$. $H(\mathbb{D})$ is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of $\mathbb{D}$. We denote by $\mathfrak{M}$ the space of all complex Borel measures on $\partial\mathbb{D}$ and let $\mathfrak{M}^*$ be the subset of $\mathfrak{M}$ consisting of probability measures. Let $\alpha > 0$ be a real number. The family $\mathcal{F}_\alpha$ of fractional Cauchy transforms is the collection of functions $f \in H(\mathbb{D})$ which admits a representation of the form

$$f(z) = \int_{\partial\mathbb{D}} \frac{1}{(1-\bar{\zeta}z)^\alpha} d\mu(\zeta) \quad (z \in \mathbb{D}) \tag{1.1}$$

for some $\mu \in \mathfrak{M}$. The principal branch is used in the power function in (1.1) and throughout the rest of the paper. The space $\mathcal{F}_\alpha$ is a Banach space with respect to the norm

$$\|f\|_{\mathcal{F}_\alpha} = \inf_{\mu\in\mathfrak{M}} \left\{ \|\mu\| \ : \ f(z) = \int_{\partial\mathbb{D}} \frac{1}{(1-\bar{\zeta}z)^\alpha} d\mu(\zeta) \right\},$$

where $\|\mu\|$ denotes the total variation of measure μ. According to the Lebesgue decomposition theorem $\mathfrak{M} = \mathfrak{M}_a + \mathfrak{M}_s$, where $\mathfrak{M}_a = \{\mu_a \in \mathfrak{M} : \mu_a << m\}$, where m is the normalized Lebesgue measure on the unit circle $\partial\mathbb{D}$, and $\mathfrak{M}_s = \{\mu_s \in \mathfrak{M} : \mu_s \perp m\}$. Thus any μ can be written as $\mu = \mu_a + \mu_s$, where $\mu_a \in \mathfrak{M}_a$, $\mu_s \in \mathfrak{M}_s$ and $\|\mu\| = \|\mu_a\| + \|\mu_s\|$. Consequently, the space $\mathcal{F}_\alpha$ may also be written as $\mathcal{F}_\alpha = (\mathcal{F}_\alpha)_a + (\mathcal{F}_\alpha)_s$, where $(\mathcal{F}_\alpha)_a$ is isometrically isomorphic to $\mathfrak{M}/\overline{H_0^1}$, the closed subspace of $\mathfrak{M}$ of absolutely continuous measures and $(\mathcal{F}_\alpha)_s$ is isomorphic to $\mathfrak{M}_s$ the closed subspace of $\mathfrak{M}$ of singular measures. If $f \in (\mathcal{F}_\alpha)_a$, then the singular part is null and the measure μ for which the integral in (1.1) holds reduces to $d\mu(e^{it}) = g(e^{it})dt$, where $g(e^{it}) \in L^1$ and dt is the Lebesgue measure on $\partial\mathbb{D}$. For more about the space $\mathcal{F}_\alpha$, we refer [1], [2] [3], [4], [8], [9] and [10]. Let

$$\eta_a(z) = \frac{a-z}{1-\bar{a}z}, \quad a, z \in \mathbb{D},$$

that is, the involutive automorphism of $\mathbb{D}$ interchanging points a and 0. Also we need the following well known identity

$$(1-|z|^2)|\eta_a'(z)| = 1-|\eta_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2} \tag{1.2}$$

The Bloch-type space $\mathcal{B}_\nu(\mathbb{D}) = \mathcal{B}_\nu$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}_\nu} := |f(0)| + b_\nu(f) = |f(0)| + \sup_{z\in\mathbb{D}} \nu(z)|f'(z)| < \infty,$$

where ν is a positive continuous function on $\mathbb{D}$ (*weight*). A weight ν is called *typical* if it is radial, i.e. $\nu(z) = \nu(|z|)$, $z \in \mathbb{D}$ and $\nu(|z|)$ decreasingly converges to 0 as $|z| \to 1$. A positive continuous function ν on the interval $[0,1)$ is called normal if there are $\delta \in [0,1)$ and τ and t, $0 < \tau < t$ such that

$$\frac{\nu(r)}{(1-r)^\tau} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r\to 1}\frac{\nu(r)}{(1-r)^\tau} = 0;$$

$$\frac{\nu(r)}{(1-r)^t} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r\to 1}\frac{\nu(r)}{(1-r)^t} = \infty.$$

If we say that a function $\nu : \mathbb{D} \to [0,\infty)$ is normal we also assume that it is radial. The little Bloch-type space $\mathcal{B}_{\nu,0}(\mathbb{D}) = \mathcal{B}_{\nu,0}$ consists of all $f \in H(\mathbb{D})$ such that

$$\lim_{|z|\to 1} \nu(z)|f'(z)| = 0.$$

With the norm $\|\cdot\|_{\mathcal{B}_\nu}$ the Bloch-type space $\mathcal{B}_\nu$ is a Banach space and the little Bloch-type space $\mathcal{B}_{\nu,0}$ is a closed subspace of the Bloch-type space $\mathcal{B}_\nu$.
Let φ be a holomorphic self-map of $\mathbb{D}$. For a non-negative integer n, we define a linear operator D_φ^n as follows:

$$D_\varphi^n f = f^{(n)} \circ \varphi, \quad f \in H(\mathbb{D}).$$

If $n = 0$, then we have $D_\varphi^n = C_\varphi$, the composition operator induced by φ, defined as $C_\varphi f = f \circ \varphi$, $f \in H(\mathbb{D})$. We recall that an operator T from a Banach space X to a Banach space Y is bounded if there exists a positive constant C such that $\|Tf\|_Y \leq C\|f\|_X$. A bounded operator $T : X \to Y$ is compact if the image of every bounded set in X is relatively compact in Y. Equivalently, $T : X \to Y$ is compact if for every bounded sequence $\{f_m\}$ in X, $\{Tf_m\}$ has a convergent sequence in Y. In [8], Hibschweiler and MacGregor proved that if $\alpha \geq 1$, then every holomorphic self-map φ of

$\mathbb{D}$ induces a bounded composition operator on $\mathcal{F}_\alpha$. In fact, Bourdon and Cima [1] proved that

$$\|C_\varphi\|_{\mathcal{F}_1\to\mathcal{F}_1} \leq \frac{2+2\sqrt{2}}{1-|\varphi(0)|}$$

which was improved to

$$\|C_\varphi\|_{\mathcal{F}_1\to\mathcal{F}_1} \leq \frac{1+2|\varphi(0)|}{1-|\varphi(0)|}$$

by Cima and Matheson [3]. Moreover, equality is attained for certain linear fractional maps.

In contrast with the situation when $\alpha \geq 1$, a self-map φ of $\mathbb{D}$ need not induce a bounded composition operator on $\mathcal{F}_\alpha$ when $0<\alpha<1$. In fact, the condition $\varphi \in \mathcal{F}_\alpha$ is necessary for C_φ to be bounded on $\mathcal{F}_\alpha$. Hibschweiler and MacGregor [8], constructed a self-map φ of $\mathbb{D}$ with $\varphi \notin \mathcal{F}_\alpha (0<\alpha<1)$. For some recent results in this area, see [2],[6],[7], [11], [13] and the references therein. In this paper, we characterize boundedness and compactness of products of composition and iterated differentation from fractional Cauchy transforms to weighted Bloch-type spaces. Throughout the paper constants are denoted by C, they are positive and not necessarily the same at each occurrence. The notation $A \asymp B$ means that there is a positive constant C such that $A/C \leq B \leq CA$.

2 Boundedness and Compactness of $D^n_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_\nu$

In this section, we characterize the boundedness and compactness of D^n_φ from the space of fractional Cauchy transforms to weighted Bloch-type spaces.

The following lemma can be found in [7], and is used throughout the rest of the paper.

Lemma 1. *Let $\alpha>0$ and $f \in H(\mathbb{D})$.*

(1) If $f \in \mathcal{F}_\alpha$ and $z \in \mathbb{D}$, then $|f(z)| \leq \|f\|_{\mathcal{F}_\alpha}/(1-|z|)^\alpha$.

(2) If $f \in \mathcal{F}_\alpha$, then $f' \in \mathcal{F}_{\alpha+1}$ and $\|f'\|_{\mathcal{F}_{\alpha+1}} \leq \alpha \|f\|_{\mathcal{F}_\alpha}$.

Theorem 2. *Let ν be a normal weight, $\alpha>0$, $n \in \mathbb{N}\cup\{0\}$ and φ a holomorphic self-map of $\mathbb{D}$. Then $D^n_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded if and only if*

$$M_1 := \sup_{\zeta\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}} < \infty. \tag{2.1}$$

Moreover, if $D^n_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded, then

$$\begin{aligned}\alpha(\alpha+1)\cdots(\alpha+n)M_1 &\leq \|D^n_\varphi\|_{\mathcal{F}_\alpha\to\mathcal{B}_\nu} \\ &\leq \alpha(\alpha+1)\cdots(\alpha+n-1)\left\{(\alpha+n)M_1 + \frac{1}{(1-|\varphi(0)|)^{n+\alpha}}\right\}.\end{aligned} \tag{2.2}$$

Proof. First, suppose that (2.1) holds. Let $f \in \mathcal{F}_\alpha$. Then there is a $\mu \in \mathfrak{M}$ such that $\|\mu\| = \|f\|_{\mathcal{F}_\alpha}$ and

$$f(z) = \int_{\partial\mathbb{D}} \frac{d\mu(\zeta)}{(1-\overline{\zeta}z)^\alpha}.$$

Thus, we have

$$f^{(n+1)}(z) = \alpha(\alpha+1)\cdots(\alpha+n) \int_{\partial\mathbb{D}} \frac{(\overline{\zeta})^{n+1}}{(1-\overline{\zeta}z)^{n+\alpha+1}} d\mu(\zeta). \tag{2.3}$$

Replacing z in (2.3) by $\varphi(z)$, using a known inequality and multiplying such obtained inequality by $\nu(z)|\varphi'(z)|$, we obtain

$$\begin{aligned}\nu(z)|\varphi'(z)||f^{(n+1)}(\varphi(z))| &\leq \alpha(\alpha+1)\cdots(\alpha+n)\int_{\partial\mathbb{D}}\frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}}d|\mu|(\zeta) \\ &\leq \alpha(\alpha+1)\cdots(\alpha+n)\sup_{\zeta\in\partial\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}}\int_{\partial\mathbb{D}}d|\mu|(\zeta) \\ &= \alpha(\alpha+1)\cdots(\alpha+n)\sup_{\zeta\in\partial\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}}\|\mu\|\end{aligned} \tag{2.4}$$

from which it follows that

$$\nu(z)|(D_\varphi^n f)'(z)| \leq \alpha(\alpha+1)\cdots(\alpha+n)\sup_{\zeta\in\partial\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}}\|f\|_{\mathcal{F}_\alpha}.$$

Taking the supremum over $z \in \mathbb{D}$, we get

$$\sup_{z\in\mathbb{D}}\nu(z)|(D_\varphi^n f)'(z)| \leq \alpha(\alpha+1)\cdots(\alpha+n)M_1\|f\|_{\mathcal{F}_\alpha}. \tag{2.5}$$

By Lemma 1, we have

$$|(D_\varphi^n f)(0)| = |f^{(n)}(\varphi(0))| \leq \frac{\|f^n\|_{\mathcal{F}_{n+\alpha}}}{(1-|\varphi(0)|)^{n+\alpha}} \leq \alpha(\alpha+1)\cdots(\alpha+n-1)\frac{\|f\|_{\mathcal{F}_\alpha}}{(1-|\varphi(0)|)^{n+\alpha}}. \tag{2.6}$$

Thus from (2.5) and (2.6), we have

$$\|D_\varphi^n f\|_{\mathcal{B}_\nu} \leq \alpha(\alpha+1)\cdots(\alpha+n-1)\left\{(\alpha+n)M_1 + \frac{1}{(1-|\varphi(0)|)^{n+\alpha}}\right\}\|f\|_{\mathcal{F}_\alpha}.$$

Hence $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded and

$$\|D_\varphi^n\|_{\mathcal{F}_\alpha\to\mathcal{B}_\nu} \leq \alpha(\alpha+1)\cdots(\alpha+n-1)\left\{(\alpha+n)M_1 + \frac{1}{(1-|\varphi(0)|)^{n+\alpha}}\right\}. \tag{2.7}$$

Next suppose that $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded. Let

$$f_\zeta(z) = \frac{1}{(1-\overline{\zeta}z)^\alpha}, \quad \zeta \in \partial\mathbb{D}. \tag{2.8}$$

Then $\|f_\zeta\|_{\mathcal{F}_\alpha} = 1$ and

$$f_\zeta^{(n+1)}(z) = \alpha(\alpha+1)\cdots(\alpha+n)\frac{(\overline{\zeta})^{n+1}}{(1-\overline{\zeta}z)^{n+\alpha+1}}.$$

From this and the boundedness of the operator $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$, we have that $\|D_\varphi^n f_\zeta\|_{\mathcal{B}_\nu} \leq \|D_\varphi^n\|_{\mathcal{F}_\alpha\to\mathcal{B}_\nu}$, for every $\zeta \in \partial\mathbb{D}$ and so

$$\alpha(\alpha+1)\cdots(\alpha+n)\sup_{\zeta\in\partial\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}} \leq \|D_\varphi^n\|_{\mathcal{F}_\alpha\to\mathcal{B}_\nu}. \tag{2.9}$$

If $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded, then from (2.7) and (2.9), inequality in (2.2) follows.

□

Theorem 3. *Let ν be a normal weight, $\alpha > 0$, $n \in \mathbb{N} \cup \{0\}$, φ a holomorphic self-map of $\mathbb{D}$ and $d\lambda(z) = dA(z)/(1-|z|^2)^2$. Then $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded if and only if*

$$L_1 := \sup_{\zeta \in \partial\mathbb{D}} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{\zeta}\varphi(z)|^{2(n+\alpha+1)}} \nu^2(z)(1-|\eta_a(z)|^2)^2 d\lambda(z) < \infty. \tag{2.10}$$

Moreover, if $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded, then asymptotic relation $L_1 \asymp M_1^2$ holds.

Proof. First assume that (2.10) holds. Since ν is normal, $\nu(a) \asymp \nu(z)$ when $z \in D(a,(1-|a|)/2) = \{|z-a| < (1-|a|)/2\}$. Also it is known that $|1-\bar{a}z| \asymp 1-|a|^2$, for $z \in D(a,(1-|a|)/2)$. Using these two facts, (1.2) and the subharmonicity of the function

$$g(z) = \frac{|\varphi'(z)|^2}{|1-\overline{\zeta}\varphi(z)|^{2(n+\alpha+1)}}$$

we obtain

$$\begin{aligned} L_1 &\geq \sup_{\zeta \in \partial\mathbb{D}} \sup_{a \in \mathbb{D}} \int_{D(a,(1-|a|)/2)} \frac{|\varphi'(z)|^2}{|1-\overline{\zeta}\varphi(z)|^{2(n+\alpha+1)}} \nu^2(z)(1-|\eta_a(z)|^2)^2 d\lambda(z) \\ &= \sup_{\zeta \in \partial\mathbb{D}} \sup_{a \in \mathbb{D}} \int_{D(a,(1-|a|)/2)} \frac{|\varphi'(z)|^2}{|1-\overline{\zeta}\varphi(z)|^{2(n+\alpha+1)}} \nu^2(z) \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dA(z) \\ &\geq \sup_{\zeta \in \partial\mathbb{D}} \sup_{a \in \mathbb{D}} \frac{\nu^2(a)|\varphi'(a)|^2}{|1-\overline{\zeta}\varphi(a)|^{2(n+\alpha+1)}} = M_1^2. \end{aligned} \tag{2.11}$$

Thus by Theorem 1, the operator $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded.
Next assume that the operator $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded. By Theorem 1, we have that (2.1) holds. From this, we have

$$L_1 \leq M_1^2 \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dA(z) = M_1^2 C < \infty. \tag{2.12}$$

The asymptotic relation $L_1 \asymp M_1^2$ follows from (2.11) and (2.12). □

Proceeding as in the proof of Theorem 2, we can easily prove the following lemma. We omit the proof.

Lemma 4. *Let $\nu : \mathbb{D} \to [0,\infty)$ be a normal weight function and $d\lambda(z) = dA(z)/(1-|z|^2)^2$. Then $f \in \mathcal{B}_\nu$ if and only if*

$$I := |f(0)|^2 + \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \nu^2(z)(1-|\eta_a(z)|^2)^2 d\lambda(z) < \infty.$$

Moreover, the following asymptotic relationship holds

$$\|f\|_{\mathcal{B}_\nu}^2 \asymp I.$$

By Lemma 1, the unit ball $B_{\mathcal{F}_\alpha}$ of $\mathcal{F}_\alpha$ is a normal family, a standard argument from Proposition 3.11 in [5] yields the proof of the next lemma.

Lemma 5. *Let ν be a normal weight, $\alpha > 0$, $n \in \mathbb{N} \cup \{0\}$ and φ a holomorphic self-map of $\mathbb{D}$. Then $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is compact if and only if for any bounded sequence $\{f_m\}_{m\in\mathbb{N}}$ in $\mathcal{F}_\alpha$ converging to zero on compact subsets of $\mathbb{D}$, we have that* $\lim_{m\to\infty} \|D_\varphi^n f_m\|_{\mathcal{B}_\nu} = 0$.

Theorem 6. *Let ν be a normal weight, $\alpha > 0$, $n \in \mathbb{N} \cup \{0\}$, φ a holomorphic self-map of $\mathbb{D}$, $d\lambda(z) = dA(z)/(1-|z|^2)^2$ and $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded. Then the following statements are equivalent:*

1. *$D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is compact.*
2. *$M_3 := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \nu^2(z)(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) < \infty$ and*

$$\lim_{r\to 1} \sup_{\zeta\in\partial\mathbb{D}} \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \frac{\nu^2(z)}{|1-\overline{\zeta}\varphi(z)|^{2(n+\alpha+1)}}(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) = 0. \quad (2.13)$$

Proof. (1) $\Rightarrow$ (2). Since $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded, for $f(z) = z^n/n! \in \mathcal{F}_\alpha$, we get

$$M_3 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \nu^2(z)(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) < \infty.$$

Let $f_m(z) = z^m$, $m \in \mathbb{N}$. It is a norm bounded sequence in $\mathcal{F}_\alpha$ converging to zero uniformly on compact subsets of $\mathbb{D}$. Hence by Lemma 2, it follows that $\|D_\varphi^n f_m\|_{\mathcal{B}_\nu} \to 0$ as $m \to \infty$. Thus for every $\epsilon > 0$, there is an $m_0 \in \mathbb{N}$ such that for $m \geq m_0$, we have

$$\Big(\prod_{j=0}^{n}(m-j)\Big)^2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\varphi(z)|^{2(m-n-1)}\nu^2(z)(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) < \epsilon. \quad (2.14)$$

From (2.14), we have that for each $r \in (0,1)$

$$r^{2(m-n-1)}\Big(\prod_{j=0}^{n}(m-j)\Big)^2 \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \nu^2(z)(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) < \epsilon. \quad (2.15)$$

Hence for $r \in \big[\prod_{j=0}^{n}(m-j)^{-\frac{1}{m-n-1}}, 1\big)$, we have

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \nu^2(z)(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) < \epsilon. \quad (2.16)$$

Let $f \in B_{\mathcal{F}_\alpha}$ and $f_t(z) = f(tz)$, $0 < t < 1$. Then $\sup_{0<t<1}\|f_t\|_{\mathcal{F}_\alpha} \leq \|f\|_{\mathcal{F}_\alpha}$, $f_t \in \mathcal{F}_\alpha$, $t \in (0,1)$ and $f_t \to f$ uniformly on compact subsets of $\mathbb{D}$ as $t \to 1$. The compactness of $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ implies that $\lim_{t\to 1}\|D_\varphi^n f_t - D_\varphi^n f\|_{\mathcal{B}_\nu} = 0$. Hence for every $\epsilon > 0$, there is a $t \in (0,1)$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_t^{(n+1)}(\varphi(z)) - f^{(n+1)}(\varphi(z))|^2\nu^2(z)(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) < \epsilon. \quad (2.17)$$

By inequalities (2.16) and (2.17), we have

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f^{(n+1)}(\varphi(z))|^2\nu^2(z)(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) \\
&\quad\leq 2\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_t^{(n+1)}(\varphi(z)) - f^{(n+1)}(\varphi(z))|^2\nu^2(z)(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) \\
&\qquad + 2\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f_t^{(n+1)}(\varphi(z))|^2\nu^2(z)(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) \\
&\quad\leq 2\varepsilon(1+\|f_t^{(n+1)}\|_\infty^2).
\end{aligned}$$

Hence for every $f \in B_{\mathcal{F}_\alpha}$, there is a $\delta_0 \in (0,1)$, $\delta_0 = \delta_0(f,\epsilon)$, such that for $r \in (\delta_0, 1)$

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f^{(n)}(\varphi(z))|^2 \nu^2(z)(1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z) < \epsilon.$$

From the compactness of $D^n_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_\nu$, we have that for every $\epsilon > 0$ there is a finite collection of functions $f_1, f_2, \dots, f_k \in B_{\mathcal{F}_\alpha}$ such that for each $f \in B_{\mathcal{F}_\alpha}$, there is a $j \in \{1,2,\dots,k\}$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(n+1)}(\varphi(z)) - f_j^{(n+1)}(\varphi(z))|^2 \nu^2(z)(1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z) < \varepsilon. \tag{2.18}$$

On the other hand, from (2.18) it follows that if $\delta := \max_{1\le j\le k} \delta_j(f_j, \varepsilon)$, then for $r \in (\delta, 1)$ and all $j \in \{1,2,\dots,k\}$ we have

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f_j^{(n+1)}(\varphi(z))|^2 \nu^2(z)(1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z) < \varepsilon. \tag{2.19}$$

From (2.18) and (2.19), we have that for $r \in (\delta, 1)$ and every $f \in B_{\mathcal{F}_\alpha}$

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f^{(n+1)}(\varphi(z))|^2 \nu^2(z)(1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z) < 4\varepsilon. \tag{2.20}$$

Applying (2.20) to the functions $f_\zeta(z) = 1/(1-\overline{\zeta}z)^\alpha$, $\zeta \in \partial\mathbb{D}$, we obtain

$$\sup_{\zeta\in\partial\mathbb{D}} \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \frac{\nu^2(z)}{|1-\overline{\zeta}\varphi(z)|^{2(n+\alpha+1)}} (1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z)$$
$$< 4\varepsilon/(\alpha(\alpha+1)\cdots(\alpha+n))^2$$

from which (2.13) follows.

(2) $\Rightarrow$ (1). Assume that $\{f_m\}_{m\in\mathbb{N}}$ is a bounded sequence in $\mathcal{F}_\alpha$, say by L, converging to 0 uniformly on compacts of $\mathbb{D}$ as $m \to \infty$. Then by the Weierstrass theorem, $f_m^{(k)}$ also converges to 0 uniformly on compacts of $\mathbb{D}$, for each $k \in \mathbb{N}$. We need to show that $\|D^n_\varphi f_m\|_{\mathcal{B}_\nu} \to 0$ as $m \to \infty$. For each $m \in \mathbb{N}$, we can find a $\mu_m \in \mathfrak{M}$ with $\|\mu_m\| = \|f_m\|_{\mathcal{F}_\alpha}$ such that

$$f_m(z) = \int_{\partial\mathbb{D}} \frac{d\mu_m(\zeta)}{(1-\overline{\zeta}z)^\alpha}. \tag{2.21}$$

Differentiating (2.21) $n+1$ times, composing such obtained equation by φ, applying Jensen's inequality, as well as the boundedness of sequence $\{f_m\}_{m\in\mathbb{N}}$, we obtain

$$|f_m^{(n+1)}(\varphi(w))|^2 \le L(\alpha(\alpha+1)\cdots(\alpha+n))^2 \int_{\partial\mathbb{D}} \frac{d|\mu_m|(\zeta)}{|1-\overline{\zeta}\varphi(w)|^{2(n+\alpha+1)}}. \tag{2.22}$$

By the second condition in (2), we have that for every $\varepsilon > 0$, there is an $r_1 \in (0,1)$ such that for $r \in (r_1, 1)$, we have

$$\sup_{\zeta\in\partial\mathbb{D}} \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \frac{\nu^2(z)}{|1-\overline{\zeta}\varphi(z)|^{2(n+\alpha+1)}} (1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z) < \varepsilon. \tag{2.23}$$

By Lemma 2, we have

$$\|D^n_\varphi f_m\|^2_{\mathcal{B}_\nu} \asymp |f_m^n(\varphi(0))|^2 + \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|\le r} |f_m^{(n+1)}(\varphi(z))|^2 (1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 \nu^2(z) d\lambda(z)$$
$$+ \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f_m^{(n+1)}(\varphi(z))|^2 (1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 \nu^2(z) d\lambda(z).$$

Using first condition in (2), (2.23), Fubini's theorem and the fact that

$$|f_m^{(n)}(\varphi(0))|^2 < \varepsilon \text{ and } \sup_{|w|\leq r} |f_m^{(n+1)}(w)|^2 < \varepsilon,$$

for sufficiently large m, say $m \geq m_0$, we have that

$$\begin{aligned}\|D_\varphi^n f_m\|_{\mathcal{B}_\nu}^2 \leq & |f_m^{(n)}(\varphi(0))|^2 \\ & + \sup_{|\varphi(z)|\leq r} |f_m^{(n+1)}(\varphi(z))|^2 \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|\leq r} (1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 \nu^2(z) d\lambda(z) \\ & + \sup_{a\in\mathbb{D}} \int_{\partial\mathbb{D}} \int_{|\varphi(z)|>r} \frac{\nu^2(z)}{|1-\overline{\zeta}\varphi(w)|^{2(n+\alpha+1)}} (1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z) d|\mu_m|(\zeta) \\ & \leq \left(1 + M_3 + \int_{\partial\mathbb{D}} d|\mu_m|(\zeta)\right)\varepsilon \\ & \leq (1+M_3+L)\varepsilon.\end{aligned}$$

Since ε is an arbitrary, the result follows by Lemma 3. □

Theorem 7. *Let ν be a normal weight, $\alpha > 0$, $n \in \mathbb{N}\cup\{0\}$ and φ a holomorphic self-map of $\mathbb{D}$. Then $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_{\nu,0}$ is bounded if and only if following conditions hold*

$$M_1 := \sup_{\zeta\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}} < \infty. \tag{2.24}$$

$$\lim_{|z|\to 1} \frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}} = 0 \tag{2.25}$$

for every $\zeta \in \partial\mathbb{D}$.

Proof. First suppose that (2.24) and (2.25) hold. By (2.25), the integrand in (2.4) tends to zero for every $\zeta \in \partial\mathbb{D}$, as $|z| \to 1$, and is dominated by the function $f(z) = M_1$. Thus by the Lebesgue convergence theorem, the integral in (2.4) tends to zero as $|z| \to 1$, implying

$$\lim_{|z|\to 1} \nu(z)|(D_\varphi^n f)'(z)| = 0.$$

Hence, for every $f \in \mathcal{F}_\alpha$ we have that $D_\varphi^n f \in \mathcal{B}_{\nu,0}$, from which the boundedness of $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_{\nu,0}$ follows. Conversely, suppose that $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_{\nu,0}$ is bounded. Then $D_\varphi^n f_\zeta \in \mathcal{B}_{\nu,0}$ for every function f_ζ, $\zeta \in \partial\mathbb{D}$, defined in (2.8), that is

$$\lim_{|z|\to 1} \frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}} = 0$$

for every $\zeta \in \partial\mathbb{D}$. Since $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_{\nu,0}$ is bounded, then $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded too. Thus by Theorem 1, (2.24) follows, as claimed. □

Theorem 8. *Let ν be a normal weight, $\alpha > 0$, $n \in \mathbb{N}\cup\{0\}$ and φ a holomorphic self-map of $\mathbb{D}$. Then $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_{\nu,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \sup_{\zeta\in\partial\mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}} = 0. \tag{2.26}$$

Proof. By a know result (see, e.g. Lemma 1 in [12], a closed set E in $\mathcal{B}_{\nu,0}$ is compact if and only if it is bounded and satisfies

$$\lim_{|z|\to 1} \sup_{f\in E} \nu(z)|f'(z)| = 0.$$

Thus the set $\{D^n_\varphi f : f \in \mathcal{F}_\alpha, \|f\|_{\mathcal{F}_\alpha} \leq 1\}$ has compact closure in $\mathcal{B}_{\nu,0}$ if and only if

$$\lim_{|z|\to 1} \sup\{\nu(z)|(D^n_\varphi f)'(z)| : f \in \mathcal{F}_\alpha, \|f\|_{\mathcal{F}_\alpha} \leq 1\} = 0. \tag{2.27}$$

Let $f \in B_{\mathcal{F}_\alpha}$, then there is a $\mu \in \mathfrak{M}$ such that $\|\mu\| = \|f\|_{\mathcal{F}_\alpha}$ and

$$f(z) = \int_{\partial\mathbb{D}} \frac{d\mu(\zeta)}{(1-\overline{\zeta}z)^\alpha}.$$

Thus we easily get that for each $f \in B_{\mathcal{F}_\alpha}$

$$\begin{aligned}\nu(z)|(D^n_\varphi f)'(z)| &\leq \alpha(\alpha+1)\cdots(\alpha+n)\|\mu\| \sup_{\zeta\in\partial\mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}} \\ &\leq \alpha(\alpha+1)\cdots(\alpha+n) \sup_{\zeta\in\partial\mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}}.\end{aligned} \tag{2.28}$$

Using (2.26) in (2.28), we get (2.27). Hence $D^n_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_{\nu,0}$ is compact. Conversely, suppose that $D^n_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_{\nu,0}$ is compact. Taking the test functions in (2.8), we can easily obtain that (2.26) follows from (2.27). ☐

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One and two-step new hybrid methods for the numerical solution of first order initial value problems

Ali Shokri
Faculty of Mathematical Science, University of Maragheh, Maragheh, Iran.
shokri@maragheh.ac.ir

Abstract

In this paper, one and two-step high order hybrid methods are presented. By adding the off-step points y_{n+v}, $(0 < \nu < 1)$, in the right hand side of the classical hybrid methods, we will discuss about the zero-stability, consistency and convergence of introduced procedures. The numerical experimentation showed that our method is considerably more efficient compared to well known methods used for the numerical solution of first order initial value problems.

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1 Introduction

Consider the initial value problems for a single first order ordinary differential equation

$$y' = f(x, y), \quad y(a) = \eta. \tag{1.1}$$

Initial value problems occur frequently in applications. The numerical solution of these kind of problems is a central task in all simulation environments for mechanical, electrical, chemical systems. There are special purpose simulation programs for application in these fields, which often require from their users a deep understanding of the basic properties of the underlying numerical methods [13, 14, 15]. Kopal in 1955, believe [10] that extrapolation and substitution methods' can be regarded as two extreme ways for a construction of numerical solutions of ordinary differential equations leaving a vast no man's land in between, the exploration of which has barely as yet begun. In this context 'extrapolation methods' means method of linear multistep type and 'substitution methods' means method of Runge-Kutta type. From discussion in some papers and books on the relative merits of linear multistep and Runge-Kutta methods, it emerged that the former class of methods, though generally the more efficient in terms of accuracy and weak stability properties for a given number of functions evaluations per step, suffered the disadvantage of requiring additional starting values and special procedures for changing steplength. These difficulties would be reduced, without sacrifice, if we could lower the stepnumber of the linear multistep methods without reducing their order. The difficulty here lies in satisfying the essential condition of zero-stability. This 'zero-stability barrier' was circumvented by the introduction, in 1964-5, of modified linear multistep formula which incorporate a function evaluation at on off-step point. Such formula, simultaneously proposed by Gragg and Stetter [6], Butcher [1], and Gear [4] were christened 'hybrid' by the last author an apt name since, whilst retaining certain linear multistep characteristics, hybrid methods share with Runge-Kutta methods the property of utilizing data at points other than the step points. Thus, we may regard the introduction of hybrid formulae as an important step into the no man's land described by Kopal.

The k-step classical hybrid methods formula [7, 8, 11] are as follows

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} + h\beta_v f_{n+v} \tag{1.2}$$

where $\alpha_k = +1$, α_0 and β_0 are not both zero, $v \notin \{0, 1, \ldots, k\}$, and also $f_{n+v} = f(x_{n+v}, y_{n+v})$. These methods are similar to linear multistep methods in predictor-corrector mode, but with one essential modification: an additional predictor is introduced at an off-step point. This means that the final (corrector) stage has an additional derivative approximation to work from. This greater generality allows the consequences of the Dahlquist barrier [3], to be avoided and it is actually possible to obtain convergent k-step methods with order $2k + 1$ up to $k = 7$. Even higher orders are available if two or more off-step points are used. The three independent discoveries of this approach were reported in [2, 3, 4, 5, 9, 12, 15, 16]. Although a flurry of activity by other authors followed, these methods have never been developed to the extent that they have been implemented in general purpose software. Recall that the formula (1.2) is zero-stable if no root of the polynomial $\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j$ has modulus greater than one and if every root with modulus one is simple. Thus Gragg and Stetter's results showed that [6], with certain exceptions, we can utilize both of new parameters v and β_v we have introduced, to raise the order of (1.2) to two above attained by linear multistep methods having the same right-hand side and the same value for k'. In this paper by utilizing parameter v in term y_{n+v}, in the right-hand side of (1.2), we prove that zero-stability property is hold.

2 k-step high order hybrid methods

For the numerical solution of the first order initial value problem (1.1), we introduce the new hybrid methods of the form

$$y_{n+1} = \sum_{j=1}^{k} a_j y_{n-j+1} + \sum_{j=1}^{v} b_j y_{n-\theta_j+1} + h \sum_{j=0}^{k} c_j f_{n-j+1} + h \sum_{j=1}^{v} d_j f_{n-\theta_j+1} \tag{2.1}$$

where a_j, b_j, c_j, d_j, $0 < \theta_j < k$ such that $\theta_j \notin \{0, 1, 2, \cdots, k\}$, $j = 1, 2, \ldots, v$ are $(2k+3v+1)$ arbitrary parameters. Formula (2.1) can only be used if we know the values of the solution $y(x)$ and $y'(x)$ at k successive points. These k values will be assumed to be given. Further, if $c_0 = 0$, this equation is refereed to as an explicit or predictor formula since y_{n+1} occurs only on one side of the equation. Also if $c_0 \neq 0$, the equation is referred to as an implicit or corrector formula since y_{n+1} occurs in both sides of the equation. In other words the unknown y_{n+1} cannot be calculated directly since it is contained within y'_{n+1}. Now with the difference equation (2.1), we can associate the difference operator L defined next.

Definition 1. Let the differential equation (1.1) have a unique solution $y(x)$ on $[a, b]$ and suppose that $y(x) \in C^{(p+1)}[a, b]$ for $p \geq 1$. Then the deference operator L for the method of (2.1) can be written as

$$\begin{aligned} L[y(x),h] &= y(x+h)-\sum_{j=1}^{k}a_jy(x+(1-j)h)-h\sum_{j=0}^{k}c_jy'(x+(1-j)h) \\ &- \sum_{j=1}^{v}\Big[b_jy(x+(1-\theta_j)h)+hd_jy'(x+(1-\theta_j)h)\Big] \end{aligned} \tag{2.2}$$

Definition 2. For the method (2.1), we define the functions $\rho(\xi)$ and $\sigma(\xi)$ as

$$\rho(\xi)=\xi^k-\sum_{j=1}^{k}a_j\xi^{k-j}-\sum_{j=1}^{v}b_j\xi^{k-\theta_j}, \quad \sigma(\xi)=\sum_{j=0}^{k}c_j\xi^{k-j} \tag{2.3}$$

and these functions so called the first and second characteristic functions, respectively.

We can assume that the functions $\rho(\xi)$ and $\sigma(\xi)$ have no common factors since, otherwise, (2.1) can be reduced to an equation of lower order. In order that the difference equation (2.1) should be useful for numerical integration, it is necessary that (2.1) be satisfied with high accuracy by the solution of the differential equation $y'=f(x,y)$, when h is small for an arbitrary function $f(x,y)$. This imposes restrictions on the coefficients a_j and b_j. We assume that the function $y(x)$ has continuous derivatives of sufficiently high order. We firstly use the Taylor series expansion to determine all the coefficients of (2.1), which can be written as

$$\begin{aligned} L[y(x),h] &= \sum_{i=0}^{\infty}\frac{h^i}{i!}y^{(i)}(x_n)-\sum_{j=1}^{k}a_j\Big[y(x_n)+\frac{(1-j)h}{1!}y^{(1)}(x_n) \\ &+ \frac{(1-j)^2h^2}{2!}y^{(2)}(x_n)+\cdots+\frac{(1-j)^qh^q}{q!}y^{(q)}(x_n)+\cdots\Big] \\ &- \sum_{j=1}^{v}\Bigg[b_j\Big(y(x_n)+\frac{(1-\theta_j)h}{1!}y^{(1)}(x_n)+\frac{(1-\theta_j)^2h^2}{2!}y^{(2)}(x_n)+\cdots \\ &+ \frac{(1-\theta_j)^qh^q}{q!}y^{(q)}(x_n)+\cdots\Big)-hd_j\Big(y'(x_n)+\frac{(1-\theta_j)h}{1!}y^{(2)}(x_n) \\ &+ \frac{(1-\theta_j)^2h^2}{2!}y^{(3)}(x_n)+\cdots+\frac{(1-\theta_j)^qh^q}{q!}y^{(q+1)}(x_n)+\cdots\Big)\Bigg] \\ &- \sum_{j=0}^{k}hc_j\Big[y'(x_n)+\frac{(1-j)h}{1!}y^{(2)}(x_n)+\frac{(1-j)^2h^2}{2!}y^{(3)}(x_n)+\cdots \\ &+ \frac{(1-j)^qh^q}{q!}y^{(q+1)}(x_n)+\cdots\Big]. \end{aligned} \tag{2.4}$$

Therefor, we have

$$\begin{aligned} L[y(x),h] &= \Big[1-\sum_{j=1}^{k}a_j-\sum_{j=1}^{v}b_j\Big]y(x_n) \\ &+ \Big[1-\sum_{j=1}^{k}(1-j)a_j-\sum_{j=1}^{v}\big((1-\theta_j)b_j+d_j\big)-\sum_{j=0}^{k}c_j\Big]hy'(x_n) \\ &+ \Big[\frac{1}{2!}-\sum_{j=1}^{k}\frac{(1-j)^2a_j}{2!}-\sum_{j=1}^{v}\Big[\frac{(1-\theta_j)^2b_j}{2!}+\frac{(1-\theta_j)d_j}{1!}\Big] \\ &- \sum_{j=0}^{k}\frac{(1-j)c_j}{1!}\Big]h^2y^{(2)}(x_n)+\cdots+\Big[\frac{1}{q!}-\sum_{j=1}^{k}\frac{(1-j)^qa_j}{q!} \\ &- \sum_{j=1}^{v}\Big[\frac{(1-\theta_j)^qb_j}{q!}+\frac{(1-\theta_j)^{(q-1)}d_j}{(q-1)!}\Big] \\ &- \sum_{j=0}^{k}\frac{(1-j)^{(q-1)}c_j}{(q-1)!}\Big]h^qy^{(q)}(x_n)+\cdots. \end{aligned}$$

Then we get

$$L[y(x),h]=C_0y(x_n)+C_1hy^{(1)}(x_n)+\cdots+C_qh^qy^{(q)}(x_n)+\cdots, \tag{2.5}$$

where

$$\begin{aligned} C_q &= \frac{1}{q!}-\sum_{j=1}^{k}\frac{(1-j)^q}{q!}a_j-\sum_{j=1}^{v}\Big[\frac{(1-\theta_j)^q}{q!}b_j+\frac{(1-\theta_j)^{(q-1)}}{(q-1)!}d_j\Big] \\ &- \sum_{j=0}^{k}\frac{(1-j)^{(q-1)}}{(q-1)!}c_j \end{aligned}$$

Definition 3. The linear multistep hybrid method (2.1) are said to be of order p if

$$C_0=C_1=C_2=\cdots=C_p=0, \qquad C_{p+1}\neq 0$$

thus for any function $y(x)\in C^{(p+2)}$ and for some nonzero constant C_{p+1}, we have

$$L[y(x),h]=-C_{p+1}h^{p+1}y^{(p+1)}(x_n)+O(h^{p+2}) \tag{2.6}$$

where $C_{p+1}/\sigma(1)$ is called the error constant.

In particular, $L[y(x),h]$ vanishes identically when $y(x)$ is polynomial whose degree is less than or equal to p.

Lemma 4. *The linear multistep hybrid method* (2.1) *is consistent if and only if*

$$\rho(1)=0, \qquad \rho'(1)=\sigma(1)+\sum_{j=1}^{k}d_j \tag{2.7}$$

Proof. We know that the general linear multistep methods are consistent if and only if they have the order of $p\geq 1$. This implies $C_0=C_1=0$. Therefore by a simple calculation, we get (2.7). □

Definition 5. The linear multistep hybrid method (2.1) is said to be consistent if it has the order of $p\geq 1$.

2.1 One-step new hybrid methods with one off-Step point

Upon choosing $k = v = 1$ in (2.1), we get

$$y_{n+1} = a_1 y_n + b_1 y_{n-\theta_1+1} + h(c_0 f_{n+1} + c_1 f_n) + h d_1 f_{n-\theta_1+1}, \tag{2.8}$$

where a_1, b_0, b_1, c_0, c_1, and $0 < \theta_1 < 1$ are 6 arbitrary parameters. In order to implement such a formula, a special predictor to estimate $y_{n-\theta_1+1}$ is necessary, we suppose that θ_1 is free parameter and by substituting $C_i = 0$, $i = 0, 1, 2, 3, 4$, we have

$$\begin{aligned}
C_0 &= 1 - a_1 - b_1 = 0 \\
C_1 &= 1 - \Big((1-\theta_1)b_1 + d_1\Big) - (c_0 + c_1) = 0 \\
C_2 &= \frac{1}{2!} - \frac{1}{2!}\Big[(1-\theta_1)^2 b_1 + 2(1-\theta_1)d_1\Big] - c_0 = 0 \\
C_3 &= \frac{1}{3!} - \frac{1}{3!}\Big[(1-\theta_1)^3 b_1 + 3(1-\theta_1)^2 d_1\Big] - \frac{1}{2!}c_0 = 0 \\
C_4 &= \frac{1}{4!} - \frac{1}{4!}\Big[(1-\theta_1)^4 b_1 + 4(1-\theta_1)^3 d_1\Big] - \frac{1}{3!}c_0 = 0
\end{aligned}$$

Now if we consider θ_1 is free parameter, we have

$$a_1 = \frac{\theta_1^3(\theta_1 - 2)}{(\theta_1 - 1)^3(\theta_1 + 1)}, \quad b_1 = \frac{2\theta_1 - 1}{(\theta_1 - 1)^3(\theta_1 + 1)}, \quad c_0 = \frac{\theta_1}{2(\theta_1 + 1)} \tag{2.9}$$

$$c_1 = \frac{\theta_1^3}{2(\theta_1 - 1)^2(\theta_1 + 1)}, \quad d_1 = \frac{\theta_1}{2(\theta_1 - 1)^2(\theta_1 + 1)}, \tag{2.10}$$

and its local truncation error is

$$\begin{aligned}
E &= \Big[\frac{1}{5!} - \frac{1}{5!}(1-\theta_1)^5 b_1 - \frac{1}{4!}(1-\theta_1)^4 d_1 - \frac{1}{4!}c_0\Big] h^5 y^{(5)}(\xi) \\
&= \frac{-\theta_1^3}{240(\theta_1 + 1)} h^5 y^{(5)}(\xi).
\end{aligned} \tag{2.11}$$

Theorem 6. *Any methods derived from* (2.8)*, under Lemma 4 conditions, are zero-stable.*

Proof. For this propose, we show that the function $\rho(\xi) = \xi - a_1 - b_1\xi^{1-\theta_1}$ has no roots other than $\xi_1 = 1$. Let $1 - \theta_1 = \nu$ then obviously $0 < \nu < 1$, and with conditions of Lemma 2.4, we can write first characteristic function $\rho(x)$ as $\rho(x) = x - a_1 - (1 - a_1)x^\nu$. Obviously $\xi_1 = 1$ is principal root of $\rho(x)$. If we suppose ρ has a root $\alpha > 1$ then ρ' must have a root β such that $1 < \beta < \alpha$. Therefore

$$\rho'(\beta) = 0 \quad \Longrightarrow 1 - \nu b_1 \beta^{\nu-1} = 0 \quad \Longrightarrow \nu b_1 \beta^{\nu-1} = 1 \quad \Longrightarrow \beta^{1-\nu} = \nu b_1$$

now since $\beta > 1$ then $\nu b_1 > 1$ hence $\nu > \frac{1}{b_1} > 1$ and this is a contradiction. Now suppose ρ has a root $0 < \alpha < 1$. then ρ' must have a root β such that $0 < \alpha < \beta < 1$. Therefore

$$\rho(\alpha) = 0 \quad \Longrightarrow \alpha - a_1 - b_1\alpha^\nu = 0 \quad \Longrightarrow b_1\alpha^\nu = \alpha - a_1 \tag{2.12}$$

But $\rho'(\beta) = 0$ then

$$\beta^{1-\nu} = \nu b_1 \tag{2.13}$$

and from (2.12) we can write

$$\nu b_1 \alpha^\nu = \nu(\alpha - a_1) \tag{2.14}$$

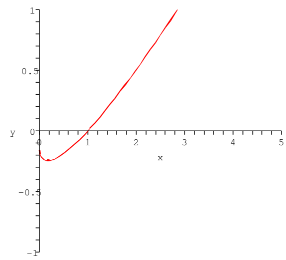

Figure 1. The first characteristic function $\rho(\xi)$, for one-step new hybrid method with $\theta_1 = \frac{1}{3}$

Therefore from (2.13) and (2.14) we have $\alpha^\nu \beta^{1-\nu} = \nu(\alpha - a_1)$. Now since $0 < \alpha, \beta < 1$ then $\nu(\alpha - a_1) < 1$ therefore $\alpha - a_1 > 1$, this means that $\alpha > 1 + a_1$ and this is a contradiction since a_1 is positive. Similarly we can show that ρ can not has negative root and this completes the proof. □

Theorem 7. *Any methods derived from* (2.8), *under Lemma 4 and Theorem 2 conditions are convergent.*

Proof. As is known, the necessary and sufficient conditions for linear multistep methods to be convergent are that they must be consistent and zero-stable. Then according to the Lemma 2.4 and Theorem 2.6, all methods generated from (2.8), are convergent. □

If we take $\theta_1 = \frac{1}{2}$, we have

$$a_1 = 1 \quad , \quad c_0 = \frac{1}{6}, \quad c_1 = \frac{1}{6}, \quad d_1 = \frac{2}{3}, \quad b_1 = 0, \tag{2.15}$$

and the method is

$$y_{n+1} = y_n + \frac{h}{6}(f_n + 4f_{n+\frac{1}{2}} + f_{n+1}), \tag{2.16}$$

which is Milne-Simpson rule, as known as, the implicit one-step classical hybrid method of order 4 and its local truncation error is

$$E = -\frac{1}{2880} h^5 y^{(5)}(\xi), \quad \xi \in (x_n, x_{n+1}).$$

By choosing $\theta_1 = \frac{1}{3}$, we have

$$a_1 = \frac{5}{32}, \quad c_0 = \frac{1}{8}, \quad c_1 = \frac{1}{32}, \quad d_1 = \frac{9}{32}, \quad b_1 = \frac{27}{32}, \tag{2.17}$$

hence the method is

$$y_{n+1} = \frac{5}{32} y_n + \frac{27}{32} y_{n+\frac{2}{3}} + \frac{h}{32}(f_n + 9f_{n+\frac{2}{3}} + 4f_{n+1}), \tag{2.18}$$

which is the implicit one-step new hybrid method of order 4. The first characteristic function $\rho(\xi)$, for this method is $\rho(\xi) = \xi - \frac{27}{32}\xi^{\frac{2}{3}} - \frac{5}{32}$, which has only one root $\xi_1 = 1$, so this method is zero-stable, and the figure of this function is shown in Figure 1. Moreover its local truncation error is $E = -\frac{1}{8640} h^5 y^{(5)}(\xi)$.

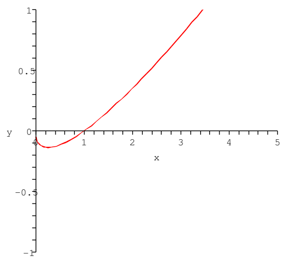

Figure 2. The first characteristic function $\rho(\xi)$, for one-step new hybrid method with $\theta_1 = \frac{1}{4}$

If we apply the Routh-Hurwitz criterion to investigate the weak stability of (2.15), the Routh-Hurwitz criterion will be clearly satisfied if and only if $\overline{h} \in (-3.41, 0)$ which required interval of absolute stability, this means that the interval of absolute stability of our new method is $(-3.41, 0)$. Similarly if we take $\theta_1 = \frac{1}{4}$ we have

$$a_1 = \frac{7}{135}, \quad b_0 = \frac{1}{10}, \quad b_1 = \frac{1}{90}, \quad c_1 = \frac{8}{45}, \quad c_2 = \frac{128}{135}, \tag{2.19}$$

then the method is

$$y_{n+1} = \frac{1}{135}(7y_n + 128y_{n+\frac{3}{4}}) + \frac{h}{90}(f_n + 16f_{n+\frac{3}{4}} + 9f_{n+1}), \tag{2.20}$$

that is implicit one-step new hybrid method of order 4. The first characteristic function $\rho(\xi)$, for this method is $\rho(\xi) = \xi - \frac{128}{135}\xi^{\frac{3}{4}} - \frac{7}{135}$, which has only one root $\xi_1 = 1$, so this method is zero-stable, and the figure of this function is shown in Figure 2. Moreover its local truncation error is $E = -\frac{1}{19200}h^5 y^{(5)}(\xi)$. Using Routh-Hurwitz criterion, its interval of absolute stability is $(-3.6, 0)$.

2.2 Two-step new hybrid methods with one off-step point

Upon choosing $k = 2$ and $v = 1$ in (3), we get

$$y_{n+1} = a_1 y_n + a_2 y_{n-1} + b_1 y_{n-\theta_1+1} + h(c_0 f_{n+1} + c_1 f_n + c_2 f_{n-1}) + h d_1 f_{n-\theta_1+1}, \tag{2.21}$$

where a_1, a_2, b_1, c_0, c_1, c_2, d_1 and $0 < \theta_1 < 1$ are 8 arbitrary parameters. In order to implement such a formula, a special predictor to estimate $y_{n-\theta_1+1}$ is necessary, we supose that θ_1 is free parameter and by substituting $C_i = 0$, $i = 0, 1, \ldots, 6$, we have

$$\begin{aligned}
C_0 &= 1 - a_1 - a_2 - b_1 = 0 \\
C_1 &= 1 + a_2 - \Big((1-\theta_1)b_1 + d_1\Big) - (c_0 + c_1 + c_2) = 0 \\
C_2 &= \frac{1}{2!} - \frac{1}{2!}\Big[a_2 + (1-\theta_1)^2 b_1 + 2(1-\theta_1)d_1\Big] - (c_0 - c_2) = 0 \\
C_3 &= \frac{1}{3!} - \frac{1}{3!}\Big[-a_2 + (1-\theta_1)^3 b_1 + 3(1-\theta_1)^2 d_1\Big] - \frac{1}{2!}(c_0 + c_2) = 0 \\
C_4 &= \frac{1}{4!} - \frac{1}{4!}\Big[a_2 + (1-\theta_1)^4 b_1 + 4(1-\theta_1)^3 d_1\Big] - \frac{1}{3!}(c_0 - c_2) = 0 \\
C_5 &= \frac{1}{5!} - \frac{1}{5!}\Big[-a_2 + (1-\theta_1)^5 b_1 + 5(1-\theta_1)^4 d_1\Big] - \frac{1}{4!}(c_0 + c_2) = 0 \\
C_6 &= \frac{1}{6!} - \frac{1}{6!}\Big[a_2 + (1-\theta_1)^6 b_1 + 6(1-\theta_1)^5 d_1\Big] - \frac{1}{5!}(c_0 - c_2) = 0,
\end{aligned}$$

now if we consider θ_1 is free parameter, we have

$$a_1 = -\frac{8\theta_1^3}{(3\theta_1+2)(\theta_1-1)^3}, \qquad a_2 = \frac{\theta_1^3(3\theta_1-8)}{(3\theta_1+2)(\theta_1-2)^3}, \tag{2.22}$$

$$b_1 = \frac{8(3\theta_1^2-6\theta_1+2)}{(3\theta_1+2)(\theta_1-1)^3(\theta_1-2)^3}, \tag{2.23}$$

$$c_0 = \frac{\theta_1}{3\theta_1+2}, \quad c_1 = \frac{4\theta_1^3}{(\theta_1-1)^2(3\theta_1+2)}, \quad c_2 = \frac{\theta_1^3}{(\theta_1-2)^2(3\theta_1+2)} \tag{2.24}$$

$$d_1 = \frac{4\theta_1}{(3\theta_1+2)(\theta_1-1)^2(\theta_1-2)^2}, \tag{2.25}$$

and its local truncation error is

$$\begin{aligned} E &= \frac{1}{7!}\left[1+a_2-\Big[(1-\theta_1)^7 b_1+7(1-\theta_1)^6 d_1+7(c_0+c_2)\Big]\right]h^7y^{(7)}(\xi) \\ &= -\frac{\theta_1^3}{1260(3\theta_1+2)}h^7y^{(7)}(\xi). \end{aligned} \tag{2.26}$$

Theorem 8. *Any methods derived from* (2.21)*, under Lemma 4 conditions, are zero-stable.*

Proof. Proving this theorem is similar to theorem 2.6. □

Theorem 9. *Any methods derived from* (2.21)*, under Lemma 4 and theorem 6 conditions are convergent.*

Proof. Proving this theorem is similar to theorem 7. □

If we take $\theta_1 = \frac{1}{2}$, we have

$$a_1 = \frac{16}{7}, \quad a_2 = \frac{13}{189}, \quad b_1 = -\frac{256}{189}, \quad d_1 = \frac{64}{63},$$

$$c_0 = \frac{1}{7}, \quad c_1 = \frac{4}{7}, \quad c_2 = \frac{1}{63}$$

and the method is

$$\begin{aligned} y_{n+1} &= \frac{1}{189}\Big(432y_n + 13y_{n-1} - 256y_{n+\frac{1}{2}}\Big) \\ &+ \frac{h}{63}(9f_{n+1}+36f_n+f_{n-1}+64f_{n+\frac{1}{2}}), \end{aligned} \tag{2.27}$$

which is the implicit two-step hybrid method of order 6. The first characteristic function $\rho(\xi)$, for this method is $\rho(\xi) = \xi^2 - \frac{432}{189}\xi + \frac{256}{189}\xi^{\frac{1}{2}} - \frac{13}{189}$, that has three roots, the principal root of which is $\xi_1 = 1$, and also $|\xi_i| < 1$, $i = 2, 3$. So this method is zero-stable, and figure of this function is shown in Figure 3. Moreover its local truncation error is $E = -\frac{1}{35280}h^7y^{(7)}(\xi)$, $\xi \in (x_{n-1}, x_{n+1})$. In the numerical experiment for (2.27), one obtains two more unknowns, $y_{n+\frac{1}{2}}$ and $y'_{n+\frac{1}{2}}$, to be solved beside y_{n+1}. For this propose, Gear [4] has used the differentiation formula given by

$$y_{n+\frac{1}{2}} = y_{n-1} + \frac{h}{8}(9f_n + 3f_{n-1}),$$

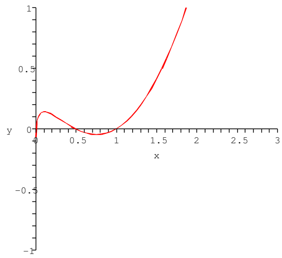

Figure 3. The first characteristic function $\rho(\xi)$, for two-step new hybrid method with $\theta_1 = \frac{1}{2}$

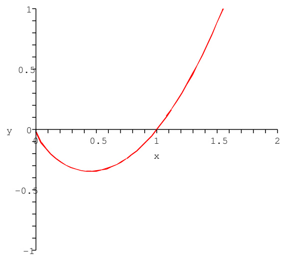

Figure 4. First characteristic function $\rho(\xi)$, for two-step new hybrid method with $\theta_1 = \frac{1}{3}$

and to calculate $y'_{n+\frac{1}{2}}$, the authors [15] have used the L-stable differentiation formula given by

$$y'_{n+\frac{1}{2}} = \frac{3}{4h}\Big(y_{n+1} - y_n\Big) - \frac{3}{4}\left(y'_n - \frac{4}{3}y'_{n+1}\right) - \frac{3h}{4}\left(\frac{1}{2}y''_n + \frac{2}{3}y''_{n+1}\right).$$

By selecting $\theta_1 = \frac{1}{3}$, we have

$$a_1 = \frac{1}{3}, \quad a_2 = \frac{7}{375}, \quad b_1 = \frac{81}{125}, \quad d_1 = \frac{9}{25}$$

$$c_0 = \frac{1}{9}, \quad c_1 = \frac{1}{9}, \quad c_2 = \frac{1}{225},$$

hence the method is

$$\begin{aligned} y_{n+1} &= \frac{1}{375}\Big(125y_n + 7y_{n-1} + 243y_{n+\frac{2}{3}}\Big) \\ &+ h(\frac{1}{9}f_{n+1} + \frac{1}{9}f_n + \frac{1}{225}f_{n-1} + \frac{9}{25}f_{n+\frac{2}{3}}), \end{aligned} \tag{2.28}$$

is the implicit two-step new hybrid method of order 6. The first characteristic function $\rho(\xi)$, for this method is $\rho(\xi) = \xi^2 - \frac{125}{375}\xi - \frac{243}{375}\xi^{\frac{2}{3}} - \frac{7}{375}$, that has only one root $\xi_1 = 1$. So this method is zero-stable, and the figure of this function is shown in Figure 4. Moreover its local truncation error is $E = -\frac{1}{102060}h^7y^{(7)}(\xi)$. If we apply the Routh-Hurwitz criterion to investigation the weak stability of (2.27), the Routh-Hurwitz criterion is clearly satisfied if and only if $\overline{h} \in (-5.21, 0)$ which required interval of absolute stability, this means that the interval of absolute stability of our new method is $(-5.21, 0)$.

x_i	Runge-Kutta	New Method (17)
1.0	0	0
2.2	-0.001373	-0.153994×10^{-7}
3.4	-0.000321	-0.933694×10^{-9}
4.6	0.000121	-0.140638×10^{-9}
5.8	0.000058	$-0.334977 \times 10^{-10}$
7.0	0.000033	$-0.105402 \times 10^{-10}$
$\vdots$	$\vdots$	$\vdots$
25.0	0.000001	$-0.462995 \times 10^{-14}$

Table 1. Absolute errors for the example 10, with $h = 0.1$, are calculated for comparison among four methods: four stage Runge-Kutta method and our new method (2.27).

x_i	Runge-Kutta	New Method (17)
1.0	0	0
2.2	-0.001373	-0.402936×10^{-9}
3.4	-0.000321	$-0.253444 \times 10^{-10}$
4.6	0.000121	$-0.387989 \times 10^{-11}$
5.8	0.000058	$-0.932727 \times 10^{-12}$
7.0	0.000033	$-0.295256 \times 10^{-12}$
$\vdots$	$\vdots$	$\vdots$
25.0	0.000001	$-0.132385 \times 10^{-15}$

Table 2. Absolute errors for the example 10, with $h = 0.025$, are calculated for comparison among four methods: four stage Runge-Kutta method and our new method (2.27).

3 Numerical Example

In this section we present some numerical results to compare our new new hybrid methods with that of other multistep methods.

Example 10. Consider the initial value problem

$$\begin{cases} y' = -5xy^2 + \frac{5}{x} - \frac{1}{x^2}, \\ y(1) = 1. \end{cases}$$

The theoretical solution of this initial value problem is $y(x) = \frac{1}{x}$. The numerical results when $h = 0.1$ are given in table 1 and as the calculations with $h = 0.025$ displayed in table 2. We compared the results of our new hybrid methods and four stage Runge-Kutta method on this problem with $h = 0.1$ and $h = 0.025$.

Example 11. Consider the initial value problem

$$\begin{cases} y' = 4x\sqrt{y}, \\ y(1) = 1. \end{cases}$$

The theoretical solution of this initial value problem is $y(x) = (1 + x^2)^2$, and our new methods, for this problem are exact.

x	y_i	New Method (2.27)
1	y_1	-1.608986324
	y_2	1.011070494
3	y_1	-1.646634125
	y_2	.9609611194
5	y_1	-1.682511516
	y_2	.9180799893
10	y_1	-1.765934684
	y_1	.8330908005

Table 3. Results of example 12, with $\mu = 65$, which are convergent for the stiff problem Van der Pol's equation.

T	h	Y	Error of (2.27)	Error of Wu's Method in [17]
50	0.05	y_1	3.312e-16	1.97e-15
		y_2	8.625e-12	2.02e-11

Table 4. Comparison of the absolute errors in the approximations obtained using the new method (2.27) and the sixth-order method of Wu et al. [17] for Example 3.4.

Example 12. Consider the van der Pol's equation

$$\begin{cases} y_1' = y_2, \\ y_2' = \mu^2((1 - y_1^2)y_2 - y_1), \end{cases}$$

with initial value $y(0) = (2, 0)^T$. We choose $\mu = 65$. We present the numerical solution of this problem using the new hybrid method (2.27) at some selected points in Table 3.

Example 13. Consider the stiff initial value problem

$$\begin{cases} y_1' = -1002y_1 + 1000y_2^2, \\ y_2' = y_1 - y_2(1 + y_2), \\ y_1(0) = 1, \quad y_2(0) = 1. \end{cases}$$

With the exact solution $y_1 = \exp(-2t)$ and $y_2 = \exp(-t)$. This equation has been solved numerically for $T = 50$ using exact starting values and the Wu's method. In the numerical experiment, we take the step lengths $h = 0.05$. In Table 4, we present the absolute errors at the end-point.

Example 14. Consider the stiff problem

$$\begin{cases} y_1' = -20y_1 - 0.25y_2 - 19.75y_3, \\ y_2' = 20y_1 - 20.25y_2 + 0.25y_3, \\ y_3' = 20y_1 - 19.75y_2 - 0.25y_3, \\ y_1(0) = 1, \quad y_2(0) = 0 \quad y_3(0) = -1. \end{cases}$$

T	h	Y	Error of (2.27)	Error of Wu's Method in [17]
50	0.005	y_1	5.26e-21	1.38e-20
		y_2	5.26e-21	1.38e-20
		y_3	5.26e-21	1.38e-20
100	0.1	y_1	6.35e-32	3.57e-31
		y_2	6.35e-32	3.57e-31
		y_3	6.35e-32	3.57e-31

Table 5. Comparison of the absolute errors in the approximations obtained using the new method (2.27) and the sixth-order method of Wu et al. [17] for Example 13.

With the exact solution

$$\begin{cases} y_1 = \frac{\left[\exp(-0.5t)+\exp(-20t)\times(\cos(20t)+\sin(20t))\right]}{2}, \\ y_2 = \frac{\left[\exp(-0.5t)+\exp(-20t)\times(\cos(20t)-\sin(20t))\right]}{2}, \\ y_3 = \frac{-\left[\exp(-0.5t)+\exp(-20t)\times(\cos(20t)-\sin(20t))\right]}{2}. \end{cases}$$

This equation has been solved numerically for $T = 50$ and $T = 100$ using exact starting values and the Wu's method. In the numerical experiment, we take the step lengths $h = 0.005$ and $h = 0.1$. In Table 5, we present the absolute errors at the end-point.

Example 15. Consider the stiff problem

$$\begin{cases} y_1' = -0.1y_1 - 49.9y_2, \\ y_2' = -50y_2, \\ y_3' = 70y_2 - 120y_3, \\ y_1(0) = 2, \quad y_2(0) = 1 \quad y_3(0) = 2. \end{cases}$$

With the exact solution

$$\begin{cases} y_1 = e^{-0.1t} + e^{-50t}, \\ y_2 = e^{-50t}, \\ y_3 = e^{-50t} + e^{-120t}. \end{cases}$$

This equation has been solved numerically for $T = 0.1$ and $T = 0.18$ using exact starting values and the Wu's method. In the numerical experiment, we take the step lengths $h = 0.001$ and $h = 0.01$. In Table 6, we present the absolute errors at the end-point.

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T	h	Y	Error of (2.27)	Error of Wu's Method in [17]
0.1	.001	y_1	2.36e-9	1.75e-7
		y_2	6.89e-10	3.59e-8
		y_3	7.21e-10	3.72e-8
0.18	.01	y_1	3.26e-8	1.64e-5
		y_2	7.26e-9	2.79e-7
		y_3	9.26e-9	2.79e-7

Table 6. Comparison of the absolute errors in the approximations obtained using the new method (2.27) and the sixth-order method of Wu et al. [17] for Example 3.6.

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