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Boundedness of the solution set for a fourth-order nonlinear differential equation with multiple deviating arguments

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Abstract

This paper deals with a fourth-order non-linear differential equation with multiple deviating arguments. Some sufficient conditions are set up for all solutions and their derivatives to be bounded. Our results are new and complement to previously known results.

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1 Introduction

We consider the nonlinear differential equation of fourth order with multiple deviating arguments

$$
\begin{aligned}
&x^{(4)}(t) + f_1(t, x(t))x^{(3)}(t) + f_2(t, x(t))x^{(2)}(t) + f_3(t, x(t))x^{(1)}(t) \\
&+g_0(t, x(t)) + \sum_{i=1}^{n} g_i(t, x(t - \tau_i(t))) = p(t) \qquad (1.1)
\end{aligned}
$$

where f_1, f_2, f_3 and $g_i (i = 0, 1, 2, ..., n)$ are continuous functions on $R^+ \times R$, $\tau_i(t) \geq 0$ $(i = 1, 2, ..., n)$ and $p(t)$ are bounded continuous functions on $R^+ = [0, +\infty)$.

Define $y(t) = \dfrac{dx(t)}{dt} + d_1 x(t)$, $z(t) = \dfrac{dy(t)}{dt} + d_2 y(t)$ and $w(t) = \dfrac{dz(t)}{dt} + d_3 z(t)$ where d_1, d_2 and d_3 are some constants. Then, we can transform Eq. (1.1) into the system, as follows,

$$\begin{aligned}
\frac{dx(t)}{dt} &= -d_1x(t) + y(t)\\
\frac{dy(t)}{dt} &= -d_2y(t) + z(t)\\
\frac{dz(t)}{dt} &= -d_3z(t) + w(t)\\
\frac{dw(t)}{dt} &= -(f_1(t,x(t)) - d_1 - d_2 - d_3)w(t)\\
&+ (-(d_1 + d_2 - f_1(t,x(t)))(d_1 + d_2 + d_3) + (d_1d_2 - f_2(t,x(t))) - d_3^2)z(t)\\
&+ ((d_1 + d_2 - f_1(t,x(t)))(d_1^2 + d_1d_2 + d_2^2) - (d_1d_2 - f_2(t,x(t)))(d_1 + d_2)\\
&- f_3(t,x(t)))y(t)\\
&+ ((f_1(t,x(t)) - d_1 - d_2)d_1^3 + (d_1d_2 - f_2(t,x(t)))d_1^2 + f_3(t,x(t))d_1)x(t)\\
&- g_0(t,x(t)) - \sum_{i=1}^{n} g_i(t,x(t-\tau_i(t)) + p(t).
\end{aligned} \tag{1.2}$$

In applied science, some practical problems are associated with higher-order nonlinear differential equations, such as nonlinear oscillations ([1]–[4]), electronic theory [5], biological model and other models ([6], [7]). Just as above, in the past few decades, the study of qualitative behaviors for higher order differential equations has been paid attention to by many scholars. And, many results relative to the stability and boundedness of solutions of higher order differential equations with delays or without delays have been obtained in view of various methods, especially, Liapunov's method (see [8]–[23] and references therein). On the other hand, some researchers have obtained their results for higher order differential equations with several deviating arguments without using Liapunov's method and Liapunov functional (see [24]–[27]). However, to the best of our knowledge, no authors have considered the boundedness of solutions of fourth order differential equations with multiple deviating arguments in non-Liapunov sense. By the way, we interpret that forming the forthcoming conditions in non-Liapunov sense for our results is easier and more useful than determining a Liapunov functional for higher order differential equations with delays. Thus, it is worthwhile to continue to investigate the boundedness of solutions of Eq. (1.1) in this case.

The main objective of this paper is to study the uniformly boundedness of solutions of (1.2). We will establish some sufficient conditions satisfying the solutions of (1.2) to be uniformly bounded. Our results are new and complement to previously known results.

2 Definition and Assumptions

We assume that $h = \max_{1\le i\le n}\left\{\sup_{t\in R}\tau_i(t)\right\} \ge 0$. Let $C([-h,0],R)$ denote the space of continuous functions $\phi : [-h,0] \to R$ with the supremum norm. It is known from ([28]–[30]) that for $g_i(i = 0,1,2,...,n), \phi, f_1, f_2, f_3, p$ and $\tau_i(t)(i = 1,2,...,n)$ continuous, given a continuous initial function $\phi \in C([-h,0],R)$ and a vector $(y_0,z_0,w_0) \in R^3$, there exists a solution of (1.2) on an interval $[0,T)$ satisfying the initial condition and satisfying (1.2) on $[0,T)$. If the solution remains bounded, then $T = +\infty$. We denote such a solution by $(x(t),y(t),z(t),w(t)) = (x(t,\phi,y_0,z_0,w_0), y(t,\phi,y_0,z_0,w_0), z(t,\phi,y_0,z_0,w_0), w(t,\phi,y_0,z_0,w_0))$ where $y(s) = y(0),\ z(s) = z(0)$ and $w(s) = w(0)$ for all $s \in [-h,0]$. Then, it follows that $(x(t),y(t),z(t),w(t))$ can be defined on $[-h,+\infty)$.

Definition. Solutions of (1.2) are called uniformly bounded (UB) if for each $B_1 > 0$ there is a $B_2 > 0$ such that $(\phi, y_0, z_0, w_0) \in C([-h,0], R) \times R^3$ and $\|\phi\| + \|y_0\| + \|z_0\| + \|w_0\| \leq B_1$ imply that $|x(t,\phi,y_0,z_0,w_0)| + |y(t,\phi,y_0,z_0,w_0)| + |z(t,\phi,y_0,z_0,w_0)| + |w(t,\phi,y_0,z_0,w_0)| \leq B_2$ for all $t \in R^+$.

In this work, we also assume that the following conditions hold:

There exist constants $K > 0$, $d_1 > 1$, $d_2 > 1$, $d_3 > 1$, $d_4 > 0$ and nonnegative constants L_i and q_i $(i = 0, 1, 2, ..., n)$ such that

i) $\left|(f_1(t,x(t)) - d_1 - d_2)d_1^3 + (d_1d_2 - f_2(t,x(t)))d_1^2 + f_3(t,x(t))d_1)u - g_0(t,u)\right| \leq L_0\,|u|$, for all $u \in R$ and $t \geq K$,

ii) $|g_1(t,u)| \leq L_1\,|u| + q_1,\ |g_2(t,u)| \leq L_2\,|u| + q_2, ..., |g_n(t,u)| \leq L_n\,|u| + q_n$ for all $u \in R$ and $t \geq K$,

iii)
$$d_4 = \inf_{t\geq K}\ (f_1(t,x(t)) - d_1 - d_2 - d_3)$$
$$-(\sup_{t\geq K}\left|(d_1 + d_2 - f_1(t,x(t)))(d_1^2 + d_1d_2 + d_2^2) - (d_1d_2 - f_2(t,x(t)))(d_1 + d_2) - f_3(t,x(t))\right|$$
$$+\sup_{t\geq K}\left|-(d_1 + d_2 - f_1(t,x(t)))(d_1 + d_2 + d_3) + (d_1d_2 - f_2(t,x(t))) - d_3^2\right|) > \sum_{i=0}^{n} L_i.$$

3 Main Results

Theorem 1. *Suppose (i)-(iii) hold. Then solutions of (1.2) are uniformly bounded.*

Proof. Let $(x(t), y(t), z(t), w(t))$ be a solution of system (1.2) with initial conditions $x(s) = \phi(s)$, $y(0) = y_0$, $z(0) = z_0$ and $w(0) = w_0$ for all $s \in [-h, 0]$ where $\phi \in C([-h,0], R)$ and $(y_0, z_0, w_0) \in R^3$.

Calculating the upper right derivatives of $|x(s)|, |y(s)|, |z(s)|$ and $|w(s)|$ along (1.2), in view of (*i*)-(*iii*), we have

$$\begin{aligned} D^+(|x(s)|)_{s=t} &= sgn(x(t))\{-d_1x(t) + y(t)\} \\ &\leq -d_1\,|x(t)| + |y(t)|, \end{aligned} \tag{3.1}$$

$$\begin{aligned} D^+(|y(s)|)_{s=t} &= sgn(y(t))\{-d_2y(t) + z(t)\} \\ &\leq -d_2\,|y(t)| + |z(t)|, \end{aligned} \tag{3.2}$$

$$\begin{aligned} D^+(|z(s)|)_{s=t} &= sgn(z(t))\{-d_3z(t) + w(t)\} \\ &\leq -d_3\,|z(t)| + |w(t)|, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&D^+(|w(s)|)_{s=t} \\
&= sgn(w(t))\{-(f_1(t,x(t)) - d_1 - d_2 - d_3)w(t) \\
&+ (-(d_1 + d_2 - f_1(t,x(t)))(d_1 + d_2 + d_3) + (d_1 d_2 - f_2(t,x(t))) - d_3^2)z(t) \\
&+ ((d_1 + d_2 - f_1(t,x(t)))(d_1^2 + d_1 d_2 + d_2^2) - (d_1 d_2 - f_2(t,x(t)))(d_1 + d_2) \\
&- f_3(t,x(t)))y(t) \\
&+ ((f_1(t,x(t)) - d_1 - d_2)d_1^3 + (d_1 d_2 - f_2(t,x(t)))d_1^2 + f_3(t,x(t))d_1)x(t)) \\
&- g_0(t,x(t)) - \sum_{i=1}^{n} g_i(t,x(t-\tau_i(t)) + p(t)\} \\
&\le \{(-\inf_{t\ge K} (f_1(t,x(t)) - d_1 - d_2 - d_3))\,|w(t)| \\
&+ \sup_{t\ge K} \left|-(d_1 + d_2 - f_1(t,x(t)))(d_1 + d_2 + d_3) + (d_1 d_2 - f_2(t,x(t))) - d_3^2\right| |z(t)| \\
&+ \sup_{t\ge K} \big|(d_1 + d_2 - f_1(t,x(t)))(d_1^2 + d_1 d_2 + d_2^2\big) \\
&- (d_1 d_2 - f_2(t,x(t)))(d_1 + d_2) - f_3(t,x(t))|\,|y(t)| \\
&+ L_0\,|x(t)| + \sum_{i=1}^{n} L_i\,|x(t-\tau_i(t))|\} + \sum_{i=0}^{n} q_i + |p(t)|\,. \qquad (3.4)
\end{aligned}$$

Let

$$M(t) = \max_{-h\le s\le t} \{\max\{|x(s)|, |y(s)|, |z(s)|, |w(s)|\}\}.$$

It is clear that $\max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\} \le M(t)$ and $M(t)$ is non-decreasing. Now, we consider the following two cases:

Case I): If there exists a sufficiently large constant $K_1 > K$ such that

$$M(t) > \max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\} \qquad (3.5)$$

for all $t \ge K_1$, then we claim that

$$M(t) \equiv M(K_1) \qquad (3.6)$$

is a constant for all $t \ge K_1$.

By contrapositive, assume (3.6) does not hold, then, there exists $t_1 \ge K_1$ such that $M(t_1) > M(K_1)$.

Here $\max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\} \le M(K_1)$ for all $-h \le t \le K_1$ and there exists $\beta \in (K_1, t_1)$ such that $\max\{|x(\beta)|, |y(\beta)|, |z(\beta)|, |w(\beta)|\} = M(t_1) \ge M(\beta)$ which contradicts (3.5). This implies that (3.6 holds. It follows that there exists $t_2 \ge K_1$ such that $\max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\} < M(t) = M(K_1)$ for all $t \ge t_2$.

Case II): There is a point $t_0 \ge K_1$ such that

$$M(t_0) = \max\{|x(t_0)|, |y(t_0)|, |z(t_0)|, |w(t_0)|\}.$$

Let $\eta = \min\left\{d_1 - 1, d_2 - 1, d_3 - 1,\ d_4 - \sum_{i=0}^{n} L_i\right\} > 0$ and $\theta = \sum_{i=0}^{n} q_i + \sup_{t\in R^+} |p(t)| + 1$ where $t \ge K_1$. Then, if $M(t_0) = \max\{|x(t_0)|, |y(t_0)|, |z(t_0)|, |w(t_0)|\} = |x(t_0)|$, then, in view of (3.1), we obtain

$$\begin{aligned} 0 &\leq D^+(|x(s)|)_{s=t_0} \leq -d_1 |x(t)| + |y(t)| \\ &\leq (1-d_1)M(t_0) \\ &< -\eta M(t_0) + \theta, \end{aligned} \tag{3.7}$$

if $M(t_0) = \max\{|x(t_0)|, |y(t_0)|, |z(t_0)|, |w(t_0)|\} = |y(t_0)|$, then, in view of (3.2), we have

$$\begin{aligned} 0 &\leq D^+(|y(s)|)_{s=t_0} \leq -d_2 |y(t)| + |z(t)| \\ &\leq (1-d_2)M(t_0) \\ &< -\eta M(t_0) + \theta, \end{aligned} \tag{3.8}$$

if $M(t_0) = \max\{|x(t_0)|, |y(t_0)|, |z(t_0)|, |w(t_0)|\} = |z(t_0)|$, then, in view of (3.3), we get

$$\begin{aligned} 0 &\leq D^+(|z(s)|)_{s=t_0} \leq -d_3 |z(t)| + |w(t)| \\ &\leq (1-d_3)M(t_0) \\ &< -\eta M(t_0) + \theta, \end{aligned} \tag{3.9}$$

if $M(t_0) = \max\{|x(t_0)|, |y(t_0)|, |z(t_0)|, |w(t_0)|\} = |w(t_0)|$, then, in view of (3.4), we get

$$\begin{aligned} 0 \leq & D^+(|w(s)|)_{s=t_0} \\ &\leq \{(-\inf_{t\geq K} (f_1(t) - d_1 - d_2 - d_3))\,|w(t)| \\ &+ \sup_{t\geq K} \left|-(d_1+d_2-f_1(t,x(t)))(d_1+d_2+d_3) + (d_1d_2 - f_2(t,x(t))) - d_3^2\right| |z(t)| \\ &+ \sup_{t\geq K} \big|(d_1+d_2-f_1(t,x(t)))(d_1^2+d_1d_2+d_2^2) \\ &- (d_1d_2 - f_2(t,x(t)))(d_1+d_2) - f_3(t,x(t))\big|\,|y(t_0)| \\ &+ L_0\,|x(t_0)| + \sum_{i=1}^{n} L_i\,|x(t_0-\tau_i(t_0))|\} + \sum_{i=0}^{n} q_i + |p(t)| \\ &\leq (\sum_{i=0}^{n} L_i - d_4)M(t_0) + \sum_{i=0}^{n} q_i + |p(t)| \\ &< -\eta M(t_0) + \theta. \end{aligned} \tag{3.10}$$

In addition, if $M(t_0) \geq \frac{\theta}{\eta}$, (3.7), (3.8), (3.9) and (3.10) imply that $M(t)$ is strictly decreasing in a small neighborhood $(t_0, t_0+\delta_0)$. This contradicts that $M(t)$ is non-decreasing. Therefore, $M(t_0) < \frac{\theta}{\eta}$ and

$$\max\{|x(t_0)|, |y(t_0)|, |z(t_0)|, |w(t_0)|\} < \frac{\theta}{\eta}. \tag{3.11}$$

For $\forall t > t_0$, by the same approach used in the proof of (3.11), we have

$$\max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\} < \frac{\theta}{\eta}, \text{ if } M(t) = \max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\}.$$

On the other hand, if $M(t) > \max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\}, t > t_0$, then, we can choose $t_0 \leq t_3 < t$ such that $M(t_3) = \max\{|x(t_3)|, |y(t_3)|, |z(t_3)|, |w(t_3)|\} < \frac{\theta}{\eta}$

and $M(s) > \max\{|x(s)|, |y(s)|, |z(s)|, |w(s)|\}$ for all $s \in (t_3, t]$. Using a similar argument as in the proof of Case (I), we can show that $M(s) \equiv M(t_3)$ is a constant, for all $s \in (t_3, t]$, which implies $\max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\} < M(t) = M(t_3) = \max\{|x(t_3)|, |y(t_3)|, |z(t_3)|, |w(t_3)|\} < \dfrac{\theta}{\eta}$.

To sum up, the solutions of (1.2) are uniformly bounded. The proof is complete. □

4 An Example

Consider the following fourth-order non-linear differential equation

$$\begin{aligned} x^{(4)}(t) &+ (18 - \frac{1}{1+t+x^2(t)})x^{(3)}(t) + (78 - \frac{4}{1+t+x^2(t)})x^{(2)}(t) \\ &+(127 - \frac{3}{1+t+x^2(t)})x^{(1)}(t) + (69 + \frac{3}{1+t+x^2(t)})x(t) + \sin x(t - |\sin t|) \\ &+ \cos x(t - 2|\sin t|) + \sin t \sin x(t - e^{|\sin t|}) + \cos t \cos x(t - e^{2|\sin t|}) = \frac{1}{1+t^2} \end{aligned} \tag{4.1}$$

Setting $y(t) = \dfrac{dx(t)}{dt} + 2x(t)$, $z(t) = \dfrac{dy(t)}{dt} + 2y(t)$ and $w(t) = \dfrac{dz(t)}{dt} + 2z(t)$ we can transform (4.1) into the following system

$$\begin{aligned} \frac{dx(t)}{dt} &= -2x(t) + y(t) \\ \frac{dy(t)}{dt} &= -2y(t) + z(t) \\ \frac{dz(t)}{dt} &= -2z(t) + w(t) \end{aligned}$$

$$\begin{aligned} \frac{dw(t)}{dt} = &-(10 - \frac{1}{1+t+x^2(t)})w(t) + (2 - \frac{2}{1+t+x^2(t)})z(t) + (1 - \frac{1}{1+t+x^2(t)})y(t) \\ &(1 - \frac{1}{1+t+x^2(t)})x(t) - \sin x(t - |\sin t|) - \cos x(t - 2|\sin t|) \\ &- \sin t \sin x(t - e^{|\sin t|}) - \cos t \cos x(t - e^{2|\sin t|}) + \frac{1}{1+t^2}. \end{aligned} \tag{4.2}$$

Then we can satisfy the assumptions $(i - iii)$:

(i) $\left|1 - \frac{1}{1+t+u^2}\right| \leq L_0|u| + q_0$ for all $t, u \in R$,

(ii) $|g_1(t,u)| = |\sin u| \leq L_1|u| + q_1, |g_2(t,u)| = |\cos u| \leq L_2|u| + q_2, |g_3(t,u)| = |\sin t \sin u| \leq L_3|u| + q_3, |g_4(t,u)| = |\cos t \cos u| \leq L_4|u| + q_4$ for all $t, u \in R$,

(iii) $d_4 = \inf\limits_{t \geq K}(10 - \frac{1}{1+t+x^2(t)}) - (\sup\limits_{t \geq K}\left|2 - \frac{2}{1+t+x^2(t)}\right| + \sup\limits_{t \geq K}\left|1 - \frac{1}{1+t+x^2(t)}\right|) > \sum\limits_{i=0}^{4} L_i$ by taking suitable L_i and q_i such as $L_0 = L_1 = L_2 = L_3 = L_4 = 1$ for appropriate q_i $(i = 0, 1, 2, 3, 4)$.Hence,all solutions of the system(4.2) are uniformly bounded.

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