

# On the Convergence of Three Step Iterative Process for Three Asymptotically Nonexpansive Multi-maps in Banach Spaces

Pankaj Kumar Jhade

Department of Mathematics, NRI Institute of Information Science & Technology, Bhopal, India  
pmathsjhade@gmail.com

Amarjeet Singh Saluja

Department of Mathematics, J H Government Postgraduate College, Betul, India  
dssaluja@rediffmail.com

---

## Abstract

In this paper, we introduced a new three-step iterative scheme with errors for finding common fixed points of three asymptotically nonexpansive multi-maps in Banach spaces and prove a strong convergence theorem of the purposed algorithm under some control conditions. Our results improved and extended many known results existing in the literature.

Received August 10, 2013

Accepted in final form January 19, 2014

Communicated with Vladimír Janiš.

**Keywords** Asymptotically nonexpansive multi-maps, three-step iteration scheme, common fixed points, strong convergence, Condition (II).

**MSC(2010)** 47H10, 47H09.

---

## 1 Introduction

Let  $D$  be a nonempty convex subset of a Banach space  $E$ . The set  $D$  is called *proximal* if for each  $x \in E$ , there exists an element  $y \in D$  such that  $\|x - y\| = d(x, D)$ , where  $d(x, D) = \inf\{\|x - z\| : z \in D\}$ . Let  $CB(D)$ ,  $K(D)$  and  $P(D)$  denote the families of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of  $D$ , respectively. The *Hausdorff metric* on  $CB(D)$  is defined by

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$$

for  $A, B \in CB(D)$ .

A single-valued map  $T : D \rightarrow D$  is called *nonexpansive* if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in D$ . A multi-valued map  $T : D \rightarrow CB(D)$  is said to be *nonexpansive* if  $H(Tx, Ty) \leq \|x - y\|$  for all  $x, y \in D$ . An element  $p \in D$  is called fixed point of  $T : D \rightarrow D$  (respectively,  $T : D \rightarrow CB(D)$ ) if  $p = Tp$  (respectively,  $p \in Tp$ ). The set of fixed points of  $T$  is denoted by  $F(T)$ .

The mapping  $T : D \rightarrow CB(D)$  is called

(i) *asymptotically nonexpansive* if there exists a sequence  $r_n \geq 1$ ,  $\lim_{n \rightarrow \infty} r_n = 1$  and

$H(T^n x, T^n y) \leq r_n \|x - y\|$  for all  $x, y \in D$  and  $n \in \mathbb{N}$ ;

(ii) *uniformly L-Lipschitzian* if there exists a constant  $L > 0$  such that  $H(T^n x, T^n y) \leq L \|x - y\|$  for all  $x, y \in D$  and  $n \in \mathbb{N}$ ;

(iii) *hemi compact* if, for any sequence  $\{x_n\}$  in  $D$  such that  $d(x_n, Tx_n) \rightarrow 0$  as  $n \rightarrow \infty$ , there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $x_{n_k} \rightarrow p \in D$ . We note that if  $D$  is compact, then every multi-valued mapping  $T : D \rightarrow CB(D)$  is hemicompact.

In 1953, Mann [8] introduced the following iteration scheme, starting from  $x_0 \in D$ , to approximate a fixed point of a nonexpansive mapping  $T$  in a Hilbert space  $H$ :

$$x_{n+1} = \alpha_n T x_n + (1 - \alpha_n) x_n \quad \text{for all } n \in \mathbb{N} \quad (1.1)$$

where  $\{\alpha_n\}$  be a sequence in  $[0,1]$  satisfies certain conditions. However, we note that Mann's iteration process (1.1) has only weak convergence, in general; for instance, see([1],[4],[11]).

The Ishikawa [5] iteration scheme, starting from  $x_0 \in D$ , is the sequence  $\{x_n\}$  defined by

$$\begin{aligned} y_n &= \beta_n T x_n + (1 - \beta_n) x_n \quad \text{for all } n \in \mathbb{N} \\ x_{n+1} &= \alpha_n T y_n + (1 - \alpha_n) x_n \quad \text{for all } n \in \mathbb{N} \end{aligned} \quad (1.2)$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  be sequences in  $[0,1]$  satisfies certain conditions.

Iterative techniques for approximating fixed points of nonexpansive single-valued mappings have been studied by various authors (see; e.g. [5], [11], [14], [18]) using the Mann iteration or the Ishikawa iteration scheme. For details on the subject, we refer the reader to Berinde [2].

Sastry and Babu [12] defined the Mann and Ishikawa iteration schemes for multi-valued mappings.

Let  $T : D \rightarrow P(D)$  be a multi-valued map and fix  $p \in F(T)$ .

**(A)** The sequence of Mann iterates is defined by  $x_0 \in D$ ,

$$x_{n+1} = \alpha_n y_n + (1 - \alpha_n) x_n \quad \text{for all } n \in \mathbb{N}$$

where  $\{\alpha_n\}$  be a sequence in  $[0,1]$  and  $y_n \in T x_n$  such that  $\|y_n - p\| = d(p, T x_n)$ .

**(B)** The sequence of Ishikawa iterates is defined by  $x_0 \in D$ ,

$$y_n = \beta_n z_n + (1 - \beta_n) x_n \quad \text{for all } n \in \mathbb{N}$$

where  $\{\beta_n\} \in [0,1]$  and  $z_n \in T x_n$  such that  $\|z_n - p\| = d(p, T x_n)$ , and

$$x_{n+1} = \alpha_n z'_n + (1 - \alpha_n) x_n \quad \text{for all } n \in \mathbb{N}$$

where  $\{\alpha_n\} \in [0,1]$  and  $z'_n \in T y_n$  such that  $\|z'_n - p\| = d(p, T y_n)$ .

Sastry and Babu [12] proved that the Mann and Ishikawa iteration schemes for a multi-valued map  $T$  with a fixed point  $p$  converges to a fixed point  $q$  of  $T$  under certain conditions. They also claimed that the fixed point  $q$  may be different from  $p$ . More precisely, they proved the following result for nonexpansive multi-valued map with compact domain.

**Theorem 1.** ([12], Theorem 5): Let  $E$  be a Hilbert space,  $D$  be a nonempty compact convex subset of  $E$ , and  $T : D \rightarrow P(D)$  be a multi-valued map with a fixed point  $p \in F(T)$ . Assume that (i)  $0 \leq \alpha_n, \beta_n < 1$ ; (ii)  $\beta_n \rightarrow 0$  and (iii)  $\sum \alpha_n \beta_n = \infty$ . Then the Ishikawa iterates  $\{x_n\}$  defined by (B) converges to a fixed point of  $T$ .

Panyanak [10] extend the above result to uniformly convex Banach spaces but the domain of  $T$  remains compact.

**Theorem 2.** ([10], Theorem 3.1): Let  $E$  be a uniformly convex Banach space,  $D$  be a nonempty compact convex subset of  $E$ , and  $T : D \rightarrow P(D)$  be a multi-valued map with a fixed point  $p \in F(T)$ . Assume that (i)  $0 \leq \alpha_n, \beta_n < 1$ ; (ii)  $\beta_n \rightarrow 0$  and (iii)  $\sum \alpha_n \beta_n = \infty$ . Then the Ishikawa iterates  $\{x_n\}$  defined by (B) converges to a fixed point of  $T$ .

Later, Song and Wang [17] noted that there was a gap in the proof of Theorem 1 (see [12], Theorem 5) and Theorem 2 (see [10], Theorem 3.1). Because the iteration  $x_n$  depends on a fixed  $p \in F(T)$  as well as  $T$ . If  $q \in F(T)$  and  $q \neq p$ , then the iteration  $x_n$  defined by  $q$  is different from the one defined by  $p$ . Therefore, one cannot derive the monotonicity of sequence  $\{\|x_n - q\|\}$  from the monotonicity of  $\{\|x_n - p\|\}$ . So the conclusion of Theorem 1 and Theorem 2 are ambiguous. They further solved/ revised the gap and also gave the affirmative answer the question using the following Ishikawa iteration scheme.

(C): Let  $D$  be a nonempty convex subset of  $E$ ,  $\alpha_n, \beta_n \in [0, 1]$  and  $\gamma_n \in (0, \infty)$  such that  $\lim_{n \rightarrow \infty} \gamma_n = 0$ . Choose  $x_0 \in D$ . Then

$$\begin{aligned} y_n &= \beta_n z_n + (1 - \beta_n)x_n \quad \text{for all } n \in \mathbb{N} \\ x_{n+1} &= \alpha_n z'_n + (1 - \alpha_n)x_n \quad \text{for all } n \in \mathbb{N} \end{aligned}$$

where  $\|z_n - z'_n\| \leq H(Tx_n, Ty_n) + \gamma_n$  and  $\|z_{n+1} - z'_n\| \leq H(Tx_{n+1}, Ty_n) + \gamma_n$  for  $z_n \in Tx_n$  and  $z'_n \in Ty_n$ .

Song and Wang [17] proved the following results. In the result, the domain of  $T$  is still compact, which is a strong condition.

**Theorem 3.** ([17], Theorem 1): Let  $E$  be a uniformly convex Banach space,  $D$  a nonempty compact convex subset of  $E$  and  $T : D \rightarrow CB(D)$  a nonexpansive multi-valued map with  $F(T) \neq \emptyset$  satisfying  $TP = \{p\}$  for any  $p \in F(T)$ . Assume that (i)  $0 \leq \alpha_n, \beta_n < 1$ ; (ii)  $\beta_n \rightarrow 0$  and (iii)  $\sum \alpha_n \beta_n = \infty$ . Then the Ishikawa iterates  $\{x_n\}$  defined by (C) converges to a fixed point of  $T$ .

Recall that a multi-valued map  $T : D \rightarrow CB(D)$  is said to satisfy Condition (I) [14] if there exists a nondecreasing function  $f : [0, \infty) \rightarrow [0, \infty)$  with  $f(0) = 0$ ,  $f(r) > 0$  for  $r \in (0, \infty)$  such that  $d(x, Tx) \geq f(d(x, F(T)))$  for all  $x \in D$ .

**Theorem 4.** ([17], Theorem 2): Let  $E$  be a uniformly convex Banach space,  $D$  a nonempty compact convex subset of  $E$  and  $T : D \rightarrow CB(D)$  a nonexpansive multi-valued map with  $F(T) \neq \emptyset$  satisfying  $TP = \{p\}$  for any  $p \in F(T)$ . Assume that  $T$  satisfies Condition (I) and  $0 \leq \alpha_n, \beta_n \in [a, b] \subset (0, 1)$ . Then the Ishikawa iterates  $\{x_n\}$  defined by (C) converges to a fixed point of  $T$ .

In 2009, Shahzad and Zegeye [15] extended and improved the results of Panyanak [10], Sastry and Babu [12] and Song and Wang [17] to quasi-nonexpansive multi-valued

maps. They also relaxed compactness of the domain of  $T$  and constructed an iteration scheme which removes the restriction of  $T$  namely  $Tp = \{p\}$  for any  $p \in F(T)$ . The results provided an affirmative answer to Panyanak [10] question in a more general setting. They introduced a new iteration as follows:

Let  $D$  be a nonempty convex subset of a Banach space  $E$  and  $\alpha_n, \alpha'_n \in [0, 1]$ . The sequence of Ishikawa iterates is defined by  $x_0 \in D$ ,

$$\begin{aligned} y_n &= \alpha'_n z'_n + (1 - \alpha'_n)x_n \quad \text{for all } n \in \mathbb{N} \\ x_{n+1} &= \alpha_n z_n + (1 - \alpha_n)x_n \quad \text{for all } n \in \mathbb{N} \end{aligned}$$

where  $T$  is a quasi-nonexpansive multi-valued map,  $z'_n \in Tx_n$  and  $z_n \in Ty_n$ .

Since 2003, the iterative schemes with error for a single-valued map in Banach spaces have been studied by many authors, see ([3], [6], [7], [9]). Motivated and inspired by Shahzad and Zegeye [15], we propose a new three-step iterative scheme for three multi-valued asymptotically nonexpansive maps in Banach spaces and prove strong convergence theorems of the proposed iteration.

## 2 Preliminaries

In this paper, we use the following iteration scheme.

Let  $D$  be a nonempty convex subset of a Banach space  $E$ .  $\alpha_n, \beta_n, \gamma_n, \alpha'_n, \beta'_n, \gamma'_n, \alpha''_n, \beta''_n, \gamma''_n \in [0, 1]$  and  $\{u_n\}, \{v_n\}, \{c_n\}$  are bounded sequences in  $D$ . Let  $T_1, T_2, T_3$  be three asymptotically nonexpansive multi-valued maps from  $D$  into  $CB(D)$ . Let  $\{x_n\}$  be the sequence defined by  $x_0 \in D$ ,

$$\begin{aligned} z_n &= \alpha''_n w_n + \beta''_n x_n + \gamma''_n u_n \quad \text{for all } n \in \mathbb{N} \\ y_n &= \alpha'_n w'_n + \beta'_n x_n + \gamma'_n v_n \quad \text{for all } n \in \mathbb{N} \\ x_{n+1} &= \alpha_n z'_n + \beta_n x_n + \gamma_n c_n \quad \text{for all } n \in \mathbb{N} \end{aligned} \tag{2.1}$$

where  $z'_n \in T_1^n y_n, w'_n \in T_2^n z_n$  and  $w_n \in T_3^n x_n$

To prove our main results, we shall make use the following definition and lemmas in the sequel.

**Definition 5.** The mappings  $T_1, T_2, T_3 : D \rightarrow CB(D)$  with  $F := \bigcap_{i=1}^3 F(T_i) \neq \phi$  are said to satisfy *Condition (II)* if there is a nondecreasing function  $f : [0, \infty) \rightarrow [0, \infty)$  with  $f(0) = 0, f(r) > 0$  for  $r \in (0, \infty)$  such that

$$\max\{d(x, T_1x), d(x, T_2x), d(x, T_3x)\} \geq f(d(x, F(T)))$$

for all  $x \in D$ .

Note that when  $T_2 = T_3 = I$ , the identity map or  $T_1 = T_2 = T_3$  Condition (II) reduces to Condition (I) of Senter and Dotson [14]. Our Condition (II) also contains Condition (A') of Khan and Fakhar-ud-din [3].

**Lemma 6.** [18]: Let  $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty$  and  $\{r_n\}_{n=1}^\infty$  be the sequences of nonnegative numbers satisfying the inequality

$$\alpha_{n+1} \leq (1 + \beta_n)\alpha_n + r_n, \quad n \geq 1.$$

If  $\sum_{n=1}^{\infty} \beta_n < \infty$  and  $\sum_{n=1}^{\infty} r_n < \infty$ , then  $\lim_{n \rightarrow \infty} \alpha_n$  exists. In particular,  $\{\alpha_n\}_{n=1}^{\infty}$  has a subsequence which converges to zero, then  $\lim_{n \rightarrow \infty} \alpha_n = 0$

**Lemma 7.** [13]: Suppose that  $E$  is a uniformly convex Banach space and  $0 < p \leq t_n \leq q < 1$  for all positive integer  $n$ . Also, suppose that  $\{x_n\}$  and  $\{y_n\}$  are two sequences of  $E$  such that  $\limsup_{n \rightarrow \infty} \|x_n\| \leq r$ ,  $\limsup_{n \rightarrow \infty} \|y_n\| \leq r$  and  $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n) y_n\| = r$  hold for some  $r \geq 0$ . Then  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ .

### 3 Main Results

Before proving our main result we shall prove the following crucial lemmas.

**Lemma 8.** Let  $E$  be a uniformly convex Banach space, and let  $D$  be a nonempty closed and convex subset of  $E$ . Let  $T_1, T_2, T_3$  be three asymptotically nonexpansive multi-maps from  $D$  into  $CB(D)$  with the sequence  $\{r_{i_n}\} \subset [1, \infty)$  satisfying  $\sum_{n=1}^{\infty} r_{i_n} < \infty$  for all  $i = 1, 2, 3$  and  $F := \bigcap_{i=1}^3 F(T_i) \neq \phi$  and  $T_i p = \{p\}$ , ( $i = 1, 2, 3$ ). Let  $\{x_n\}$  be the sequence defined by (2.1), where  $\alpha_n, \beta_n, \gamma_n, \alpha'_n, \beta'_n, \gamma'_n, \alpha''_n, \beta''_n, \gamma''_n$  be real sequences in  $(0, 1)$  such that  $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$  and  $\{u_n\}, \{v_n\}, \{c_n\}$  are bounded sequences in  $D$  with  $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty, \sum_{n=1}^{\infty} \gamma''_n < \infty$ . Then

(i)  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists for all  $p \in F$ ;

(ii) there exists a constant  $M > 0$  such that

$$\|x_{n+m} - p\| \leq M \|x_n - p\| + M \sum_{k=n}^{n+m-1} b_k$$

for all  $n, m \geq 1$  and  $p \in F$ , where  $M = e^{3 \sum_{k=n}^{n+m-1} r_k}$ .

*Proof.* (i) Let  $p \in F$  be the common fixed point of  $\{T_i\}$ , ( $i = 1, 2, 3$ ). Since  $\{u_n\}, \{v_n\}, \{c_n\}$  are bounded sequences in  $D$ , we can put

$$M \geq \max \left\{ \sup_{n \geq 1} \|u_n - p\|, \sup_{n \geq 1} \|v_n - p\|, \sup_{n \geq 1} \|c_n - p\| \right\}$$

Then  $M$  is a finite number for each  $n \in N$ . For each  $n \geq 1$ , let  $r_n = \max\{r_{i_n} : i = 1, 2, 3\}$ . Thus we have  $r_n \geq 0$ ,  $\lim_{n \rightarrow \infty} r_{i_n} = 0$  and

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n z'_n + \beta_n x_n + \gamma_n c_n - p\| \\ &\leq \alpha_n \|z'_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|c_n - p\| \\ &\leq \alpha_n d(z'_n, T_1^n p) + \beta_n \|x_n - p\| + \gamma_n \|c_n - p\| \\ &\leq \alpha_n H(T_1^n y_n, T_1^n p) + \beta_n \|x_n - p\| + \gamma_n \|c_n - p\| \\ &\leq \alpha_n r_n \|y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|c_n - p\| \end{aligned} \quad (3.1)$$

Similarly, we have

$$\|y_n - p\| \leq \alpha'_n r_n \|z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \quad (3.2)$$

and

$$\|z_n - p\| \leq \alpha''_n r_n \|x_n - p\| + \beta''_n \|x_n - p\| + \gamma''_n \|u_n - p\| \quad (3.3)$$

Substituting (3.3) in (3.2), we get

$$\begin{aligned}
\|y_n - p\| &\leq \alpha'_n \alpha''_n r_n^2 \|x_n - p\| + \alpha'_n \beta''_n r_n \|x_n - p\| \\
&\quad + \alpha'_n \gamma''_n r_n \|u_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\
&= (1 - \beta'_n - \gamma'_n) \alpha''_n r_n^2 \|x_n - p\| + \beta'_n \|x_n - p\| \\
&\quad + (1 - \beta'_n - \gamma'_n) \beta''_n r_n^2 \|x_n - p\| + m_n \\
&\leq (1 - \beta'_n) \alpha''_n r_n^2 \|x_n - p\| + \beta'_n r_n^2 \|x_n - p\| \\
&\quad + (1 - \beta'_n) \beta''_n r_n^2 \|x_n - p\| + m_n \\
&\leq (1 - \beta'_n) r_n^2 \|x_n - p\| + \beta'_n r_n^2 \|x_n - p\| + m_n \\
&= r_n^2 \|x_n - p\| + m_n
\end{aligned} \tag{3.4}$$

where  $m_n = \alpha'_n \gamma''_n r_n \|u_n - p\| + \gamma'_n \|v_n - p\|$

Note that  $\sum_{n=1}^{\infty} m_n < \infty$  as  $\sum_{n=1}^{\infty} \gamma'_n < \infty$  and  $\sum_{n=1}^{\infty} \gamma''_n < \infty$ .

Substituting (3.4) in (3.1), we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n r_n^3 \|x_n - p\| + \alpha_n r_n m_n \\
&\quad + \beta_n \|x_n - p\| + \gamma_n \|c_n - p\| \\
&\leq (\alpha_n + \beta_n) r_n^3 \|x_n - p\| + b_n \\
&= r_n^3 \|x_n - p\| + b_n
\end{aligned} \tag{3.5}$$

where  $b_n = \alpha_n r_n m_n + \gamma_n \|c_n - p\|$ .

Since  $\sum_{n=1}^{\infty} r_n < \infty$ ,  $\sum_{n=1}^{\infty} m_n < \infty$  and  $\sum_{n=1}^{\infty} \gamma_n < \infty$ , which implies that  $\sum_{n=1}^{\infty} b_n < \infty$ .

It follows that from Lemma (6) that  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists for all  $p \in F$ . This proved that the first part of the lemma.

(ii) Since  $1 + x \leq e^x$  for all  $x > 0$ . Then from (3.5)

$$\begin{aligned}
\|x_{n+m} - p\| &\leq r_{n+m-1}^3 \|x_{n+m-1} - p\| + b_{n+m-1} \\
&\leq e^{3r_{n+m-1}} \|x_{n+m-1} - p\| + b_{n+m-1} \\
&\leq e^{3r_{n+m-1}} [e^{3r_{n+m-2}} \|x_{n+m-2} - p\| + b_{n+m-2}] + b_{n+m-1} \\
&\leq e^{3(r_{n+m-1} + r_{n+m-2})} \|x_{n+m-2} - p\| + e^{3r_{n+m-1}} b_{n+m-2} + b_{n+m-1} \\
&\leq e^{3(r_{n+m-1} + r_{n+m-2})} \|x_{n+m-2} - p\| + e^{3r_{n+m-1}} [b_{n+m-2} + b_{n+m-1}] \\
&\vdots \\
&\leq e^{3 \sum_{k=n}^{n+m-1} r_k} \|x_n - p\| + e^{3 \sum_{k=n}^{n+m-1} r_k} \sum_{k=n}^{n+m-1} b_k \\
&\leq M \|x_n - p\| + M \sum_{k=n}^{n+m-1} b_k
\end{aligned}$$

where  $M = e^3 \sum_{k=n}^{n+m-1} r_k$ .

This completes the proof of the lemma. □

**Lemma 9.** *Let  $E$  be a uniformly convex Banach space, and let  $D$  be a nonempty closed and convex subset of  $E$ . Let  $T_1, T_2, T_3$  be three asymptotically nonexpansive multi-maps from  $D$  into  $CB(D)$  with the sequence  $\{r_{i_n}\} \subset [1, \infty)$  satisfying  $\sum_{n=1}^{\infty} r_{i_n} < \infty$  for all  $i = 1, 2, 3$  and  $F := \bigcap_{i=1}^3 F(T_i) \neq \emptyset$  and  $T_i p = \{p\}, (i = 1, 2, 3)$ . Let  $\{x_n\}$  be the sequence defined by (2.1) with the following restrictions:*

(i)  $0 < \alpha \leq \alpha_n, \alpha'_n, \alpha''_n \leq 1 - \alpha$  for some  $\alpha \in (0, 1)$  and for all  $n \geq n_0, \exists n_0 \in \mathbb{N}$ ;

(ii)  $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty,$  and  $\sum_{n=1}^{\infty} \gamma''_n < \infty$ .

Then  $\lim_{n \rightarrow \infty} \|z'_n - x_n\| = \lim_{n \rightarrow \infty} \|w'_n - x_n\| = \lim_{n \rightarrow \infty} \|w_n - x_n\| = 0$  for all  $n \in \mathbb{N}$ .

*Proof.* Let  $p \in F$ . It follows from Lemma (8) that  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists for all  $n \in \mathbb{N}$ . Let  $\lim_{n \rightarrow \infty} \|x_n - p\| = c$  for some  $c \geq 0$ . For each  $n \geq 1$ , let  $r_n = \max\{r_{i_n} : i = 1, 2, 3\}$ . Taking the *limsup* of (3.4), we obtain

$$\limsup_{n \rightarrow \infty} \|y_n - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = c \tag{3.6}$$

So

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|z'_n - p\| &\leq \limsup_{n \rightarrow \infty} d(z'_n, T_1^n p) \\ &\leq \limsup_{n \rightarrow \infty} H(T_1^n y_n, T_1^n p) \\ &\leq r_n \|y_n - p\| \\ &\leq c \end{aligned} \tag{3.7}$$

Next, we consider

$$\limsup_{n \rightarrow \infty} \|z'_n - p + \gamma_n(c_n - x_n)\| \leq \limsup_{n \rightarrow \infty} \|z'_n - p\| + \gamma_n \|c_n - x_n\| \tag{3.8}$$

It follows from (3.7) that

$$\limsup_{n \rightarrow \infty} \|z'_n - p + \gamma_n(c_n - x_n)\| \leq c \tag{3.9}$$

By the Triangle inequality

$$\limsup_{n \rightarrow \infty} \|x_n - p + \gamma_n(c_n - x_n)\| \leq c \tag{3.10}$$

Moreover, we note that

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} \|x_{n+1} - p\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n z'_n + \beta_n x_n + \gamma_n(c_n - (1 - \alpha_n)p - \alpha_n p)\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n z'_n - \alpha_n p + \alpha_n \gamma_n c_n - \alpha_n \gamma_n x_n + (1 - \alpha_n)x_n \\ &\quad - (1 - \alpha_n)p - \gamma_n x_n + \gamma_n c_n - \alpha_n \gamma_n c_n + \alpha_n \gamma_n x_n\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n(z'_n - p + \gamma_n(c_n - x_n)) \\ &\quad + (1 - \alpha_n)(x_n - p + \gamma_n(c_n - x_n))\| \end{aligned} \tag{3.11}$$

It follows from (3.9), (3.10) and Lemma (7) that

$$\lim_{n \rightarrow \infty} \|z'_n - x_n\| = 0$$

Next, we prove that  $\lim_{n \rightarrow \infty} \|w'_n - x_n\| = 0$

For each  $n \geq 1$ ,

$$\begin{aligned} \|x_n - p\| &\leq \|z'_n - x_n\| + \|z'_n - p\| \\ &\leq \|z'_n - x_n\| + d(z'_n, T_1^n p) \\ &\leq \|z'_n - x_n\| + H(T_1^n y_n, T_1^n p) \\ &\leq \|z'_n - x_n\| + r_n \|y_n - p\| \end{aligned} \quad (3.12)$$

Since  $\lim_{n \rightarrow \infty} \|z'_n - x_n\| = 0 = \lim_{n \rightarrow \infty} r_n$ , it follows from (3.6) and (3.12) that

$$c = \lim_{n \rightarrow \infty} \|x_n - p\| \liminf_{n \rightarrow \infty} \|y_n - p\| \leq \limsup_{n \rightarrow \infty} \|y_n - p\| \leq c \quad (3.13)$$

Hence  $\lim_{n \rightarrow \infty} \|y_n - p\| = c$

Observe that

$$\|z_n - p\| \leq r_n \|x_n - p\| + \gamma''_n \|v_n - p\|$$

By the boundedness of  $\{v_k\}$  and  $\lim_{n \rightarrow \infty} r_n = 0 = \lim_{n \rightarrow \infty} \gamma''_n$ , we have

$$\limsup_{n \rightarrow \infty} \|z_n - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| \leq c$$

and so

$$\limsup_{n \rightarrow \infty} \|w'_n - p\| \leq \limsup_{n \rightarrow \infty} r_n \|z_n - p\| \leq c$$

Next, we consider

$$\|w'_n - p + \gamma'_n(u_n - x_n)\| \leq \|w'_n - p\| + \gamma'_n \|u_n - x_n\| \quad (3.14)$$

Taking *limsup* on both sides, we get

$$\limsup_{n \rightarrow \infty} \|w'_n - p + \gamma'_n(u_n - x_n)\| \leq c$$

By the Triangle inequality, we see that

$$\limsup_{n \rightarrow \infty} \|x_n - p + \gamma'_n(u_n - x_n)\| \leq c$$

Since  $\lim_{n \rightarrow \infty} \|y_n - p\| = c$ , we obtain

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} \|x_n - p\| \\ &= \lim_{n \rightarrow \infty} \|\alpha'_n w'_n + \beta'_n x_n + \gamma'_n u_n - p\| \\ &= \lim_{n \rightarrow \infty} \|\alpha'_n (w'_n - p + \gamma'_n(u_n - x_n)) + (1 - \alpha'_n)(x_n - p + \gamma'_n(u_n - x_n))\| \end{aligned} \quad (3.15)$$

By Lemma (7), we obtain  $\lim_{n \rightarrow \infty} \|w'_n - x_n\| = 0$

Similarly, by using the same argument as in the proof above, we have  $\lim_{n \rightarrow \infty} \|w_n - x_n\| = 0$ , for all  $n \in \mathbb{N}$ . This completes the proof.  $\square$



**Theorem 10.** Let  $E$  be a uniformly convex Banach space, and let  $D$  be a nonempty closed and convex subset of  $E$ . Let  $T_1, T_2, T_3$  be three asymptotically nonexpansive multi-maps from  $D$  into  $CB(D)$  with the sequence  $\{r_{i_n}\} \subset [1, \infty)$  satisfying  $\sum_{n=1}^{\infty} r_{i_n} < \infty$  for all  $i = 1, 2, 3$  and  $F := \bigcap_{i=1}^3 F(T_i) \neq \phi$  and  $T_i p = \{p\}, (i = 1, 2, 3)$ . Let  $\{x_n\}$  be the sequence defined by (2.1) with the following restrictions:

(i)  $0 < \alpha \leq \alpha_n, \alpha'_n, \alpha''_n \leq 1 - \alpha$  for some  $\alpha \in (0, 1)$  and for all  $n \geq n_0, \exists n_0 \in \mathbb{N}$ ;

(ii)  $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty,$  and  $\sum_{n=1}^{\infty} \gamma''_n < \infty$ .

If  $\{T_i\}, (i = 1, 2, 3)$  satisfying Condition (II), then the sequence  $\{x_n\}$  converges strongly to a common fixed point of  $F$ .

*Proof.* From Lemma (9), we have

$$\lim_{n \rightarrow \infty} \|z'_n - x_n\| = \lim_{n \rightarrow \infty} \|w'_n - x_n\| = \lim_{n \rightarrow \infty} \|w_n - x_n\| = 0 \quad (3.16)$$

Also,

$$d(x_n, T_3^n x_n) \leq \|w_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

Since  $\{x_n\}, \{u_n\}$  are bounded, so is  $\{u_n - w_n\}$ . Now, let  $K = \sup_{n \in \mathbb{N}} \|u_n - w_n\|$ . By assumption and (3.16), we get

$$\begin{aligned} \|z_n - w_n\| &\leq \|\alpha''_n w_n + \beta''_n x_n + \gamma''_n u_n - w_n\| \\ &\leq \beta''_n \|x_n - w_n\| + \gamma''_n \|u_n - w_n\| \\ &\leq \beta''_n \|x_n - w_n\| + \gamma''_n K \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned} \quad (3.17)$$

It follows from (3.16) and (3.17) that

$$\|z_n - x_n\| \leq \|z_n - w_n\| + \|w_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (3.18)$$

Again from (3.16) and (3.18), we have

$$\begin{aligned} d(x_n, T_2^n x_n) &\leq d(x_n, T_2^n z_n) + H(T_2^n z_n, T_2^n x_n) \\ &\leq \|x_n - w'_n\| + r_n \|z_n - x_n\| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

Now, since  $\{y_n\}, \{v_n\}$  are bounded, so is  $\{v_n - w'_n\}$ . Now, let  $K = \sup_{n \in \mathbb{N}} \|v_n - w'_n\|$ . By assumption and (3.16), we get

$$\begin{aligned} \|y_n - w'_n\| &\leq \|\alpha'_n w'_n + \beta'_n x_n + \gamma'_n v_n - w'_n\| \\ &\leq \beta'_n \|x_n - w'_n\| + \gamma'_n \|v_n - w'_n\| \\ &\leq \beta'_n \|x_n - w'_n\| + \gamma'_n K \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned} \quad (3.19)$$

It follows from (3.16) and (3.19) that

$$\|y_n - x_n\| \leq \|y_n - w'_n\| + \|w'_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (3.20)$$

Therefore from (3.20) and (3.16), we get

$$\begin{aligned} d(x_n, T_1^n x_n) &\leq d(x_n, T_1^n y_n) + H(T_1^n y_n, T_1^n x_n) \\ &\leq \|x_n - z'_n\| + r_n \|y_n - x_n\| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

Now since  $T_1, T_2, T_3$  satisfy Condition (II), we have  $d(x_n, F) \rightarrow 0$ . Thus there is a subsequence  $\{x_{n_j}\}$  of  $\{x_n\}$  and a sequence  $p_j \subset F$  such that

$$\|x_{n_j} - p_j\| \leq 2^{-j} \quad (3.21)$$

Set  $M = e^3 \sum_{k=n}^{n+m-1} r_k$  and write  $n_{j+1} = n_j + l$  for some  $l \geq 1$ . Then we have by (3.5)

$$\begin{aligned} \|x_{n_{j+1}} - p_j\| &= \|x_{n_j+l} - p_j\| \\ &\leq M \|x_{n_j} - p_j\| + M \sum_{k=n_j}^{n_j+l-1} b_k \\ &< \frac{M}{2^j} + M \sum_{k=n_j}^{n_j+l-1} b_k \end{aligned}$$

Next we shall show that  $\{p_j\}$  is Cauchy sequence in  $D$ .

Note that

$$\begin{aligned} \|p_{j+1} - p_j\| &\leq \|p_{j+1} - x_{n_{j+1}}\| + \|x_{n_{j+1}} - p_j\| \\ &< \frac{1}{2^{j+1}} + \frac{M}{2^j} + M \sum_{k=n_j}^{n_j+l-1} b_k \\ &< \frac{2M+1}{2^{j+1}} + M \sum_{k=n_j}^{n_j+l-1} b_k \end{aligned}$$

This implies that  $\{p_j\}$  is Cauchy sequence in  $D$ . Assume that  $p_j \rightarrow p$  as  $j \rightarrow \infty$ .

Since  $d(p_j, T_i^n p) \leq H(T_i^n p, T_i^n p_j) \leq r_n \|p - p_j\|$  for all  $i = 1, 2, 3$  and  $p_j \rightarrow p$  as  $j \rightarrow \infty$ .

It follows that  $d(p_j, T_i^n p) = 0$  for all  $i = 1, 2, 3$  and thus  $p \in F$ . It implies by (3.21) that  $\{x_{n_j}\}$  converges strongly to  $p$ . Since  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists, it follows that  $\{x_n\}$  converges strongly to  $p$ . This completes the proof of the theorem.  $\square$

The main result of this paper holds under the assumption that  $Tp = \{p\}$  for all  $p \in F$ . This condition was introduced by Shahzad and Zegeye [15]. The following examples give an example of nonexpansive multi-map  $T$  which satisfies the property that  $Tp = \{p\}$  for all  $p \in F := \bigcap_{i=1}^3 F(T_i)$  and  $Tx$  is not a singleton for all  $x \notin F$ .

**Example 11.** Consider  $D = [0, 1] \times [0, 1]$  with the usual norm. Define  $T : D \rightarrow CB(D)$  by

$$T(x, y) = \begin{cases} \{(x, 0)\}, & \text{if } x \neq 0, y = 0; \\ \{(0, y)\}, & \text{if } x = 0, y \neq 0; \\ \{(x, 0), (0, y)\}, & \text{if } x, y \neq 0; \\ \{(0, 0)\}, & \text{if } x, y = 0. \end{cases}$$

**Example 12.** Consider  $D = [0, 1]$  with the usual norm. Define  $T : D \rightarrow CB(D)$  by

$$Tx = \left[ \frac{x+1}{2}, 1 \right]$$

**Example 13.** Consider  $D = [0, 1] \times [0, 1]$  with the usual norm. Define  $T : D \rightarrow CB(D)$  by

$$T(x, y) = \{x\} \times \left[ \frac{y+1}{2}, 1 \right]$$

### Acknowledgements

We would like to extend our sincerest thanks to the anonymous referee for the exceptional review of this work. The suggestions and recommendations in the report increased the quality of our paper.

### References

- [1] H.H. Bauschke, E. Matoušková and S. Reich, Projection and proximal point methods: convergence results and counterexamples, *Nonlinear Anal.* **56** (2004), 715-738.
- [2] V. Berinde, Iterative approximation of fixed points, *Lecture Notes in Mathematics*, vol. **1912** Springer, Berlin, 2007.
- [3] H. Fukhar-ud-din and S.H. Khan, Convergence of iterates with errors of asymptotically quasi-nonexpansive mappings and applications, *J. Math. Anal. Appl.* **328** (2007), 821-829.
- [4] A. Genal, J. Lindenstrass, An example concerning fixed points, *Israel J. Math.* **22** (1975), 81-86.
- [5] S. Ishikawa, Fixed points and iteration of a nonexpansive mapping in a Banach space, *Proc. Amer. Math. Soc.* **59** (1976), 65-71.
- [6] S.H. Khan and H. Fukhar-ud-din, Weak and strong convergence of a scheme with errors for two nonexpansive mappings *Nonlinear Anal.* **61** (2005), 1295-1301.
- [7] D. Lei and L. Shenghong, Ishikawa iteration process with errors for nonexpansive mappings in uniformly convex Banach spaces *Internat. J. Math. & Math. Sci.* **24** (2000), 49-53.
- [8] W.R. Mann, Mean value methods in iteration *Proc. Amer. Math. Soc.* **4** (1953), 506-510.
- [9] K. Nammanee, M.A. Noor and S. Suantai, Convergence criteria of modified Noor iterations with errors for asymptotically nonexpansive mappings, *J. Math. Anal. Appl.* **314** (2006), 320-334.
- [10] B. Panyanak, Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces, *Comput. Math. Appl.* **54** (2007), 872-877.
- [11] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces *J. Math. Anal. Appl.* **67** (1979), 274-276.
- [12] K.P.R. Sastry and G.V.R. Babu, Convergence of Ishikawa iterates for a multivalued mappings with fixed point, *Czechoslovak Math. J.* **55** (2005), 817-826.
- [13] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, *Bull. Austr. Math. Soc.* **43** (1991), 153-159.
- [14] H.F. Senter and W.G. Dotson, Approximating fixed points of nonexpansive mappings, *Proc. Amer. Math. Soc.* **44** (1974), 375-380.
- [15] N. Shahzad and H. Zegeye, On Mann and Ishikawa Iteration schemes for multivalued maps in Banach spaces, *Nonlinear Anal.* **71** (2009), 838-844.
- [16] C. Shiau K.K. Tan and C.S. Wong, Quasi-nonexpansive multivalued maps and selection, *Fund. Math.* **87** (1975), 109-119.

- [17] Y. Song and W. Wang, Erratum to "Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces" [Comput. Math. Appl. 54(2007), 872-877], *Comput. Math. Appl.* **55** (2005), 2999-3002.
- [18] K.K. Tan and H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, *J. Math. Anal. Appl.* **178** (1993), 301-308.