Products of composition and iterated differentiation operators from fractional Cauchy transforms to weighted Bloch-type spaces

Ajay K. Sharma
School of Mathematics, Shri Mata Vaishno Devi University, Kakryal, Katra-182320, J&K, India
aksju_76@yahoo.com

Ram Krishan
School of Mathematics, Shri Mata Vaishno Devi University, Kakryal, Katra-182320, J&K, India
ramk123.verma@gmail.com

Abstract
We consider products of composition and iterated differentiation operators from the space of fractional Cauchy transforms to weighted Bloch-type spaces and little weighted Bloch-type spaces. Upper and lower bounds for norm of these operators are computed and compactness is completely characterized.

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1 Introduction and Preliminaries

Let \( \mathbb{D} \) be the open unit disk in the complex plane \( \mathbb{C} \), \( \partial \mathbb{D} \) its boundary, \( dA(z) \) the normalized area measure on \( \mathbb{D} \) (i.e. \( A(\mathbb{D}) = 1 \)) and \( H^\infty \) the space of all bounded holomorphic functions on \( \mathbb{D} \) with the norm \( \| f \|_{\infty} = \sup_{z \in \mathbb{D}} |f(z)| \). Let \( H(\mathbb{D}) \) the class of all holomorphic functions on \( \mathbb{D} \). \( H(\mathbb{D}) \) is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of \( \mathbb{D} \). We denote by \( \mathcal{M} \) the space of all complex Borel measures on \( \partial \mathbb{D} \) and let \( \mathcal{M}^* \) be the subset of \( \mathcal{M} \) consisting of probability measures. Let \( \alpha > 0 \) be a real number. The family \( \mathcal{F}_\alpha \) of fractional Cauchy transforms is the collection of functions \( f \in H(\mathbb{D}) \) which admits a representation of the form

\[
f(z) = \int_{\partial \mathbb{D}} \frac{1}{(1 - \zeta z)^\alpha} d\mu(\zeta) \quad (z \in \mathbb{D})
\]

(1.1)

for some \( \mu \in \mathcal{M} \). The principal branch is used in the power function in (1.1) and throughout the rest of the paper. The space \( \mathcal{F}_\alpha \) is a Banach space with respect to the norm

\[
\| f \|_{\mathcal{F}_\alpha} = \inf_{\mu \in \mathcal{M}} \left\{ \| \mu \| : f(z) = \int_{\partial \mathbb{D}} \frac{1}{(1 - \zeta z)^\alpha} d\mu(\zeta) \right\},
\]

where \( \| \mu \| \) denotes the total variation of measure \( \mu \). According to the Lebesgue decomposition theorem \( \mathcal{M} = \mathcal{M}_a + \mathcal{M}_s \), where \( \mathcal{M}_a = \{ \mu_a \in \mathcal{M} : \mu_a << m \} \), where \( m \) is the...
normalized Lebesgue measure on the unit circle $\partial \mathbb{D}$, and \( \mathcal{M}_s = \{ \mu_s \in \mathcal{M} : \mu_s \perp m \} \). Thus any \( \mu \) can be written as \( \mu = \mu_a + \mu_s \), where \( \mu_a \in \mathcal{M}_a \), \( \mu_s \in \mathcal{M}_s \) and \( \| \mu \| = \| \mu_a \| + \| \mu_s \| \). Consequently, the space \( \mathcal{F}_\alpha \) may also be written as \( \mathcal{F}_\alpha = (\mathcal{F}_\alpha)_a + (\mathcal{F}_\alpha)_s \), where \( (\mathcal{F}_\alpha)_a \) is isometrically isomorphic to \( \mathcal{M}/H_0^1 \), the closed subspace of \( \mathcal{M} \) of absolutely continuous measures and \( (\mathcal{F}_\alpha)_s \) is isomorphic to \( \mathcal{M}_s \) the closed subspace of \( \mathcal{M} \) of singular measures.

If \( f \in (\mathcal{F}_\alpha)_a \), then the singular part is null and the measure \( \mu \) for which the integral in (1.1) holds reduces to \( d\mu(e^{it}) = g(e^{it})dt \), where \( g(e^{it}) \in L^1 \) and \( dt \) is the Lebesgue measure on \( \partial \mathbb{D} \). For more about the space \( \mathcal{F}_\alpha \), we refer \([1],[2],[3],[4],[8],[9]\) and \([10]\).

Let 
\[
\eta_a(z) = \frac{a-z}{1-\bar{a}z}, \quad a, z \in \mathbb{D},
\]
that is, the involutive automorphism of \( \mathbb{D} \) interchanging points \( a \) and \( 0 \). Also we need the following well known identity
\[
(1 - |z|^2)|\eta'_a(z)| = 1 - |\eta_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2} \tag{1.2}
\]

The Bloch-type space \( \mathcal{B}_\nu(\mathbb{D}) = \mathcal{B}_\nu \) consists of all \( f \in H(\mathbb{D}) \) such that
\[
\|f\|_{\mathcal{B}_\nu} := |f(0)| + b_\nu(f) = |f(0)| + \sup_{z \in \mathbb{D}} \nu(z)|f'(z)| < \infty,
\]
where \( \nu \) is a positive continuous function on \( \mathbb{D} \) (weight). A weight \( \nu \) is called \textit{typical} if it is radial, i.e. \( \nu(z) = \nu(|z|) \), \( z \in \mathbb{D} \) and \( \nu(|z|) \) decreasingly converges to \( 0 \) as \( |z| \to 1 \). A positive continuous function \( \nu \) on the interval \([0,1] \) is called normal if there are \( \delta \in (0,1) \) and \( \tau \) and \( t \), \( 0 < \tau < t \) such that
\[
\frac{\nu(r)}{(1-r)^\tau} \text{ is decreasing on } [\delta,1] \text{ and } \lim_{r \to 1} \frac{\nu(r)}{(1-r)^\tau} = 0;
\]
\[
\frac{\nu(r)}{(1-r)^t} \text{ is increasing on } [\delta,1] \text{ and } \lim_{r \to 1} \frac{\nu(r)}{(1-r)^t} = \infty.
\]

If we say that a function \( \nu : \mathbb{D} \to [0,\infty) \) is normal we also assume that it is radial. The little Bloch-type space \( \mathcal{B}_{\nu,0}(\mathbb{D}) = \mathcal{B}_{\nu,0} \) consists of all \( f \in H(\mathbb{D}) \) such that
\[
\lim_{|z| \to 1} \nu(z)|f'(z)| = 0.
\]

With the norm \( \| \cdot \|_{\mathcal{B}_\nu} \) the Bloch-type space \( \mathcal{B}_\nu \) is a Banach space and the little Bloch-type space \( \mathcal{B}_{\nu,0} \) is a closed subspace of the Bloch-type space \( \mathcal{B}_\nu \).

Let \( \varphi \) be a holomorphic self-map of \( \mathbb{D} \). For a non-negative integer \( n \), we define a linear operator \( D^n_\varphi \) as follows:
\[
D^n_\varphi f = f^{(n)} \circ \varphi, \quad f \in H(\mathbb{D}).
\]
If \( n = 0 \), then we have \( D^0_\varphi = C_\varphi \), the composition operator induced by \( \varphi \), defined as \( C_\varphi f = f \circ \varphi \), \( f \in \bar{H}(\mathbb{D}) \). We recall that an operator \( T \) from a Banach space \( X \) to a Banach space \( Y \) is bounded if there exists a positive constant \( C \) such that \( \|Tf\|_Y \leq C\|f\|_X \). A bounded operator \( T : X \to Y \) is compact if the image of every bounded set in \( X \) is relatively compact in \( Y \). Equivalently, \( T : X \to Y \) is compact if for every bounded sequence \( \{f_m\} \) in \( X \), \( \{Tf_m\} \) has a convergent sequence in \( Y \). In \([8]\), Hibschweiler and MacGregor proved that if \( \alpha \geq 1 \), then every holomorphic self-map \( \varphi \) of
\( \mathbb{D} \) induces a bounded composition operator on \( \mathcal{F}_\alpha \). In fact, Bourdon and Cima \([1]\) proved that
\[
\|C_\varphi\|_{\mathcal{F}_1 \to \mathcal{F}_1} \leq \frac{2 + 2\sqrt{2}}{1 - |\varphi(0)|}
\]
which was improved to
\[
\|C_\varphi\|_{\mathcal{F}_1 \to \mathcal{F}_1} \leq \frac{1 + 2|\varphi(0)|}{1 - |\varphi(0)|}
\]
by Cima and Matheson \([3]\). Moreover, equality is attained for certain linear fractional maps.
In contrast with the situation when \( \alpha \geq 1 \), a self-map \( \varphi \) of \( \mathbb{D} \) need not induce a bounded composition operator on \( \mathcal{F}_\alpha \) when \( 0 < \alpha < 1 \). In fact, the condition \( \varphi \in \mathcal{F}_\alpha \) is necessary for \( C_\varphi \) to be bounded on \( \mathcal{F}_\alpha \). Hibschweiler and MacGregor \([8]\), constructed a self-map \( \varphi \) of \( \mathbb{D} \) with \( \varphi \notin \mathcal{F}_\alpha \) \( (0 < \alpha < 1) \). For some recent results in this area, see \([2],[6],[7],[11],[13]\) and the references therein. In this paper, we characterize boundedness and compactness of products of composition and iterated differentiation from fractional Cauchy transforms to weighted Bloch-type spaces. Throughout the paper constants are denoted by \( C \), they are positive and not necessarily the same at each occurrence. The notation \( A \asymp B \) means that there is a positive constant \( C \) such that \( A/C \leq B \leq CA \).

2 Boundedness and Compactness of \( D^\alpha_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_\nu \)

In this section, we characterize the boundedness and compactness of \( D^\alpha_\varphi \) from the space of fractional Cauchy transforms to weighted Bloch-type spaces.

The following lemma can be found in \([7]\), and is used throughout the rest of the paper.

**Lemma 1.** Let \( \alpha > 0 \) and \( f \in H(\mathbb{D}) \).

1. If \( f \in \mathcal{F}_\alpha \) and \( z \in \mathbb{D} \), then \( |f(z)| \leq \|f\|_{\mathcal{F}_\alpha}/(1 - |z|)^\alpha \).

2. If \( f \in \mathcal{F}_\alpha \), then \( f' \in \mathcal{F}_{\alpha + 1} \) and \( \|f'\|_{\mathcal{F}_{\alpha + 1}} \leq \alpha \|f\|_{\mathcal{F}_\alpha} \).

**Theorem 2.** Let \( \nu \) be a normal weight, \( \alpha > 0 \), \( n \in \mathbb{N} \cup \{0\} \) and \( \varphi \) a holomorphic self-map of \( \mathbb{D} \). Then \( D^\alpha_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_\nu \) is bounded if and only if
\[
M_1 := \sup_{\zeta \in \partial \mathbb{D}} \sup_{z \in \mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1 - \zeta \varphi(z)|^{n+\alpha+1}} < \infty.
\] (2.1)

Moreover, if \( D^\alpha_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_\nu \) is bounded, then
\[
\alpha(\alpha + 1) \cdots (\alpha + n)M_1 \leq \|D^\alpha_\varphi\|_{\mathcal{F}_\alpha \to \mathcal{B}_\nu}
\leq \alpha(\alpha + 1) \cdots (\alpha + n - 1) \left\{ \alpha(\alpha + n)M_1 + \frac{1}{(1 - |\varphi(0)|)^{\alpha+1}} \right\}.
\] (2.2)

**Proof.** First, suppose that (2.1) holds. Let \( f \in \mathcal{F}_\alpha \). Then there is a \( \mu \in \mathcal{M} \) such that \( \|\mu\| = \|f\|_{\mathcal{F}_\alpha} \) and
\[
f(z) = \int_{\partial \mathbb{D}} \frac{d\mu(\zeta)}{(1 - \zeta z)^\alpha}.
\]
Thus, we have
\[
f^{(n+1)}(z) = \alpha(\alpha + 1) \cdots (\alpha + n) \int_{\partial \mathbb{D}} \frac{(\zeta)^{n+1}}{(1 - \zeta z)^{n+\alpha+1}} d\mu(\zeta).
\] (2.3)
Replacing $z$ in (2.3) by $\varphi(z)$, using a known inequality and multiplying such obtained inequality by $\nu(z)|\varphi'(z)|$, we obtain

$$\nu(z)|\varphi'(z)||f^{(n+1)}(\varphi(z))| \leq \alpha(\alpha + 1) \cdots (\alpha + n) \int_{\partial D} \frac{\nu(z)|\varphi'(z)|}{|1 - \overline{\varphi(z)}|^{n+\alpha+1}} d\mu(\zeta)$$  \hspace{1cm} (2.4)

$$\leq \alpha(\alpha + 1) \cdots (\alpha + n) \sup_{\zeta \in \partial D} \sup_{z \in D} \frac{\nu(z)|\varphi'(z)|}{|1 - \overline{\varphi(z)}|^{n+\alpha+1}} \int_{\partial D} d\mu(\zeta)$$

$$= \alpha(\alpha + 1) \cdots (\alpha + n) \sup_{\zeta \in \partial D} \sup_{z \in D} \frac{\nu(z)|\varphi'(z)|}{|1 - \overline{\varphi(z)}|^{n+\alpha+1}} \|\mu\|$$

from which it follows that

$$\nu(z)|(D^nf)'(z)| \leq \alpha(\alpha + 1) \cdots (\alpha + n) \sup_{\zeta \in \partial D} \sup_{z \in D} \frac{\nu(z)|\varphi'(z)|}{|1 - \overline{\varphi(z)}|^{n+\alpha+1}} \|f\|_{F_\alpha}.$$  \hspace{1cm} (2.5)

Taking the supremum over $z \in \mathbb{D}$, we get

$$\sup_{z \in \mathbb{D}} \nu(z)|(D^nf)'(z)| \leq \alpha(\alpha + 1) \cdots (\alpha + n) M_1 \|f\|_{F_\alpha}. \hspace{1cm} (2.6)$$

By Lemma 1, we have

$$|(D^n\varphi f)(0)| = |f^{(n)}(\varphi(0))| \leq \left\|f^n\right\|_{F_{n+\alpha}} \leq \alpha(\alpha + 1) \cdots (\alpha + n - 1) \frac{\|f\|}{(1 - |\varphi(0)|)^{n+\alpha}}.$$

Thus from (2.5) and (2.6), we have

$$\|D^nf\|_{B_\nu} \leq \alpha(\alpha + 1) \cdots (\alpha + n - 1) \left\{(\alpha + n) M_1 + \frac{1}{(1 - |\varphi(0)|)^{n+\alpha}}\right\} \|f\|_{F_\alpha}.$$

Hence $D^nf : F_\alpha \rightarrow B_\nu$ is bounded and

$$\|D^nf\|_{F_\alpha \rightarrow B_\nu} \leq \alpha(\alpha + 1) \cdots (\alpha + n - 1) \left\{(\alpha + n) M_1 + \frac{1}{(1 - |\varphi(0)|)^{n+\alpha}}\right\}. \hspace{1cm} (2.7)$$

Next suppose that $D^nf : F_\alpha \rightarrow B_\nu$ is bounded. Let

$$f_\zeta(z) = \frac{1}{(1 - \overline{\zeta}z)^\alpha}, \quad \zeta \in \partial \mathbb{D}. \hspace{1cm} (2.8)$$

Then $\|f_\zeta\|_{F_\alpha} = 1$ and

$$f_\zeta^{(n+1)}(z) = (\alpha + 1) \cdots (\alpha + n) \frac{(\zeta)^{n+1}}{(1 - \zeta z)^{n+\alpha+1}}.$$

From this and the boundedness of the operator $D^nf : F_\alpha \rightarrow B_\nu$, we have that $\|D^nf_\zeta\|_{B_\nu} \leq \|D^nf\|_{F_\alpha \rightarrow B_\nu}$, for every $\zeta \in \partial \mathbb{D}$ and so

$$\alpha(\alpha + 1) \cdots (\alpha + n) \sup_{\zeta \in \partial D} \sup_{z \in D} \frac{\nu(z)|\varphi'(z)|}{|1 - \overline{\varphi(z)}|^{n+\alpha+1}} \leq \|D^nf\|_{F_\alpha \rightarrow B_\nu}. \hspace{1cm} (2.9)$$

If $D^nf : F_\alpha \rightarrow B_\nu$ is bounded, then from (2.7) and (2.9), inequality in (2.2) follows. \qed
Theorem 3. Let $\nu$ be a normal weight, $\alpha > 0$, $n \in \mathbb{N} \cup \{0\}$, $\varphi$ a holomorphic self-map of $\mathbb{D}$ and $d\lambda(z) = dA(z)/(1 - |z|^2)^2$. Then $D^n_\varphi : F_\alpha \to B_\nu$ is bounded if and only if

$$L_1 := \sup_{\zeta \in \partial \mathbb{D}} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1 - \zeta \varphi(z)|^2(n+\alpha+1)} \nu^2(z)(1 - |\eta_a(z)|^2)^2 d\lambda(z) < \infty. \quad (2.10)$$

Moreover, if $D^n_\varphi : F_\alpha \to B_\nu$ is bounded, then asymptotic relation $L_1 \asymp M_1^2$ holds.

Proof. First assume that (2.10) holds. Since $\nu$ is normal, $\nu(a) \asymp \nu(z)$ when $z \in D(a, (1 - |a|)/2) = \{|z - a| < (1 - |a|)/2\}$. Also it is known that $|1 - \bar{a}z| \asymp 1 - |a|^2$, for $z \in D(a, (1 - |a|)/2)$. Using these two facts, (1.2) and the subharmonicity of the function

$$g(z) = \frac{|\varphi'(z)|^2}{|1 - \zeta \varphi(z)|^2(n+\alpha+1)}$$

we obtain

$$L_1 \geq \sup_{\zeta \in \partial \mathbb{D}} \sup_{a \in \mathbb{D}} \int_{D(a,(1-|a|)/2)} \frac{|\varphi'(z)|^2}{|1 - \zeta \varphi(z)|^2(n+\alpha+1)} \nu^2(z)(1 - |\eta_a(z)|^2)^2 d\lambda(z)$$

$$= \sup_{\zeta \in \partial \mathbb{D}} \sup_{a \in \mathbb{D}} \int_{D(a,(1-|a|)/2)} \frac{|\varphi'(z)|^2}{|1 - \zeta \varphi(z)|^2(n+\alpha+1)} \nu^2(z)(1 - |a|^2)^2 \frac{dA(z)}{|1 - az|^2}$$

$$\geq \sup_{\zeta \in \partial \mathbb{D}} \sup_{a \in \mathbb{D}} \frac{\nu^2(a)|\varphi'(a)|^2}{|1 - \zeta \varphi(a)|^2(n+\alpha+1)} = M_1^2. \quad (2.11)$$

Thus by Theorem 1, the operator $D^n_\varphi : F_\alpha \to B_\nu$ is bounded.

Next assume that the operator $D^n_\varphi : F_\alpha \to B_\nu$ is bounded. By Theorem 1, we have that (2.1) holds. From this, we have

$$L_1 \leq M_1^2 \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} dA(z) = M_1^2 C < \infty. \quad (2.12)$$

The asymptotic relation $L_1 \asymp M_1^2$ follows from (2.11) and (2.12).

Proceeding as in the proof of Theorem 2, we can easily prove the following lemma. We omit the proof.

Lemma 4. Let $\nu : \mathbb{D} \to [0, \infty)$ be a normal weight function and $d\lambda(z) = dA(z)/(1 - |z|^2)^2$. Then $f \in B_\nu$ if and only if

$$I := |f(0)|^2 + \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \nu^2(z)(1 - |\eta_a(z)|^2)^2 d\lambda(z) < \infty.$$

Moreover, the following asymptotic relationship holds

$$\|f\|_{B_\nu}^2 \asymp I.$$

By Lemma 1, the unit ball $B_{F_\alpha}$ of $F_\alpha$ is a normal family, a standard argument from Proposition 3.11 in [5] yields the proof of the next lemma.

Lemma 5. Let $\nu$ be a normal weight, $\alpha > 0$, $n \in \mathbb{N} \cup \{0\}$ and $\varphi$ a holomorphic self-map of $\mathbb{D}$. Then $D^n_\varphi : F_\alpha \to B_\nu$ is compact if and only if for any bounded sequence $\{f_m\}_{m \in \mathbb{N}}$ in $F_\alpha$ converging to zero on compact subsets of $\mathbb{D}$, we have that $\lim_{m \to \infty} \|D^n_\varphi f_m\|_{B_\nu} = 0$. 

Theorem 6. Let $\nu$ be a normal weight, $\alpha > 0$, $n \in \mathbb{N} \cup \{0\}$, $\varphi$ a holomorphic self-map of $\mathbb{D}$, $d\lambda(z) = dA(z)/(1 - |z|^2)^2$ and $D^n_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded. Then the following statements are equivalent:

1. $D^n_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is compact.

2. $M_3 := \sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} \nu^2(z)(1 - |\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) < \infty$

and

$$\lim_{r \to 1} \sup_{\varphi(z) \in \partial \mathbb{D}} \sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} \nu^2(z) (1 - |\eta_a(z)|^{2(n+\alpha+1)}) (1 - |\varphi(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) = 0. \quad (2.13)$$

Proof. (1) $\Rightarrow$ (2). Since $D^n_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded, for $f(z) = z^n/n! \in \mathcal{F}_\alpha$, we get

$$M_3 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \nu^2(z)(1 - |\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) < \infty.$$ 

Let $f_m(z) = z^m$, $m \in \mathbb{N}$. It is a norm bounded sequence in $\mathcal{F}_\alpha$ converging to zero uniformly on compact subsets of $\mathbb{D}$. Hence by Lemma 2, it follows that $\|D^n_{\varphi_m}f_m\|_{\mathcal{B}_\nu} \to 0$ as $m \to \infty$. Thus for every $\epsilon > 0$, there is an $m_0 \in \mathbb{N}$ such that for $m \geq m_0$, we have

$$\left(\prod_{j=0}^{n} (m - j)\right)^2 \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\varphi(z)|^{2(m-n-1)} \nu^2(z)(1 - |\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) < \epsilon. \quad (2.14)$$

From (2.14), we have that for each $r \in (0, 1)$

$$r^{2(m-n-1)} \left(\prod_{j=0}^{n} (m - j)\right)^2 \sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} \nu^2(z)(1 - |\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) < \epsilon. \quad (2.15)$$

Hence for $r \in \left[\prod_{j=0}^{n} (m - j)^{-\frac{1}{m-n-1}}, 1\right)$, we have

$$\sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} \nu^2(z)(1 - |\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) < \epsilon. \quad (2.16)$$

Let $f \in B_{\mathcal{F}_\alpha}$ and $f_t(z) = f(tz)$, $0 < t < 1$. Then $\sup_{0 < t < 1} \|f_t\|_{\mathcal{F}_\alpha} \leq \|f\|_{\mathcal{F}_\alpha}$, $f_t \in \mathcal{F}_\alpha$, $t \in (0, 1)$ and $f_t \to f$ uniformly on compact subsets of $\mathbb{D}$ as $t \to 1$. The compactness of $D^n_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ implies that $\lim_{t \to 1} \|D^n_{\varphi_t}f_t - D^n_{\varphi}f\|_{\mathcal{B}_\nu} = 0$. Hence for every $\epsilon > 0$, there is a $t \in (0, 1)$ such that

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_t^{(n+1)}(\varphi(z)) - f^{(n+1)}(\varphi(z))|^2 \nu^2(z)(1 - |\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) < \epsilon. \quad (2.17)$$

By inequalities (2.16) and (2.17), we have

$$\sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} |f_t^{(n+1)}(\varphi(z))|^2 \nu^2(z)(1 - |\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z)$$

$$\leq 2 \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_t^{(n+1)}(\varphi(z)) - f^{(n+1)}(\varphi(z))|^2 \nu^2(z) (1 - |\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z)$$

$$+ 2 \sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} |f_t^{(n+1)}(\varphi(z))|^2 \nu^2(z)(1 - |\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z)$$

$$\leq 2\varepsilon (1 + \|f_t^{(n+1)}\|_\infty^2).$$
Hence for every \( f \in B_{F_\alpha} \), there is a \( \delta_0 \in (0,1) \), \( \delta_0 = \delta_0(f,\epsilon) \), such that for \( r \in (\delta_0,1) \)
\[
\sup_{a \in \mathbb{D}} \int_{|\varphi(z)|>r} |f^{(n)}(\varphi(z))|^2 \nu^2(z)(1 - |\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z) < \epsilon.
\]

From the compactness of \( D_\varphi^n : \mathcal{F}_\alpha \to B_\nu \), we have that for every \( \epsilon > 0 \) there is a finite collection of functions \( f_1, f_2, \ldots, f_k \in B_{F_\alpha} \) such that for each \( f \in B_{F_\alpha} \), there is a \( j \in \{1, 2, \ldots, k\} \) such that
\[
\sup_{a \in \mathbb{D}} \int_{|\varphi(z)|>r} |f^{(n+1)}(\varphi(z)) - f_j^{(n+1)}(\varphi(z))|^2 \nu^2(z)(1 - |\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z) < \epsilon. \tag{2.18}
\]

On the other hand, from (2.18) it follows that if \( \delta := \max_{1 \leq j \leq k} \delta_j(f_j,\epsilon) \), then for \( r \in (\delta,1) \) and all \( j \in \{1, 2, \ldots, k\} \) we have
\[
\sup_{a \in \mathbb{D}} \int_{|\varphi(z)|>r} |f^{(n+1)}(\varphi(z))|^2 \nu^2(z)(1 - |\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z) < \epsilon. \tag{2.19}
\]

From (2.18) and (2.19), we have that for \( r \in (\delta,1) \) and every \( f \in B_{F_\alpha} \),
\[
\sup_{a \in \mathbb{D}} \int_{|\varphi(z)|>r} |f^{(n+1)}(\varphi(z))|^2 \nu^2(z)(1 - |\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z) < 4\epsilon. \tag{2.20}
\]

Applying (2.20) to the functions \( f_\zeta(z) = 1/(1 - \zeta z)^\alpha \), \( \zeta \in \partial \mathbb{D} \), we obtain
\[
\sup_{\zeta \in \partial \mathbb{D}} \sup_{a \in \mathbb{D}} \int_{|\varphi(z)|>r} \nu^2(z) \frac{\nu^2(z)}{|1 - \zeta \varphi(z)|^2(n+\alpha+1)(1 - |\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z)} < 4\epsilon/(\alpha(\alpha+1) \cdots (\alpha+n))^2
\]
from which (2.13) follows.

(2) \(\Rightarrow\) (1). Assume that \( \{f_m\}_{m \in \mathbb{N}} \) is a bounded sequence in \( \mathcal{F}_\alpha \), say by \( L \), converging to 0 uniformly on compacts of \( \mathbb{D} \) as \( m \to \infty \). Then by the Weierstrass theorem, \( f_m^{(k)} \) also converges to 0 uniformly on compacts of \( \mathbb{D} \), for each \( k \in \mathbb{N} \). We need to show that \( \|D_{\varphi}^n f_m\|_{B_\nu} \to 0 \) as \( m \to \infty \). For each \( m \in \mathbb{N} \), we can find a \( \mu_m \in \mathfrak{M} \) with \( \|\mu_m\| = \|f_m\|_{\mathcal{F}_\alpha} \) such that
\[
f_m(z) = \int_{\partial \mathbb{D}} \frac{d\mu_m(\zeta)}{(1 - \zeta z)^\alpha}. \tag{2.21}
\]

Differentiating (2.21) \( n + 1 \) times, composing such obtained equation by \( \varphi \), applying Jensen’s inequality, as well as the boundedness of sequence \( \{f_m\}_{m \in \mathbb{N}} \), we obtain
\[
|f_m^{(n+1)}(\varphi(w))|^2 \leq L(\alpha(\alpha+1) \cdots (\alpha+n))^2 \int_{\partial \mathbb{D}} \frac{d|\mu_m||\zeta)}{(1 - \zeta \varphi(w)|^2(n+\alpha+1)). \tag{2.22}
\]

By the second condition in (2), we have that for every \( \epsilon > 0 \), there is an \( r_1 \in (0,1) \) such that for \( r \in (r_1,1) \), we have
\[
\sup_{\zeta \in \partial \mathbb{D}} \sup_{a \in \mathbb{D}} \int_{|\varphi(z)|>r} \nu^2(z) \frac{\nu^2(z)}{|1 - \zeta \varphi(z)|^2(n+\alpha+1)(1 - |\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z)} < \epsilon. \tag{2.23}
\]

By Lemma 2, we have
\[
\|D_{\varphi}^n f_m\|^2_{B_\nu} = |f_m^{(0)}(\varphi(0))|^2 + \sup_{a \in \mathbb{D}} \int_{|\varphi(z)| \leq r} |f_m^{(n+1)}(\varphi(z))|^2 (1 - |\eta_a(z)|^2)^2 |\varphi'(z)|^2 \nu^2(z) d\lambda(z)
+ \sup_{a \in \mathbb{D}} \int_{|\varphi(z)|>r} |f_m^{(n+1)}(\varphi(z))|^2 (1 - |\eta_a(z)|^2)^2 |\varphi'(z)|^2 \nu^2(z) d\lambda(z).
\]
Using first condition in (2), (2.23), Fubini’s theorem and the fact that
\[ |f_m^{(n)}(\varphi(0))|^2 < \varepsilon \quad \text{and} \quad \sup_{|w| \leq r} |f_m^{(n+1)}(w)|^2 < \varepsilon, \]
for sufficiently large \( m \), say \( m \geq m_0 \), we have that
\[
\|D_{m}^{n}f_m\|_{B_{\nu}}^2 \leq |f_m^{(n)}(\varphi(0))|^2 \\
+ \sup_{|\varphi(z)| \leq r} |f_m^{(n+1)}(\varphi(z))|^2 \sup_{a \in \mathbb{D}} \int_{|\varphi(z)| \leq r} (1 - |\eta_a(z)|^2)^2 |\varphi'(z)|^2 \nu^2(d\lambda(z)) \\
+ \sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} |1 - \zeta \varphi(w)|^2(2n+1)(1 - |\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z) d|\mu_m|(\zeta) \\
\leq \left(1 + M_3 + \int_{\partial \mathbb{D}} d|\mu_m|(\zeta)\right)\varepsilon \\
\leq (1 + M_3 + L)\varepsilon.
\]
Since \( \varepsilon \) is an arbitrary, the result follows by Lemma 3.

\[ \square \]

**Theorem 7.** Let \( \nu \) be a normal weight, \( \alpha > 0 \), \( n \in \mathbb{N} \cup \{0\} \) and \( \varphi \) a holomorphic self-map of \( \mathbb{D} \). Then \( D_{\varphi}^n : \mathcal{F}_\alpha \to B_{\nu,0} \) is bounded if and only if following conditions hold

\[
M_1 := \sup_{\zeta \in \partial \mathbb{D}} \sup_{z \in \mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1 - \zeta \varphi(z)|^{n+\alpha+1}} < \infty. \tag{2.24}
\]

\[
\lim_{|z| \to 1} \frac{\nu(z)|\varphi'(z)|}{|1 - \zeta \varphi(z)|^{n+\alpha+1}} = 0 \tag{2.25}
\]

for every \( \zeta \in \partial \mathbb{D} \).

**Proof.** First suppose that (2.24) and (2.25) hold. By (2.25), the integrand in (2.4) tends to zero for every \( \zeta \in \partial \mathbb{D} \), as \( |z| \to 1 \), and is dominated by the function \( f(z) = M_1 \). Thus by the Lebesgue convergence theorem, the integral in (2.4) tends to zero as \( |z| \to 1 \), implying

\[
\lim_{|z| \to 1} \nu(z)|(D_{\varphi}^n f)'(z)| = 0.
\]

Hence, for every \( f \in \mathcal{F}_\alpha \) we have that \( D_{\varphi}^n f \in B_{\nu,0} \), from which the boundedness of \( D_{\varphi}^n : \mathcal{F}_\alpha \to B_{\nu,0} \) follows. Conversely, suppose that \( D_{\varphi}^n : \mathcal{F}_\alpha \to B_{\nu,0} \) is bounded. Then \( D_{\varphi}^n f_\zeta \in B_{\nu,0} \) for every function \( f_\zeta, \zeta \in \partial \mathbb{D} \), defined in (2.8), that is

\[
\lim_{|z| \to 1} \frac{\nu(z)|\varphi'(z)|}{|1 - \zeta \varphi(z)|^{n+\alpha+1}} = 0
\]

for every \( \zeta \in \partial \mathbb{D} \). Since \( D_{\varphi}^n : \mathcal{F}_\alpha \to B_{\nu,0} \) is bounded, then \( D_{\varphi}^n : \mathcal{F}_\alpha \to B_{\nu} \) is bounded too. Thus by Theorem 1, (2.24) follows, as claimed.

\[ \square \]

**Theorem 8.** Let \( \nu \) be a normal weight, \( \alpha > 0 \), \( n \in \mathbb{N} \cup \{0\} \) and \( \varphi \) a holomorphic self-map of \( \mathbb{D} \). Then \( D_{\varphi}^n : \mathcal{F}_\alpha \to B_{\nu,0} \) is compact if and only if

\[
\lim_{|z| \to 1} \sup_{\zeta \in \partial \mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1 - \zeta \varphi(z)|^{n+\alpha+1}} = 0. \tag{2.26}
\]
Proof. By a know result (see, e.g. Lemma 1 in [12], a closed set $E$ in $B_{\nu,0}$ is compact if and only if it is bounded and satisfies

$$\lim_{|z| \to 1} \sup_{f \in E} \nu(z)|f'(z)| = 0.$$  

Thus the set $\{D^n_{\varphi}f : f \in F_\alpha, \|f\|_{F_\alpha} \leq 1\}$ has compact closure in $B_{\nu,0}$ if and only if

$$\lim_{|z| \to 1} \sup_{f \in F_\alpha, \|f\|_{F_\alpha} \leq 1} \nu(z)|D^n_{\varphi}f'(z)| = 0. \tag{2.27}$$

Let $f \in B_{F_\alpha}$, then there is a $\mu \in M$ such that $\|\mu\| = \|f\|_{F_\alpha}$ and

$$f(z) = \int_{\partial \mathcal{D}} \frac{d\mu(\zeta)}{(1 - \zeta z)^\alpha}.$$ 

Thus we easily get that for each $f \in B_{F_\alpha}$

$$\nu(z)|D^n_{\varphi}f'(z)| \leq \alpha(\alpha + 1) \cdots (\alpha + n) \|\mu\| \sup_{\zeta \in \partial \mathcal{D}} \frac{\nu(z)|\varphi'(z)|}{|1 - \overline{\zeta}\varphi(z)|^{n+\alpha+1}}$$

$$\leq \alpha(\alpha + 1) \cdots (\alpha + n) \sup_{\zeta \in \partial \mathcal{D}} \frac{\nu(z)|\varphi'(z)|}{|1 - \overline{\zeta}\varphi(z)|^{n+\alpha+1}}. \tag{2.28}$$

Using (2.26) in (2.28), we get (2.27). Hence $D^n_{\varphi} : F_\alpha \to B_{\nu,0}$ is compact. Conversely, suppose that $D^n_{\varphi} : F_\alpha \to B_{\nu,0}$ is compact. Taking the test functions in (2.8), we can easily obtain that (2.26) follows from (2.27). \qed

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References

